Separating Functional Computation from Relations

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Introduction

Logical foundations of arithmetic usually start with a quantificational logic of relations.

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Application: We wish to extended the Abella theorem prover to have conventional notations, e.g. $(3 * x) + 2 \leq 10$, instead of

$$\exists x_1.\ \mathit{times}\ 3\ x\ x_1 \wedge \exists X_2.\ \mathit{plus}\ x_1\ 2\ x_2 \wedge \mathit{lesseq}\ x_2\ 10$$

We are willing to change the parser and proof automation, but not the logic.

Earlier approaches

- Enhance the equality theory (e.g., Troelstra) : primitive recursive functions are black-boxes and all computation instances (e.g. $23 + 756 = 779$) are added as ground equations.

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If R is an $n+1$-ary predicate such that

$$\forall \bar{x}.([\exists y.R(\bar{x}, y)] \wedge \forall y \forall z [R(\bar{x}, y) \supset R(\bar{x}, z) \supset y = z])$$

then there exists a n-ary function f_R s.t. $f_R(\bar{x}) = y$ *iff* $R(\bar{x}, y)$. Church formally wrote this using the choice operator ι:

$$\lambda x_1 \ldots \lambda x_n.\iota(\lambda y.R(x_1, \ldots, x_n, y))$$

A new design

We want a new "rule" such that :

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We want to achieve this goal in a purely logical, proof-search oriented setting. We use the following two ideas.

- A focused proof system to synthesize such rules
- A term representation that helps to translate arithmetic expression into expressions involving predicate

Focusing: a top-level perspective

- Proof-search in Gentzen's sequent calculus suffer from a great deal of non-determinancy and redundancy.
- A focused proof system guides proof construction by distinguishing between invertible and non-invertible rules.
- Such proofs contain an alternation of two phases: the negative / invertible / "don't care" phase and the positive / non-invertible / "don't know" phase.
- Focused proof systems have two kinds of sequents to build these two phases.

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2. An ambiguity of polarity arises with singletons. If $P(\cdot)$ is a singleton, then,

   $$\forall x[P(x) \supset Q(x)] \equiv \exists x[P(x) \wedge Q(x)] \equiv Q(\epsilon P)$$

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   It is then always possible to position P in the negative phase.
3. Ultimately: focusing in logic (not arithmetic) can define administrative normal forms, a term representation which can connect functions-as-constructors to functions-as-relations.

The propositional fragment

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A polarized formula P is positive if it is a positive atomic formula or its top-level logical connective is either t^+, f, $\wedge^+$, or $\vee$.

A polarized formula N is negative if it is a negative atomic formula or its top-level logical connective is either t^-, $\wedge^-$, or $\supset$.

The propositional fragment

Negative Phase Introduction Rules

$$\frac{\Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow t^+, \Theta \vdash \Delta_1 \Uparrow \Delta_2} \quad \frac{\Gamma \Uparrow \cdot \vdash B_1 \Uparrow \cdot \quad \Gamma \Uparrow \cdot \vdash B_2 \Uparrow \cdot}{\Gamma \Uparrow \cdot \vdash B_1 \wedge^- B_2 \Uparrow \cdot} \quad \frac{\Gamma \Uparrow B_1 \vdash B_2 \Uparrow \cdot}{\Gamma \Uparrow \cdot \vdash B_1 \supset B_2 \Uparrow \cdot}$$

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Positive Phase Introduction Rules

$$\frac{\Gamma \Downarrow \cdot \vdash B_1 \Downarrow \cdot \qquad \Gamma \Downarrow B_2 \vdash \cdot \Downarrow E}{\Gamma \Downarrow B_1 \supset B_2 \vdash \cdot \Downarrow E} \qquad \frac{\Gamma \Downarrow \cdot \vdash B_1 \Downarrow \cdot \qquad \Gamma \Downarrow \cdot \vdash B_2 \Downarrow \cdot}{\Gamma \Downarrow \cdot \vdash B_1 \wedge^+ B_2 \Downarrow \cdot}$$

$$\frac{\Gamma \Downarrow \cdot \vdash B_i \Downarrow \cdot}{\Gamma \Downarrow \cdot \vdash B_1 \vee B_2 \Downarrow \cdot}\ i \in \{1, 2\} \qquad \frac{\Gamma \Downarrow B_i \vdash \cdot \Downarrow E}{\Gamma \Downarrow B_1 \wedge^- B_2 \vdash \cdot \Downarrow E}\ i \in \{1, 2\}$$

