Focusing and Polarization in Linear, Intuitionistic, and Classical Logics

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Abstract

A focused proof system provides a normal form to cut-free proofs in which the application of invertible and non-invertible inference rules is structured. Within linear logic, the focused proof system of Andreoli provides an elegant and comprehensive normal form for cut-free proofs. Within intuitionistic and classical logics, there are various different proof systems in the literature that exhibit focusing behavior. These focused proof systems have been applied to both the *proof search* and the *proof normalization* approaches to computation. We present a new, focused proof system for intuitionistic logic, called *LJF*, and show how other intuitionistic proof systems can be mapped into the new system by inserting logical connectives that prematurely stop focusing. We also use *LJF* to design a focused proof system *LKF* for classical logic. Our approach to the design and analysis of these systems is based on the completeness of focusing in linear logic and on the notion of polarity that appears in Girard's LC and LU proof systems.

1. Introduction

Cut-elimination provides an important normal form for sequent calculus proofs. But what normal forms can we uncover about the structure of cut-free proofs? Since cut-free proofs play important roles in the foundations of computation, such normal forms might find a range of applications in the proof normalization foundations for functional programming or in the proof search foundations of logic programming.

1.1. About focusing

Andreoli's *focusing* proof system for linear logic (the *triadic* proof system of [1]) provides a normal form for cut-free proofs in linear logic. Although we describe this system, here called *LLF,* in more detail in Section 2, we highlight two aspect of focusing proofs here. First, linear logic connectives can be divided into the *asynchronous* connectives, whose right-introduction rules are invertible, and the *synchronous* connectives, whose right introduction rules are not (generally) invertible. The search for a focused proof can capitalize on this classification by applying (reading inference rules from conclusion to premise) all invertible rules in any order (without the need for backtracking) and by applying a chain of invertible rules that focuses on a given formula and its positive subformulas. Such a chain of applications, usually called a *focus*, terminates when it reaches an asynchronous formula. Proof search can then alternate between applications of asynchronous introduction rules and chains of synchronous introduction rules.

A second aspect of focusing proofs is that the synchronous/asynchronous classification of non-atomic formulas must be extended to atomic formulas. The arbitrary assignment of positive (synchronous) and negative (asynchronous) *bias* to atomic formulas can have a major impact on, not the existence of focused proofs, but the shape of focused proofs. For example, consider the Horn clause specification of the Fibonacci series:

$$fib(0,0) \wedge fib(1,1) \wedge \forall n \forall f \forall f'[fib(n,f) \supset fib(n+1,f') \supset fib(n+2,f+f')].$$

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If all atomic formulas are given a negative bias, then there exists only one focused proof of $\mathit{fib}(n, f_n)$: this one can be classified as a "backward chaining" proof and its size is exponential in n. On the other hand, if all atomic formulas are given a positive bias, then there is an infinite number of focused proofs all of which are classified as "forward chaining" proofs: the smallest such proof is of size linear in n.

1.2. Results

The contributions of this paper are the following. First, we introduce in Section 5 a new focusing proof system *LJF* and show that it is sound and complete for intuitionistic logic. Notable features of *LJF* are that it allows for atoms of different bias and it contains two versions of conjunction: while these conjunctions are logically equivalent, they are affected by focusing differently. Furthermore, in Section 6, we show that *LJF* satisfies cut-elimination.Second, in Section 7, we show how several other focusing proof systems can be captured in *LJF*, in the sense of *full completeness* (one-to-one correspondence between proofs in different systems). One should note that while there are many focusing proof systems for intuitionistic logic in the literature, *LJF* appears to be the first to provide a single (intuitionistic) framework for capturing many of them. Third, in Section 8, we use *LJF* to derive *LKF*, a focusing system for classical logic.

1.3. Methodology and Related work

There is a number of sequent calculus proof systems known to be complete for intuitionistic logic that exhibit characteristics of focusing. Some of these proof systems fix globally on either forward chaining or backward chaining. The early work on *uniform proofs* [2] and the *LJT* proof system [3] are both backward chaining calculi (all atoms have negative bias) while the *LJQ* calculus [3, 4] selects the global preference to be forward chaining (all atoms have positive bias). The PhD theses of Howe [5] and Chaudhuri [6] also explored various focusing proof systems for both linear and intuitionistic logics. Less has been published about systems that allow for mixing bias on atoms. While the λRCC proof system of Jagadeesan, Nadathur, and Saraswat [7] is not a focusing system explicitly, it does allow for two polarities of atoms (agents and constraints are positive and goals are negative) and for mixing both forward chaining and backward chaining in a superset of the hereditary Harrop fragment of intuitionistic logic: forward chaining is used to model constraint propagation and backward chaining is used to model goal-directed search.

We are interested in providing a flexible and unifying framework that can collect together important aspects of many of these proof systems. There are several ways to motivate and validate the design of such a system. One approach stays entirely within intuitionistic logic and works directly with invertibility and permutability of inference rules. Such an approach has been taken in many papers, such as [2, 8, 4]. Our approach uses linear logic, with its exponential operators ! and ?, as a unifying framework for looking at intuitionistic (and classical) logic. The fact that Andreoli's focused system was defined for full linear logic provides us with a convenient platform for exploring the issues around focusing and polarity. We translate intuitionistic logic into linear logic, then show that proof systems for intuitionistic logic match focused proofs of the translated image (Section 3). A crucial aspect of understanding focusing in intuitionistic logic is provided by identifying the precise relationship between Andreoli's notion of polarity with Girard's notion of polarity found in the LC [9] and LU [10] systems (see Section 4).

Another system concerning polarity and focusing is found in the work of Danos, Joinet, and Schellinx [11, 12]. Many techniques that they developed, such as *inductive decorations,* are used throughout our analysis. Our work diverges from theirs in the adaptation of Andreoli's system (LLF) as our main instrument of construction. The LK_p^η system of [12] describes focused proofs for classical logic. Its connections to polarization and focusing were further explored and extended by Laurent, Quatrini, and de Falco [13] using *polarized proof nets.* It may be tempting to speculate that the best way to arrive at a notion of intuitionistic focusing is by simple modifications to these systems, such as restricting them to single-conclusion sequents. Closer examination, however, reveals intricate issues concerning this approach. For example, the notion of classical *polarity* appears to be distinct from and *contrary* to intuitionistic polarity, especially at the level of atoms (see Sections 4 and 8). Resolving this issue would be central to finding systems that support combined forward and backward chaining. Although the relationship between LK_p^η and our systems is interesting, we chose for this work to derive intuitionistic focusing from focusing in linear logic as opposed to classical logic.

