Handout for Lecture 4

(text from the paper [2])
Dale Miller, 15 April 2014

The inference rules for the LKF focused proof system [1] for classical logic are given in Figure 1.

Structural Rules

$$\frac{\vdash \Theta, C \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, C}\ \mathit{Store} \qquad \frac{\vdash \Theta \Uparrow N}{\vdash \Theta \Downarrow N}\ \mathit{Release}$$

$$\frac{\vdash P, \Theta \Downarrow P}{\vdash P, \Theta \Uparrow \cdot}\ \mathit{Focus} \qquad \frac{}{\vdash \neg P, \Theta \Downarrow P}\ \mathit{Id}\ (\text{literal } P)$$

Introduction of negative connectives

$$\frac{}{\vdash \Theta \Uparrow \Gamma, t^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A \quad \vdash \Theta \Uparrow \Gamma, B}{\vdash \Theta \Uparrow \Gamma, A \wedge^- B}$$

$$\frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, f^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A, B}{\vdash \Theta \Uparrow \Gamma, A \vee^- B} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A}{\vdash \Theta \Uparrow \Gamma, \forall x A}$$

Introduction of positive connectives

$$\frac{}{\vdash \Theta \Downarrow t^+} \qquad \frac{\vdash \Theta \Downarrow A \quad \vdash \Theta \Downarrow B}{\vdash \Theta \Downarrow A \wedge^+ B}$$

$$\frac{\vdash \Theta \Downarrow A_i}{\vdash \Theta \Downarrow A_1 \vee^+ A_2} \qquad \frac{\vdash \Theta \Downarrow A[t/x]}{\vdash \Theta \Downarrow \exists x A}$$

Figure 1: The focused proof system LKF for classical logic. Here, P is positive, N is negative, C is a positive formula or a negative literal, Θ consists of positive formulas and negative literals, and x is not free in Θ, Γ. Endsequents have the form $\vdash \cdot \Uparrow \Gamma$.

Sequents for LKF are divided into *negative sequents* $\vdash \Theta \Uparrow \Gamma$ and *positive sequents* $\vdash \Theta \Downarrow B$, where Θ and Γ are multisets of formulas and B is a formula. (These sequents are formally one-sided sequents: formulas on the left of $\Uparrow$ and $\Downarrow$ are *not* negated as they are in two-sided sequents.) Notice that in this focused proof system, we have reused the term "structural rule" for a different set of rules which formally contains instances of weakening (Id) and contraction ($Focus$). Notice also that in any proof that has a conclusion of the form $\vdash \cdot \Uparrow B$, the only formulas that are to the left of an $\Uparrow$ or $\Downarrow$ occurring in that proof are either positive formulas or negative literals: it is only these formulas that are weakened (in the Id rule). The only formulas contracted (in the $Focus$ rule) are positive formulas. Thus, although linear logic is not used here directly, non-atomic negative formulas are treated linearly in the sense that they are never duplicated nor weakened in an LKF proof.

Let B be a formula of first-order logic. By a *polarization* of B we mean a formula, say B', where all the propositional connectives are replaced by *polarized versions* of the same connective and where all atomic formulas are assigned either a positive or negative polarity. Thus, an occurrence of the disjunction $\vee$ is replaced by an occurrence of either $\vee^+$ or $\vee^-$; similarly with $\wedge$ and with the logical constants for true t and false f. For simplicity, we shall assume that polarization for atomic formulas is a global assignment to all atomic formulas. Properly speaking, focused proof systems contain *polarized* formulas and not simply formulas.

Theorem LKF is sound and complete for classical logic. More precisely, let B be a first order formula and let B' be a polarization of B. Then B is provable in classical logic if and only if there is an LKF proof of $\vdash \cdot \Uparrow B'$ [1].

Notice that polarization does not affect provability but it does affect the shape of possible LKF proofs. To illustrate an application of the correctness of LKF, we show how it provides a direct proof the following theorem.

Herbrand's Theorem Let B is quantifier-free formula and let $\bar{x}$ be a (non-empty) list of variables containing the free variables of B. The formula $\exists\bar{x}B$ is classically provable if and only if there is a list of substitutions $\theta_1, \ldots, \theta_m$ ($m \geq 1$), all with domain $\bar{x}$, such that the (quantifier-free) disjunction $B\theta_1 \vee \cdots \vee B\theta_m$ is provable (*i.e.*, tautologous).

Proof. The converse direction is straightforward. Thus, assume that $\exists\bar{x}B$ is provable. Let B' be the result of polarizing all occurrences of propositional connectives negatively. By the completeness of LKF, there is an LKF proof Ξ of $\vdash \exists\bar{x}B \Uparrow \cdot$. The only sequents of the form $\vdash \Theta \Uparrow \cdot$ in Ξ are such that Θ is equal to $\{\exists\bar{x}B'\} \cup \mathcal{L}$ for $\mathcal{L}$ a multiset of literals. Such a sequent can only be proved by a *Decide* rule by focusing on either a positive literal in $\mathcal{L}$ or the original formula $\exists\bar{x}B'$: in the latter case, the synchronous phase above it provides a substitution for all the variables in $\bar{x}$. One only needs to collect all of these substitutions into a list $\theta_1, \ldots, \theta_m$ and then show that the proof Ξ is essentially also a proof of $\vdash B'\theta_1 \vee^+ \cdots \vee^+ B'\theta_m \Uparrow \cdot$. QED.

References

[1] C. Liang and D. Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, 2009.

[2] D. Miller. Finding unity in computational logic. In *ACM-BCS-Visions Conference*, Apr. 2010. Available from the author's website.