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Structural rules

$$\frac{\Gamma, N \Downarrow N \vdash \cdot \Downarrow E}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow E}\ D_l \qquad \frac{C, \Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow C, \Theta \vdash \Delta_1 \Uparrow \Delta_2}\ S_l \qquad \frac{\Gamma \Uparrow P \vdash \cdot \Uparrow E}{\Gamma \Downarrow P \vdash \cdot \Downarrow E}\ R_l$$

$$\frac{\Gamma \Downarrow \cdot \vdash P \Downarrow \cdot}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P}\ D_r \qquad \frac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E}{\Gamma \Uparrow \cdot \vdash E \Uparrow \cdot}\ S_r \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \cdot}{\Gamma \Downarrow \cdot \vdash N \Downarrow \cdot}\ R_r$$

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Interlude: Bipoles

A bipole is a derivation whose conclusion and premises are all border sequents (of the form $\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E$):

$$
\cfrac{\cfrac{\cfrac{\cfrac{\Gamma, N, \mathcal{N} \Uparrow \cdot \vdash \cdot \Uparrow E}{\cdots}}{\Gamma, N \Uparrow P \vdash \cdot \Uparrow E}\; R_l}{\cfrac{\Gamma, N \Downarrow P \vdash \cdot \Downarrow E}{\cdots}}}{\cfrac{\Gamma, N \Downarrow N \vdash \cdot \Downarrow E}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow E}\; D_l}
$$

Negative phase

Positive phase

These are the synthetic inference rules.

Examples of fixed point definitions

Declare the primitive type i and constants $z : i$ and $s : i \to i$.
z, $(s\ z)$, $(s\ (s\ z))$, $(s\ (s\ (s\ z)))$ are abbreviated by $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ etc.

As a Horn clause theory

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nat z.
nat (s X) :- nat X.
plus z X X.
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As fixed point definitions

$$\text{nat} = \mu\lambda N\lambda n(n = \mathbf{0} \vee \exists n'(n = s\ n' \wedge^{+} N\ n'))$$

$$\begin{aligned} plus = & \mu\lambda P\lambda n\lambda m\lambda p.(n = \mathbf{0} \wedge^{+} m = p) \vee \\ & \exists n'\exists p'(n = s\ n' \wedge^{+} p = s\ p' \wedge^{+} P\ n'\ m\ p') \end{aligned}$$

Rules for quantification, term equality and fix-point

Typed first-order quantification rules

$$\frac{\Sigma \vdash t : \tau \qquad \Sigma : \Gamma \Downarrow [t/x]B \vdash \cdot \Downarrow E}{\Sigma : \Gamma \Downarrow \forall x_\tau . B \vdash \cdot \Downarrow E} \qquad \frac{y : \tau, \Sigma : \Gamma \Uparrow \cdot \vdash [y/x]B \Uparrow \cdot}{\Sigma : \Gamma \Uparrow \cdot \vdash \forall x_\tau . B \Uparrow \cdot}$$

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Equality rules [Girard, Schroeder-Heister]

$$\frac{\Sigma\theta : \Gamma\theta \Uparrow \Theta\theta \vdash \Delta_1\theta \Uparrow \Delta_2\theta}{\Sigma : \Gamma \Uparrow s = t, \Theta \vdash \Delta_1 \Uparrow \Delta_2} \dagger \quad \frac{}{\Sigma : \Gamma \Uparrow s = t, \Theta \vdash \Delta_1 \Uparrow \Delta_2} \ddagger \quad \frac{}{\Sigma : \Gamma \Downarrow \cdot \vdash t = t \Downarrow \cdot}$$

Provisos: (†) θ is the mgu of s and t. (‡) t and s are not unifiable.