In Section 8, we show how a classical focusing calculus can then be derived from the new intuitionistic system.

Focusing proof systems have been applied in a number of settings. The earliest work on focusing in linear logic was motivated, in part, by logic programming: Andreoli's original focusing paper [1] was used to justify the design of the LO logic programming language [14]. The specialized proof systems for the Lolli [15] and Forum [16] logic programming languages were justified by the completeness of *uniform proofs*, which can be seen as a particular kinds of focused proofs in which all atoms are given negative polarity. Chaudhuri, Pfenning, and Price in [17] restrict Andreoli's focusing proof system for full linear logic to the intuitionistic fragment and then show that adopting different global bias assignment for atoms leads to either SLD-resolution or hyper-resolution on Horn clauses. The papers [3, 11, 12, 13] are motivated by foundational issues in function programming and the λ-calculus. Also, Levy [18] presents focus-style proof systems for typing in the λ-calculus and Curien and Herbelin [19] (among others) have noted the relationship between forward chaining and call-by-value evaluation and between backward chaining and call-by-name evaluation.

While this paper is mostly concerned with first-order logics, we believe that most of our results can be extended to include second-order quantification as well. Although this paper, which is an extended version of the conference paper [20], is mostly self-contained, certain proofs are not presented in full detail in order to save space: missing details can be found in [21].

2. Focusing in Linear Logic

We summarize the key results from [1] on focusing proofs for linear logic.

A *literal* is either an atomic formula or the linear negation of an atomic formula. A linear logic formula is in *negation normal form* if it does not contain occurrences of $\multimap$ and if all negations have atomic scope. If K is literal, then $K^\perp$ denotes its complement: in particular, if K is $A^\perp$ then $K^\perp$ is A.

Connectives in linear logic are either *asynchronous* or *synchronous*. The asynchronous connectives are $\bot$, $⅋$, $?$, $\top$, $\&$, and $\forall$ while the synchronous connectives are their de Morgan dual, namely, $\mathbf{1}$, $\otimes$, $!$, $\mathbf{0}$, $\oplus$, and $\exists$. Asynchronous connectives are those where the right-introduction rule is always invertible. Formally, a formula in negation normal form is of three kinds: literal, asynchronous (*i.e.*, its top-level connective is asynchronous), and synchronous (*i.e.*, its top-level connective is synchronous).

As mentioned in Section 1.1, the classification of non-atomic formulas as asynchronous or synchronous is pushed to literals by assigning a fixed but arbitrary *bias* to atoms: an atom given a *negative bias* is linked to asynchronous behavior while an atom given *positive bias* is linked to synchronous behavior. In Andreoli's original presentation of focusing [1], all atoms were classified as "positive" and their negations "negative." Girard made a similar assignment for LC [9]. In a classical setting, such a choice is possible since classical negation simply flips bias. In intuitionistic systems, however, a more natural treatment is to assign an arbitrary bias directly to atoms. This bias of atoms is extended to literals: negating a negative atom yields a positive literal and negating a positive atom yields a negative literal.

The focusing proof system LLF for linear logic, presented in Figure 1, contains two kinds of sequents. In the sequent $\Psi\colon \Delta \Uparrow L$, the "zones" Ψ and Δ are multisets and L is a list. This sequent encodes the usual one-sided sequent $\vdash ?\Psi, \Delta, L$ (here, we assume the natural coercion of lists into multisets). This sequent will also satisfy the invariant that requires Δ to contain only literals and synchronous formulas. In the sequent $\Psi\colon \Delta \Downarrow F$, the zone Ψ is a multiset of formulas and Δ is a multiset of literals and synchronous formulas, and F is a single formula. Notice that the bias of literals is explicitly referred to in the $[R \Uparrow]$ and initial rules: in particular, in the initial rules, the literal on the right of the $\Downarrow$ must be positive.

The following theorem, which was proved in [1], states that LLF is sound and complete for linear logic.

Theorem 1. *Let F be a formula of linear logic. The F is provable in linear logic if and only if the sequent $\cdot\colon\cdot \Uparrow F$ is provable in LLF.*

It is a simple consequence of this theorem that changes to the bias assigned to atoms does not affect provability of a linear logic formula: it can, however, affect the structure of focused proofs.

Asynchronous phase

$$\frac{\vdash \Psi\colon \Delta \Uparrow L}{\vdash \Psi\colon \Delta \Uparrow \bot, L}\ [\bot] \qquad \frac{\vdash \Psi\colon \Delta \Uparrow F, G, L}{\vdash \Psi\colon \Delta \Uparrow F \parr G, L}\ [\parr] \qquad \frac{\vdash \Psi, F\colon \Delta \Uparrow L}{\vdash \Psi\colon \Delta \Uparrow ?F, L}\ [?]$$

$$\frac{}{\vdash \Psi\colon \Delta \Uparrow \top, L}\ [\top] \qquad \frac{\vdash \Psi\colon \Delta \Uparrow F, L \quad \vdash \Psi\colon \Delta \Uparrow G, L}{\vdash \Psi\colon \Delta \Uparrow F \,\&\, G, L}\ [\&] \qquad \frac{\vdash \Psi\colon \Delta \Uparrow B[y/x], L}{\vdash \Psi\colon \Delta \Uparrow \forall x.B, L}\ [\forall]$$

$$\frac{\vdash \Psi\colon \Delta, F \Uparrow L}{\vdash \Psi\colon \Delta \Uparrow F, L}\ [R\Uparrow] \quad \text{provided that } F \text{ is not asynchronous}$$

Synchronous phase

$$\frac{}{\vdash \Psi\colon \cdot \Downarrow \mathbf{1}}\ [\mathbf{1}] \qquad \frac{\vdash \Psi\colon \Delta_1 \Downarrow F \quad \vdash \Psi\colon \Delta_2 \Downarrow G}{\vdash \Psi\colon \Delta_1, \Delta_2 \Downarrow F \otimes G}\ [\otimes] \qquad \frac{\vdash \Psi\colon \cdot \Uparrow F}{\vdash \Psi\colon \cdot \Downarrow\ !F}\ [!]$$

$$\frac{\vdash \Psi\colon \Delta \Downarrow F_1}{\vdash \Psi\colon \Delta \Downarrow F_1 \oplus F_2}\ [\oplus_l] \qquad \frac{\vdash \Psi\colon \Delta \Downarrow F_2}{\vdash \Psi\colon \Delta \Downarrow F_1 \oplus F_2}\ [\oplus_r] \qquad \frac{\vdash \Psi\colon \Delta \Downarrow B[t/x]}{\vdash \Psi\colon \Delta \Downarrow \exists x.B}\ [\exists]$$

$$\frac{\vdash \Psi\colon \Delta \Uparrow F}{\vdash \Psi\colon \Delta \Downarrow F}\ [R\Downarrow] \quad \text{provided that } F \text{ is either asynchronous or a negative literal}$$

Identity and Decide rules

$$\text{If } K \text{ a positive literal:} \quad \frac{}{\vdash \Psi\colon K^\bot \Downarrow K}\ [I_1] \qquad \frac{}{\vdash \Psi, K^\bot\colon \cdot \Downarrow K}\ [I_2]$$

$$\text{If } F \text{ is not a negative literal:} \quad \frac{\vdash \Psi\colon \Delta \Downarrow F}{\vdash \Psi\colon \Delta, F \Uparrow \cdot}\ [D_1] \qquad \frac{\vdash \Psi, F\colon \Delta \Downarrow F}{\vdash \Psi, F\colon \Delta \Uparrow \cdot}\ [D_2]$$

Figure 1: The focused proof system LLF for linear logic

3. Translating Intuitionistic Logic

The most well-known translation of intuitionistic logic into linear logic is likely the "$(\cdot)^\circ$" translation given by Girard in [22]: there, the intuitionistic implication $A \supset B$ is translated into the linear logic formula $!A \multimap B$ (atoms are also translated to atoms). Unfortunately, this translation rules out modeling certain kinds of proofs as a focused proof (in particular, backward chaining proofs) and it forces atoms to have negative polarity. For example, the intuitionistic sequent $a, a \supset b, b \supset c \vdash_I c$ has two different proofs corresponding to forward chaining from a ($\supset L$ with $a \supset b$) and backward chaining from c ($\supset L$ with $b \supset c$). Using the above translation on this intuitionistic sequent yields the focused sequent $a^\bot, !a \otimes b^\bot, !b \otimes c^\bot; c \Uparrow \cdot$ for which there is only one focused proof: using $[D_2]$ with $!b \otimes c^\bot$ leads to a proof (corresponding to backward chaining) while using $[D_2]$ on $!a \otimes b^\bot$ leads to a failed proof attempt. This translation of intuitionistic logic into linear logic is closely related to the LJT proof system for backward chaining proof search (see Section 7).

The translation we will use to derive a unified focusing system for intuitionistic logic (called LJF in Section 5) is inspired by the polarized translation of LU. For the purpose of providing completeness proofs later on, we first present a translation, given in Table 1, that maps unfocused LJ proofs to focused linear logic proofs. As is customary, intuitionistic negation is defined as $A \supset \mathit{false}$. The translation induces a bijection between arbitrary LJ proofs and LLF proofs of the translated image in the following sense. First notice that this translation is *asymmetric*: the intuitionistic formula A is translated using A^1 if it occurs on the right-side of an LJ sequent and as A^0 if it occurs on the left-side. Since this translation is used to capture cut-free proofs, such distinctions are not problematic. Since the left-hand side of a sequent in LJ will be negated when translated to a one-sided linear logic sequent, $(B^0)^\bot$ is also shown.

The liberal use of ! in this translation *throttles* focusing in the following sense. In contrast to unfocused sequent calculus where inference rules are applied independently of each other, focused proof systems organize

B	B^1	B^0	$(B^0)^\perp$
atom Q	Q	Q	$Q^\perp$
true	1	$\top$	0
false	0	0	$\top$
$P \wedge Q$	$!(P^1 \,\&\, Q^1)$	$!P^0 \,\&\, !Q^0$	$?(P^0)^\perp \oplus ?(Q^0)^\perp$
$P \vee Q$	$!P^1 \oplus !Q^1$	$!P^0 \oplus !Q^0$	$?(P^0)^\perp \,\&\, ?(Q^0)^\perp$
$P \supset Q$	$!(?(P^0)^\perp \parr Q^1)$	$!P^1 \multimap !Q^0$	$!P^1 \otimes ?(Q^0)^\perp$
$\exists x P$	$\exists x !P^1$	$\exists x !P^0$	$\forall x ?(P^0)^\perp$
$\forall x P$	$!\forall x P^1$	$\forall x !P^0$	$\exists x ?(P^0)^\perp$

Table 1: The 0/1 translation used to encode LJ proofs into linear logic.

a proof into alternating phases of asynchronous and synchronous rules. Focusing on the formula $!P$, however, forces the immediate release of the focus and returns the sequent to an unfocused sequent. Furthermore, the translation of every formula B^1 is synchronous or atomic at the outer-most layer. These characteristics of the translation ensures that a focused proof is always returned to an unfocused sequent after the introduction of each corresponding intuitionistic connective, thus mimicking an unfocused sequent calculus.

Although nominally a multiset, the unbounded context of a LLF sequent is, in fact, treated additively. In mapping intuitionistic logic to linear logic, the left-hand side contexts of sequents are also treated additively. The contexts never decrease from conclusion to premise. We will not be able to directly account for certain known optimizations for LJ proofs: for example, the left introduction rule for implication can be optimized so that the introduced implication is maintained in one premise but not the other (*c.f.*, [23]). However, as our translation will remain faithful to intuitionistic provability, one can expect other refinements to remain admissible in the new intuitionistic sequent calculus that we derive. We therefore consider such optimizations as an orthogonal issue.

Figure 2 provides a sequent calculus proof system for intuitionistic logic in which the left-hand-side of the sequent arrow is a set of formulas. In each left-introduction rule, the principal formula is assumed to also be a side-formula: in this way, the principle formula is retained in all premises. This variant of LJ bares close resemblances to the "*G3i*" calculus [24] particularly since contraction and weakening are not explicit inference rules. In Figure 2, the additive version of $\wedge L$ is used instead of the following multiplicative rule

$$\frac{A, B, \Gamma \vdash_I R}{A \wedge B, \Gamma \vdash_I R}\ \wedge L.$$

Such variations are common in intuitionistic calculi. One of the results of this paper is the further clarification of these variations. We will later discuss when one version of $\wedge$ might be preferable to the other and eventually offer a proof system that includes both.