Fixed point rules

$$\frac{\Sigma : \Gamma \Uparrow B(\mu B)\bar{t}, \Delta \vdash \cdot \Uparrow E}{\Sigma : \Gamma \Uparrow \mu B\, \bar{t}, \Delta \vdash \cdot \Uparrow E}\ unfoldL \qquad \frac{\Sigma : \Gamma \Downarrow \cdot \vdash B(\mu B)\bar{t} \Downarrow \cdot}{\Sigma : \Gamma \Downarrow \cdot \vdash \mu B\, \bar{t} \Downarrow \cdot}\ unfoldR$$

The polarity ambiguity of singleton sets

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A proof of $\Sigma : \Gamma \Uparrow \cdot \vdash \forall x.P(x) \supset Q(x) \Uparrow \cdot$ computes the value that satisfies P, starting with proving $y, \Sigma : \Gamma \Uparrow P(y) \vdash Q(y) \Uparrow \cdot$. The completed phase has the premise $\Sigma : \Gamma \Uparrow \cdot \vdash \cdot \Uparrow Q(t)$.

Example

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Using unfoldL yields

$$x, \Sigma : \Gamma \Uparrow ((\mathbf{2} = \mathbf{0} \wedge^{+} \mathbf{3} = x) \vee \exists n' \exists x' (\mathbf{2} = s\ n' \wedge^{+} x = s\ x' \wedge^{+} \mathit{plus}\ n'\ \mathbf{3}\ x')) \vdash \cdot \Uparrow (Q\ x).$$

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(1) $x, \Sigma : \Gamma \Uparrow ((\mathbf{2} = \mathbf{0} \wedge^{+} \mathbf{3} = x) \vdash \cdot \Uparrow (Q\ x)$ is proved immediately.

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The negative phase terminates with the border premise

$$\Sigma : \Gamma \Uparrow \cdot \vdash \cdot \Uparrow (Q\ \mathbf{5})$$

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Phases as abstractions

There are two challenges to making abstractions of negative phases.

1. Since there may be many paths to compute the same functional value, the premises of a negative phase may *repeat the same sequents many times*. We can identify the premises of a negative phase as set of border sequents.

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2. There are *many ways to build a negative phase* but all constructions yield the same border sequents. We will simply ignore how a phase is constructed.

This latter challenge also holds in confluent rewriting systems: after finding one path to a normal form, no other paths need to be considered.

Need for suspensions

Suspension allows some mixing of functional and symbolic computation. For example, let *times* be

$$\mu\lambda T\lambda n\lambda m\lambda p((n = \mathbf{0}\wedge^{+} p = \mathbf{0})\vee\exists n'\exists p'(n = s\ n'\wedge^{+} T\ n'\ m\ p'\wedge^{+} plus\ p'\ m\ p))$$

To prove $(0 \times (x + 1)) + y = y$, we prove the formula

$$\forall u.\ times\ \mathbf{0}\ (s\ x)\ u \supset \forall v.\ plus\ u\ y\ v \supset v = y$$

$$y, u, v, \Sigma : \cdot \Uparrow times\ \mathbf{0}\ (s\ x)\ u,\ plus\ u\ y\ v \vdash v = y \Uparrow \cdot$$

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Schedule the *times* predicate before the *plus* predicate.

Treating the *times* predicate causes the instantiation of u.

Then schedule the *plus* predicate.

Then the negative phase ends with $y, \Sigma : \cdot \Uparrow \cdot \vdash \cdot \Uparrow y = y$.

In general: Suspend *plus* and *times* if their first argument is an eigenvariable.

Suspension restrictions

$\mathcal{S}$ is defined at the mathematics level over the $(\mu B \bar{t})$ expression.