The following proposition relates focused proofs under the 0/1 translation with intuitionistic proofs. Note that since R^1 is always synchronous or atomic, $\vdash (\Gamma^0)^\perp :\Uparrow R^1$ is interchangeable with $\vdash (\Gamma^0)^\perp : R^1 \Uparrow$.

Proposition 2. *Let $(\Gamma^0)^\perp$ be the multiset $\{(D^0)^\perp \mid D \in \Gamma\}$. The focused proofs of $\vdash (\Gamma^0)^\perp :\Uparrow R^1$ are in bijective correspondence with the proofs of $\Gamma \vdash_I R$.*

Proof A bijective mapping between proofs can be described by a simple recursion on structure of proofs. We show the following representative cases.

$$\frac{}{A,\Gamma \vdash_I A}\ ID,\ A \text{ atomic} \qquad \frac{}{false,\Gamma \vdash_I R}\ falseL \qquad \frac{}{\Gamma \vdash_I true}\ trueR$$

$$\frac{A_i,\Gamma \vdash_I R}{A_1 \wedge A_2,\Gamma \vdash_I R}\ \wedge L \qquad \frac{\Gamma \vdash_I A \quad \Gamma \vdash_I B}{\Gamma \vdash_I A \wedge B}\ \wedge R$$

$$\frac{A,\Gamma \vdash_I R \quad B,\Gamma \vdash_I R}{A \vee B,\Gamma \vdash_I R}\ \vee L \qquad \frac{\Gamma \vdash_I A_i}{\Gamma \vdash_I A_1 \vee A_2}\ \vee R$$

$$\frac{\Gamma \vdash_I A \quad B,\Gamma \vdash_I R}{A \supset B,\Gamma \vdash_I R}\ \supset L \qquad \frac{\Gamma, A \vdash_I B}{\Gamma \vdash_I A \supset B}\ \supset R$$

$$\frac{A[t/x],\Gamma \vdash_I R}{\forall x A,\Gamma \vdash_I R}\ \forall L \qquad \frac{\Gamma \vdash_I A}{\Gamma \vdash_I \forall y A}\ \forall R \qquad \frac{A,\Gamma \vdash_I R}{\exists y A,\Gamma \vdash_I R}\ \exists L \qquad \frac{\Gamma \vdash_I A[t/x]}{\Gamma \vdash_I \exists x A}\ \exists R$$

Figure 2: A proof system based on "*G3i*" for intuitionistic logic. In $\forall R$ and $\exists L$, y is not free in the conclusion.

1. $((D_1 \wedge D_2)^0)^\perp = (!D_1^0 \& !D_2^0)^\perp = ?(D_1^0)^\perp \oplus ?(D_2^0)^\perp \ \in \ (\Gamma^0)^\perp$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash (\Gamma^0)^\perp, (D_i^0)^\perp : R^1 \Uparrow}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(D_i^0)^\perp}\,[?]}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D_i^0)^\perp}\,[R \Downarrow]}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D_1^0)^\perp \oplus ?(D_2^0)^\perp}\,[\oplus]}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow}\,[D_2] \qquad \longleftrightarrow \qquad \dfrac{\Gamma, D_i \vdash_I R}{\Gamma, D_1 \wedge D_2 \vdash_I R}\ \wedge L$$

2. $(G_1 \wedge G2)^1 = !(G_1^1 \,\&\, G_2^1)$.

$$\dfrac{\dfrac{\dfrac{\vdash (\Gamma^0)^\perp ; \Uparrow G_1^1 \quad \vdash (\Gamma^0)^\perp ; \Uparrow G_2^1}{\vdash (\Gamma^0)^\perp : \Uparrow G_1^1 \,\&\, G_2^1}\,[\&]}{\vdash (\Gamma^0)^\perp : \Downarrow !(G_1^1 \,\&\, G_2^1)}\,[!]}{\vdash (\Gamma^0)^\perp : !(G_1^1 \,\&\, G_2^1) \Uparrow}\,[D_1] \qquad \longleftrightarrow \qquad \dfrac{\Gamma \vdash_I G_1 \quad \Gamma \vdash_I G_2}{\Gamma \vdash_I G_1 \wedge G_2}\ \wedge R$$

3. $((G \supset D)^0)^\perp = (!G^1 \multimap !D^0)^\perp = !G^1 \otimes ?(D^0)^\perp \ \in \ (\Gamma^0)^\perp$:

$$\dfrac{\dfrac{\dfrac{\vdash (\Gamma^0)^\perp : \Uparrow G^1}{\vdash (\Gamma^0)^\perp : \Downarrow !G^1}\,[!] \quad \dfrac{\dfrac{\vdash (\Gamma^0)^\perp, (D^0)^\perp : R^1 \Uparrow}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(D^0)^\perp}\,[?]}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D^0)^\perp}\,[R \Downarrow]}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow !G^1 \otimes ?(D^0)^\perp}\,[\otimes]}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow}\,[D_2] \qquad \longleftrightarrow \qquad \dfrac{\Gamma \vdash_I G \quad \Gamma, D \vdash_I R}{\Gamma, D \supset G \vdash_I R}\ \supset L$$

4. $(D \supset G)^1 = !(!D^0 \multimap G^1) = !(G^1 \parr ?(D^0)^\perp)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash (D^0)^\perp, (\Gamma^0)^\perp : \Uparrow G^1}{\vdash (\Gamma^0)^\perp : \Uparrow G^1, ?(D^0)^\perp}\,[?]}{\vdash (\Gamma^0)^\perp : \Uparrow G^1 \parr ?(D^0)^\perp}\,[\parr]}{\vdash (\Gamma^0)^\perp : \Downarrow !(G^1 \parr ?(D^0)^\perp)}\,[!]}{\vdash (\Gamma^0)^\perp : !(G^1 \parr ?(D^0)^\perp) \Uparrow}\,[D_1] \qquad \longleftrightarrow \qquad \dfrac{D, \Gamma \vdash_I G}{\Gamma \vdash_I D \supset G}\ \supset R$$

5. $(\mathit{false}^0)^\perp = 0^\perp = \top \ \in \ (\Gamma^0)^\perp$:

$$\dfrac{\dfrac{\dfrac{}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow \top}\,[\top]}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow \top}\,[R \Downarrow]}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow}\,[D_2] \qquad \longleftrightarrow \qquad \dfrac{}{\Gamma, \mathit{false} \vdash_I R}\ \mathit{falseL}$$

F	F^q (right)	F^j (left)
atom C	C	C
false	0	0
$A \wedge B$	$A^q \otimes B^q$	$!A^j \otimes !B^j$
$A \vee B$	$A^q \oplus B^q$	$!A^j \oplus !B^j$
$A \supset B$	$(!A^j \multimap B^q) \otimes 1$	$A^q \multimap !B^j$

Figure 3: The j/q translation for LJQ'.