Examples

1. The μ-expression contains more than 100 symbols
2. The first term in the list $\bar{t}$ is an eigenvariable

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We need a restriction to enforce determinancy

> ($*$) *For all μ-expressions $(\mu B \bar{t})$ and for all substitutions θ defined on the eigenvariables free in that expression, if $\mathcal{S}$ holds for $(\mu B \bar{t})\theta$ then $\mathcal{S}$ holds for $(\mu B \bar{t})$.*

Suspensions during the positive phase

A suspension predicate $\mathcal{S}$ is defined only on μ-expressions.

Suspensions during the positive phase

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$\Downarrow$-sequents need a new multiset zone Ω.

$$\Gamma \Downarrow \Theta; \Omega \vdash \Delta_1 \Downarrow \Delta_2.$$

Formulas in Ω are not "stored" just "suspended".

Only the decide, release, and initial rules deal with this context. It only exists in the positive phase.

Term representation using the $\lambda\kappa$-calculus (Brock-Nannestad, Guenot & Gustafsson)

$$\begin{array}{rrcl} \textit{Terms}: & t, u & ::= & \lambda x.t \mid x\ k \mid \uparrow p \\ \textit{Values}: & p, q & ::= & x \mid \downarrow t \\ \textit{Continuations}: & k & ::= & \varepsilon \mid p :: k \mid \kappa x.t \end{array}$$

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$$

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$$
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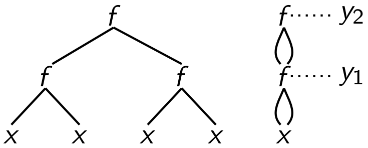
   With some syntactic sugar :

$$\lambda x_1 \dots \lambda x_n .\ \textbf{name}\ y = h\ (p_1, \dots, p_m)\ \textbf{in}\ t$$

Example: ANF and sharing



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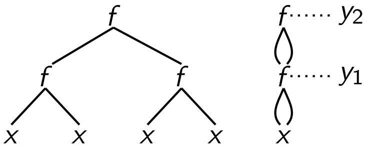

$f : i \ \rightarrow \ i \ \rightarrow \ i$ and $x : i$
When i is negative:

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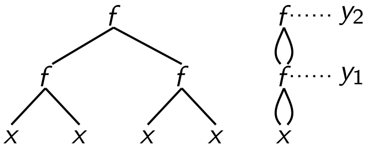

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$$f\ (f\ (x,\ x),\ f\ (x,\ x))$$

When i is positive:

$$f\ x :: x :: \kappa y_1.(f\ y_1 :: y_1 :: \kappa y_2.y_2)$$

$$\textbf{name}\ y_1 = (f\ x\ x)\ \textbf{in}\ \textbf{name}\ y_2 = (f\ y_1\ y_1)\ \textbf{in}\ y_2$$

Mixed term representations

Add the binary infix term constructor $+$ of type $i \to i \to i$.

The expression $P(2+2)$ can be presented as :

$$\textbf{name}\ u = (s\ z)\ \textbf{in}\ \textbf{name}\ v = (s\ u)\ \textbf{in}\ \textbf{name}\ x = v + v\ \textbf{in}\ P(x)$$

We now have a mix of

- uninterpreted term constructors (e.g., z and s) and
- interpreted term constructors ($+$) which will be interpreted by predicates.

Interpreting term constructors

The formal introduction of a new interpreted binary term constructor such as $+ : i \to i \to i$ must be tied to a 3-ary μ-expression R and a formal proof that R encodes a function:

$$\forall x, y([\exists z. R(x, y, z)] \wedge \forall z \forall z' [R(x, y, z) \supset R(x, y, z') \supset z = z']).$$

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$$\frac{\Sigma : \Gamma \Uparrow R_f\ \bar{x}\ y, B, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Sigma : \Gamma \Uparrow \textbf{name}\ z = f\ \bar{x}\ \textbf{in}\ B, \Theta \vdash \Delta_1 \Uparrow \Delta_2} \qquad \frac{\Sigma : \Gamma \Uparrow R_f\ \bar{x}\ y, \Theta \vdash B \Uparrow \cdot}{\Sigma : \Gamma \Uparrow \Theta \vdash \textbf{name}\ z = f\ \bar{x}\ \textbf{in}\ B \Uparrow \cdot}$$