In each case the focus arrows of the remaining premises of the LLF proofs always point upwards ($\Uparrow$), indicating a termination of focus. The translation ensures that all formulas G^1 are synchronous or atomic, and thus $\vdash (\Gamma^0)^\perp :\Uparrow G^1$ is provable if and only if $\vdash (\Gamma^0)^\perp : G^1 \Uparrow$ is provable (via the reaction rule $R\Uparrow$). Thus the inductive hypothesis applies to the remaining premises. Every right rule of LJ is initiated by a D_1 decision rule and every left rule is initiated by a D_2 rule.

The greatly constrained choices of inference rules in the focused proof system can be used to establish that this mapping is bijective. □

An important characteristic of the 0/1 translation is that the bias of atoms does not affect the structure of proofs: bias of atoms play a greater role in translations that insert fewer !'s. We also note that this translation procedure remains bijective even in the presence of 0. Even a proof such as

$$\frac{\mathit{false} \vdash_I A \quad \mathit{false}, B \vdash_I C}{\mathit{false}, A \supset B \vdash_I C} \supset L$$

has *exactly one* corresponding focused proof. The ! exponential forces contexts to be split in the "right way," preventing occurrences of multiple-conclusion sequents even when 0 is present in the context (see case 3 above). Dealing with 0 will cause some problem in other translations (see, for example, Lemma 5).

The 0/1 translation forms a starting point in establishing the completeness of other proof systems. These systems can be seen as *induced* from alternative translations of intuitionistic logic. Consider, for example, the LJQ' proof system presented in [4]. A translation to LLF proofs for this system is given in Figure 3. The linear implication $P \multimap Q$ is defined as the usual $P^\perp \parr Q$. *All atoms must be given positive bias* for this translation. Notice that the translation for implication on the left-hand side is of the form $A \multimap !B$, which is complementary to Girard's original translation: $!A \multimap B$. The "$\otimes 1$" device is a way to control the structure of focusing proofs without affecting provability. Another consequence of having only positive atoms is that conjunction is translated using $\otimes$ instead of &. We shall return to LJQ' in Section 7, in the context of a generalized focusing framework for intuitionistic logic.

With minor changes, Girard's original translation of intuitionistic logic in [22] induces the complement to LJQ' called LJT [3] (which was derived from LKT [11]). As already noted, all atoms must be given negative bias for this translation. For a more extended treatment of LJT, see [21].

Given a translation such as that of LJQ', one can give a completeness proof for the system using a *"grand tour"* through linear logic as follows:

1. Show that a proof under the 0/1 translation can be converted into a proof under the new translation. This usually follows from cut-elimination in linear logic.
2. Define a mapping between proofs in the new system (such as LJQ') and LLF proofs of its translation.
3. Show soundness of the new system with respect to LJ. This is usually trivial. The "tour" is now complete, since proofs in LJ map to proofs under the 0/1 translation.

Figure 4 further illustrates the strategy. The label l/r represents the translation for some arbitrary intuitionistic sequent system, which is indicated by $\vdash_O$. The arrow on the left-hand-side depicts Proposition 2; the top arrow depicts Step 1, the right-hand-side rule depicts Step 2, and the bottom arrow depicts Step 3.

$$
\begin{tikzcd}
\overset{0/1}{\vdash\Uparrow} \arrow[r] \arrow[d, leftrightarrow] & \overset{l/r}{\vdash\Uparrow} \arrow[d, leftrightarrow] \\
\vdash_I & \vdash_O \arrow[l]
\end{tikzcd}
$$

Figure 4: Grand tour through linear logic

An intuitionistic system that contains atoms of both positive and negative bias is λRCC [7]. Two special cases of the $\supset L$ rule are distinguished involving $E \supset D$ for positive atom E and $G \supset A$ for negative atom A. Each rule requires that the complementary atom (E on the left, A on the right) is present when applied, thus terminating one branch of the proof. One can translate these special cases using forms $E \multimap !D'$ and $!G' \multimap A$, respectively, in linear logic. The strategy outlined above can then be used to not only prove its completeness but also extend it with more aggressive focusing features.

Our interest here is not the construction of individual systems but the building of a unifying framework for focusing in intuitionistic logic. Such a task requires a closer examination of *polarity* and its connection to focusing.

4. Permeable Formulas and their Polarity

In order to construct a general focusing scheme for intuitionistic logic, the non-linear (exponential) aspects of proofs need special attention, especially in light of the fact that the [!] rule stops a bottom-up construction of focused application of synchronous rules (the arrow $\Downarrow$ in the conclusion flips to $\Uparrow$ in the premise).

For our purposes here, a particularly flexible way to deal with the exponentials in the translations of intuitionistic formulas is via the notion of *permeation* that is used in LU [10]. In particular, there are essentially three grades of *permeation*. The formula B is *left-permeable* if $B \equiv\, !B$, is *right-permeable* if $B \equiv\, ?B$, and *neutral* otherwise. Within sequent calculus proofs, a formula is left-permeable if it admits structural rules on the left and right-permeable if it admits structural rules on the right. An example of a left-permeable formula is $\exists x!A$. All left-permeable formulas are synchronous and all right-permeables are asynchronous. In the LU system, both the left and right sides of sequents contain two zones — one that treats formulas linearly and one that permits structural rules. A left-permeable (resp., right-permeable) formula is allowed to move between both zones on the left (right). In addition, LU introduces atoms that are inherently left or right-permeable or neutral: one can simulate LU in “regular” linear logic by translating left-permeable atoms A as $!A$ and right-permeable ones as $?A$.

To preserve the focusing characteristics of permeable atoms as positively or negatively biased atoms, we use the following asymmetrical translation. The superscript -1 indicates the left-side translation and $+1$ indicates the right-side translation:

$P^{-1} = !P$ and $P^{+1} = P$, for left-permeable atom P.
$N^{-1} = N$ and $N^{+1} = ?N$, for right-permeable atom N.
$B^{-1} = B^{+1} = B$, for neutral atom B.

The ! rule of LLF causes a loss of focus in all circumstances: the main reason we use asymmetric translations is that they can insert fewer occurrences of !. The translation of positive atoms above preserves *permeation* *on the left* while allowing for *focus on the right.* That is, left-permeable atoms can now be interpreted meaningfully as positively biased atoms in focused proofs, and dually for right-permeable atoms. Furthermore, the permeation of positive atoms is *"one-way only:"* they cannot be selected for focus again once they enter the non-linear context. In other words, this characteristic of the focusing sequent calculus allows asynchronous formulas to be fully decomposed, up to and including atoms.

Intuitionistic logic uses the left-permeable and neutral formulas and atoms. LU defines a translation for intuitionistic logic so that all synchronous formulas are left-permeable.

The final element of intuitionistic polarity is that *neutral atoms should be assigned negative bias in focused proofs.* Neutral atoms that are introduced into the left context (e.g. by a $\supset L$ rule) must immediately end that branch of the proof in an identity rule. Otherwise, the unique *stoup* is lost when multiple non-permeable atoms accumulate in the linear context.

The LU and LLF systems serve as a convenient platform for the unified characterization of polarity and focusing in all three logics. We can now understand the terminology of "positive" and "negative" formulas in each logic as follows:

Linear logic: *Positive* formulas are synchronous formulas and positively biased neutral atoms. *Negative* formulas are asynchronous formulas and negatively biased neutral atoms.

Intuitionistic logic: *Positive* formulas are left-permeable formulas (including left-permeable atoms). *Negative* formulas are asynchronous neutral formulas and negatively biased neutral atoms.

Classical logic: *Positive* formulas are left-permeable formulas. *Negative* formulas are right-permeable formulas.

5. The *LJF* Sequent Calculus

The asymmetrical translation of atoms unifies LU's notion of polarity with the positive/negative duality of LLF proofs. We can now use a LU-based translation of intuitionistic logic that will induce a new, *focused* sequent calculus for intuitionistic logic, one that is even more sensitive to the polarity of formulas than LLF.

We first make another adjustment on the LU translation of intuitionistic conjunction. Instead of using $\&$ or $\otimes$ depending on the polarities of the subformulas, we introduce two versions of the intuitionistic conjunction: $\wedge^+$ and $\wedge^-$ (Danos *et. al.* used similar connectives [12]). These connectives are equivalent in intuitionistic logic in terms of provability but differ in their impact on the structure of focused proofs.[1] The use of two conjunctions means that the top-level structure of a formula completely determines its polarity. Using two conjunctives provides some flexibility in capturing different approaches to focusing. For example, Chaudhuri [6, Section 6.4] provides a treatment of conjunction that interprets occurrences of conjunctions in an asynchronous phase as asynchronous and in a synchronous phase as synchronous: such a treatment can be captured by mapping $\wedge$ explicitly to either $\wedge^-$ or $\wedge^+$ depending the structure of the formula in which they occur.

Polarity in intuitionistic logic is defined as follows.

Definition 3. Atoms in LJF are arbitrarily divided between those that are positive and those that are negative. *Positive formulas* are of the following forms: positive atoms, true, false, $A \wedge^+ B$, $A \vee B$ and $\exists x A$. *Negative formulas* are among negative atoms, $A \wedge^- B$, $A \supset B$ and $\forall x A$.

Notice that the classification of atoms as positive and negative is done on possibly open atomic formulas since eigenvariables of the proof may appear free in them. In the most general approach to assigning bias to atoms, it might well be the case that an atom and a substitution instance of an atom can be of different

[1] The reader may notice that there can also be two versions of true. It is possible to use the linear constant $\top$ to represent true, but this is unnecessary since $\mathsf{false} \supset \mathsf{false}$ can be used instead. Insisting on dual versions of all constants is more suitable for classical logic (see Section 8).

$P^{+1} = P$, $P^{-1} = !P$	for positive atom P
$N^{+1} = N^{-1} = N$	for negative atom N
$true^{+1} = true^{-1} = 1$	$false^{+1} = false^{-1} = 0$
$(P \wedge^+ Q)^{-1} = P^{-1} \otimes Q^{-1}$	$(A \wedge^+ B)^{+1} = A^{+1} \otimes B^{+1}$
$(P \wedge^+ N)^{-1} = P^{-1} \otimes !N^{-1}$	
$(N \wedge^+ P)^{-1} = !N^{-1} \otimes P^{-1}$	$(A \wedge^- B)^{-1} = A^{-1} \,\&\, B^{-1}$
$(N \wedge^+ M)^{-1} = !N^{-1} \otimes !M^{-1}$	$(A \wedge^- B)^{+1} = A^{+1} \,\&\, B^{+1}$
$(P \vee Q)^{-1} = P^{-1} \oplus Q^{-1}$	$(A \vee B)^{+1} = A^{+1} \oplus B^{+1}$
$(P \vee N)^{-1} = P^{-1} \oplus !N^{-1}$	
$(N \vee P)^{-1} = !N^{-1} \oplus Q^{-1}$	
$(N \vee M)^{-1} = !N^{-1} \oplus !M^{-1}$	
$(P \supset B)^{+1} = P^{-1} \multimap B^{+1}$	$(A \supset B)^{-1} = A^{+1} \multimap B^{-1}$
$(N \supset B)^{+1} = !N^{-1} \multimap B^{+1}$	
$(\exists x P)^{-1} = \exists x P^{-1}$	$(\exists x A)^{+1} = \exists x A^{+1}$
$(\exists x N)^{-1} = \exists x !N^{-1}$	
$(\forall x A)^{-1} = \forall x A^{-1}$	$(\forall x A)^{+1} = \forall x A^{+1}$

Figure 5: The –1/ +1 translation of intuitionistic logic (–1 for left, +1 for right). Here, P, Q represent positive formulas; N, M represent negative formulas; and A, B represent arbitrary formulas.

polarities. It would seem that the *stability* of bias assignment under (first-order) substitution is a natural expectation but it is not a requirement for the description of focused proofs. We shall assume such stability, however, in the proof of cut-elimination in the next section. For more about possible approaches to assigning bias to atomic formulas, see [25, 26].

Our full translation of intuitionistic logic extends the –1/+1 translation of atoms. It remains asymmetrical and eliminates nearly all occurrences of ! on the right-hand side. The translation is shown in Figure 5. All positive formulas are translated into left-permeable formulas on the left-hand side. The positive formulas of intuitionistic logic (and classical logic) can be given linear treatments on the left.