Conclusion

$$\dfrac{\dfrac{\dfrac{\vdash Q(\mathbf{5})}{\mathit{plus}\ 2\ 3\ x \vdash Q(x)}\ \text{Negative Phase}}{\vdash \textbf{name}\ x = 2+3\ \textbf{in}\ Q(x)}\ \text{Interpret}}{\vdash Q(2+3)}\ \text{Parse/Translate}$$

Conclusion

We have presented a treatment of functional computation based on relations providing:

- a method for moving expressions denoting embedded computation into naming-combinators of the logic (ANF normal form)
- a mean of organizing introduction rules so that functional computations can be identified as one specific phase of computation (the negative phase).

Possible future work:

- Treat more datatypes than numerals; also higher-order expressions.
- Extend this project to include "functional-up-to-equivalence".
- Design this into Abella. See: LFMTP 2018 paper by Chaudhuri, Gérard, and M.

Thank you

$$\frac{y, \Sigma : \Gamma \Uparrow R_f\ \bar{x}\ y, B, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Sigma : \Gamma \Uparrow \textbf{name}\ y = f\ \bar{x}\ \textbf{in}\ B, \Theta \vdash \Delta_1 \Uparrow \Delta_2} \qquad \frac{y, \Sigma : \Gamma \Uparrow R_f\ \bar{x}\ y, \Theta \vdash B \Uparrow \cdot}{\Sigma : \Gamma \Uparrow \Theta \vdash \textbf{name}\ y = f\ \bar{x}\ \textbf{in}\ B \Uparrow \cdot}$$

$$\frac{\Sigma : \Gamma \Uparrow \cdot \vdash \textbf{name}\ x = f\ \bar{x}\ \textbf{in}\ B \Uparrow \cdot}{\Sigma : \Gamma \Downarrow \cdot \vdash \textbf{name}\ x = f\ \bar{x}\ \textbf{in}\ B \Downarrow \cdot} \qquad \frac{\Sigma : \Gamma \Uparrow \textbf{name}\ x = t\ \textbf{in}\ B \vdash \cdot \Uparrow \Delta}{\Sigma : \Gamma \Downarrow \textbf{name}\ x = t\ \textbf{in}\ B \vdash \cdot \Downarrow \Delta}$$

Figure : Introduction rules for interpreted constructors

The incorporation of the *naming* context Ψ.

Name binding rules: the variable x is not bound in Σ nor in Ψ.

$$\frac{\Sigma : x := t, \Psi; \Gamma \Uparrow B, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Sigma : \Psi; \Gamma \Uparrow \textbf{name}\ x = t\ \textbf{in}\ B, \Theta \vdash \Delta_1 \Uparrow \Delta_2} \qquad \frac{\Sigma : x := t, \Psi; \Gamma \Uparrow \cdot \vdash B \Uparrow \cdot}{\Sigma : \Psi; \Gamma \Uparrow \cdot \vdash \textbf{name}\ x = t\ \textbf{in}\ B \Uparrow \cdot}$$

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Positive phase quantifier rules

$$\frac{\Sigma, \Sigma(\Psi) \Uparrow \cdot \vdash t : \tau \Uparrow \cdot \quad \Sigma : \Psi; \Gamma \Downarrow [t/x]B \vdash \cdot \Downarrow E}{\Sigma : \Psi; \Gamma \Downarrow \forall x_\tau . B \vdash \cdot \Downarrow E}$$

$$\frac{\Sigma, \Sigma(\Psi) \Uparrow \cdot \vdash t : \tau \Uparrow \cdot \quad \Sigma : \Psi; \Gamma \Downarrow \cdot \vdash [t/x]B \Downarrow \cdot}{\Sigma : \Psi; \Gamma \Downarrow \cdot \vdash \exists x_\tau . B \Downarrow \cdot}$$