The +1/–1 translation induces the *LJF* sequent calculus displayed in Figure 6. Sequents in *LJF* can be interpreted as follows:

1. The sequent $[\Gamma], \Theta \longrightarrow \mathcal{R}$ is an *unfocused sequent*. Here, Γ and Θ are both multisets and Γ contains only negative formulas and positive atoms. The symbol $\mathcal{R}$ denotes either the formula R or the "bracketed" formula $[R]$. End sequents of LJF proofs usually have the form $[], \Theta \longrightarrow R$.
2. The sequent $[\Gamma] \longrightarrow [R]$ is a special case of the previous sequent in which Θ is empty (and, hence, not written) and $\mathcal{R}$ is of the form $[R]$. Such sequents denote the end of the asynchronous phase: proof search continues with the selection of a focus.
3. The sequent $[\Gamma] \xrightarrow{A} [R]$ represents *left-focusing* on the formula A. Provability of this sequent is related to the provability of $\Gamma, A \vdash_I R$.
4. The sequent $[\Gamma] -_A\!\!\rightarrow$ represents *right-focusing* on the formula A. Provability of this sequent is related to provability of the sequent $\Gamma \vdash_I A$.

Intuitionistic focused proofs are more structured than LLF proofs since the polarities of intuitionistic logic observe stronger invariances. While the non-linear context of LLF sequents can contain both synchronous and asynchronous formulas, the translation of an intuitionistic sequent into a LLF sequent is such that positive formulas, which are asynchronous on the left, are placed in the linear context (because P^{-1} is always equivalent to $!P^{-1}$ for positive P).

Decision and Reaction Rules

$$\frac{[N,\Gamma] \xrightarrow{N} [R]}{[N,\Gamma] \longrightarrow [R]}\ Lf \qquad \frac{[\Gamma] -_P\!\!\rightarrow}{[\Gamma] \longrightarrow [P]}\ Rf \qquad \frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]}\ R_l \qquad \frac{[\Gamma] \longrightarrow N}{[\Gamma] -_N\!\!\rightarrow}\ R_r$$

$$\frac{[C,\Gamma],\Theta \longrightarrow \mathcal{R}}{[\Gamma],\Theta,C \longrightarrow \mathcal{R}}\ []_l \qquad \frac{[\Gamma],\Theta \longrightarrow [D]}{[\Gamma],\Theta \longrightarrow D}\ []_r$$

Initial Rules

$$\frac{}{[P,\Gamma] -_P\!\!\rightarrow}\ I_r,\ \text{atomic } P \qquad \frac{}{[\Gamma] \xrightarrow{N} [N]}\ I_l,\ \text{atomic } N$$

Introduction Rules

$$\frac{}{[\Gamma],\Theta, false \longrightarrow \mathcal{R}}\ falseL \qquad \frac{[\Gamma],\Theta \longrightarrow \mathcal{R}}{[\Gamma],\Theta, true \longrightarrow \mathcal{R}}\ trueL \qquad \frac{}{[\Gamma] -_{true}\!\!\rightarrow}\ trueR$$

$$\frac{[\Gamma] \xrightarrow{A_i} [R]}{[\Gamma] \xrightarrow{A_1 \wedge^- A_2} [R]}\ \wedge^- L \qquad \frac{[\Gamma],\Theta \longrightarrow A \quad [\Gamma],\Theta \longrightarrow B}{[\Gamma],\Theta \longrightarrow A\ \wedge^-\ B}\ \wedge^- R$$

$$\frac{[\Gamma],\Theta, A, B \longrightarrow \mathcal{R}}{[\Gamma],\Theta, A\ \wedge^+\ B \longrightarrow \mathcal{R}}\ \wedge^+ L \qquad \frac{[\Gamma] -_A\!\!\rightarrow \quad [\Gamma] -_B\!\!\rightarrow}{[\Gamma] -_{A\ \wedge^+ B}\!\!\rightarrow}\ \wedge^+ R$$

$$\frac{[\Gamma],\Theta, A \longrightarrow \mathcal{R} \quad [\Gamma],\Theta, B \longrightarrow \mathcal{R}}{[\Gamma],\Theta, A \vee B \longrightarrow \mathcal{R}}\ \vee L \qquad \frac{[\Gamma] -_{A_i}\!\!\rightarrow}{[\Gamma] -_{A_1 \vee A_2}\!\!\rightarrow}\ \vee R$$

$$\frac{[\Gamma] -_A\!\!\rightarrow \qquad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]}\ \supset L \qquad \frac{[\Gamma],\Theta, A \longrightarrow B}{[\Gamma],\Theta \longrightarrow A \supset B}\ \supset R$$

$$\frac{[\Gamma],\Theta, A \longrightarrow \mathcal{R}}{[\Gamma],\Theta, \exists y A \longrightarrow \mathcal{R}}\ \exists L \qquad \frac{[\Gamma] -_{A[t/x]}\!\!\rightarrow}{[\Gamma] -_{\exists x A}\!\!\rightarrow}\ \exists R \qquad \frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]}\ \forall L \qquad \frac{[\Gamma],\Theta \longrightarrow A}{[\Gamma],\Theta \longrightarrow \forall y A}\ \forall R$$

Figure 6: The Intuitionistic Sequent Calculus LJF. Here, P is positive, N is negative, C is a negative formula or positive atom, and D a positive formula or negative atom. Other formulas are arbitrary. Also, y is not free in Γ, Θ, or R.

Like LLF, a key characteristic of *LJF* is the assignment of polarity to atoms. To illustrate the effect of these assignments on the structure of focused proofs, consider the sequent $a, a \supset b, b \supset c \vdash c$ where a, b and c are atoms. This sequent can be proved either by *forward chaining* through the clause $a \supset b$, or *backward chaining* through the clause $b \supset c$. Assume that atoms a and b are assigned positive polarity and that c is assigned negative polarity. This assignment effectively adopts the forward chaining strategy, reflected in the following *LJF* proof segment (here, Γ is the multiset $\{a, a \supset b, b \supset c\}$):

$$
\dfrac{\dfrac{}{[\Gamma] -_{a}\!\!\rightarrow}\,I_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[b,\Gamma] -_{b}\!\!\rightarrow}\,I_r \qquad \dfrac{}{[b,\Gamma] \xrightarrow{c} [c]}\,I_l}{[b,\Gamma] \xrightarrow{b\supset c} [c]}\,\supset L}{[b,\Gamma] \longrightarrow [c]}\,Lf}{[\Gamma], b \longrightarrow [c]}\,[]_l}{[\Gamma] \xrightarrow{b} [c]}\,R_l}{[\Gamma] \xrightarrow{a\supset b} [c]}\,\supset L
$$

The polarities of a and c do not fundamentally affect the structure of the proof in this example, while assigning negative polarity to atom b restricts the proof to use the backward chaining strategy:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\Gamma] -_{a}\!\!\rightarrow}\,I_r \qquad \dfrac{}{[\Gamma] \xrightarrow{b} [b]}\,I_l}{[\Gamma] \xrightarrow{a\supset b} [b]}\,\supset L}{[\Gamma] \longrightarrow [b]}\,Lf}{[\Gamma] \longrightarrow b}\,[]_r}{[\Gamma] -_{b}\!\!\rightarrow}\,R_r \qquad \dfrac{}{[\Gamma] \xrightarrow{c} [c]}\,I_l}{[\Gamma] \xrightarrow{b\supset c} [c]}\,\supset L
$$

The availability of both $\wedge^-$ and $\wedge^+$ offers important design choices for proof search strategies. For example, a focused proof search of the sequent

$$[a, a \supset b, b \supset c, c \supset d, c \supset e] \longrightarrow d \wedge^- e,$$

requires the right-side asynchronous $\wedge^-$ to be decomposed immediately into d and e. Thus, the derivation of c from the first three formulas in the context cannot be shared between these subgoals. Using $d \wedge^+ e$ will delay the decomposition of the goal until it is selected for focus at another time, making a forward chaining strategy possible (and, hence, c can be added to the left-context before processing the conjunctive right-hand-side). On the other hand, using asynchronous formulas on the right-hand side is consistent with the uniform proof model of logic programming interpreters [2].

To prove the correctness of LJF we follow the "grand tour" strategy. First we show that a proof under the 0/1 translation can be transformed to a proof under the +1/–1 translation. Let $\vdash_{LL}$ represent sequents in the unfocused linear logic sequent calculus.

Proposition 4. *Let Γ, R consist of a multiset of intuitionistic formulas. Let Γ_2^{-1} consist of all positive formulas in Γ^{-1} and Γ_1^{-1} contain all negative formulas in Γ^{-1}. If $!\Gamma^0 \vdash_{LL} R^1$ is provable then $!\Gamma_1^{-1}, \Gamma_2^{-1} \vdash_{LL} R^{+1}$ is also provable.*

Proof For every positive intuitionistic formula P, the linear logic formula P^{-1} is left permeable (in particular, P^{-1} is provability equivalent to $!P^{-1}$). We now argue that for every intuitionistic formula F, we have $F^1 \vdash_{LL} F^{+1}$ and $!F^{-1} \vdash_{LL} !F^0$. The proof is by mutual induction on the size of formulas. We show the following representative cases.

1. $!(P \vee N)^{-1} \vdash_{LL} !(P \vee N)^0$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{P^{-1} \vdash_{LL} !P^{-1} \quad !P^{-1} \vdash_{LL} !P^0}{P^{-1} \vdash_{LL} !P^0}\,Cut
}{P^{-1} \vdash_{LL} !P^0 \oplus !N^0}\,\oplus R
\qquad
\dfrac{!N^{-1} \vdash_{LL} !N^0}{!N^{-1} \vdash_{LL} !P^0 \oplus !N^0}\,\oplus R
}{P^{-1} \oplus !N^{-1} \vdash_{LL} !P^0 \oplus !N^0}\,\oplus L
}{
\dfrac{!(P^{-1} \oplus !N^{-1}) \vdash_{LL} !P^0 \oplus !N^0}{!(P^{-1} \oplus !N^{-1}) \vdash_{LL} !(!P^0 \oplus !N^0)}\,!R
}\,!L
$$

Here P is positive and N negative, and the proof must appeal to the permeability of P^{-1}. The remaining premises follow from inductive hypotheses.

2. $!(A \supset B)^{-1} \vdash_{LL} !(A \supset B)^0$: we prove this case in two stages, joined by a cut:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{A^{+1} \vdash_{LL} A^{+1} \quad B^{-1} \vdash_{LL} B^{-1}}{A^{+1} \multimap B^{-1}, A^{+1} \vdash_{LL} B^{-1}}\,\multimap L
}{!(A^{+1} \multimap B^{-1}), A^{+1} \vdash_{LL} B^{-1}}\,!L
}{!(A^{+1} \multimap B^{-1}), !A^{+1} \vdash_{LL} B^{-1}}\,!L
}{!(A^{+1} \multimap B^{-1}), !A^{+1} \vdash_{LL} !B^{-1}}\,!R
}{!(A^{+1} \multimap B^{-1}) \vdash_{LL} !A^{+1} \multimap !B^{-1}}\,\multimap R
}{!(A^{+1} \multimap B^{-1}) \vdash_{LL} !(!A^{+1} \multimap !B^{-1})}\,!R
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
!B^{-1} \vdash_{LL} !B^0 \quad
\dfrac{
\dfrac{A^1 \vdash_{LL} A^{+1}}{!A^1 \vdash_{LL} A^{+1}}\,!L
}{!A^1 \vdash_{LL} !A^{+1}}\,!R
}{!A^{+1} \multimap !B^{-1}, !A^1 \vdash_{LL} !B^0}\,\multimap L
}{!A^{+1} \multimap !B^{-1} \vdash_{LL} !A^1 \multimap !B^0}\,\multimap R
}{!(!A^{+1} \multimap !B^{-1}) \vdash_{LL} !A^1 \multimap !B^0}\,!L
}{!(!A^{+1} \multimap !B^{-1}) \vdash_{LL} !(!A^1 \multimap !B^0)}\,!R
}{!(A^{+1} \multimap B^{-1}) \vdash_{LL} !(!A^1 \multimap !B^0)}\,Cut
$$

The premises are either trivially provable or follow from inductive hypotheses.

Other cases are similar. The proposition holds by cut-elimination in the linear sequent calculus. □

The next step in the completeness proof constructs a mapping between LJF proofs and LLF proofs of its image in linear logic. The possible presence of 0 causes some additional difficulty and slightly compromises the perfect bijection between proofs.

Lemma 5. *If there is no proof of* $!\Gamma^{-1} \vdash_{LL} 0$ *and if* $!\Gamma^{-1} \vdash_{LL} \Delta^{+1}$ *then* Δ *contains exactly one formula.*

Proof Proceed by contradiction. Assume there is such a proof with a non-singleton right-hand side and consider such a proof of minimal height. By inspecting the premises of each possible inference figure for the final rule of the proof, we see that in each case at least one premise will also have a non-singleton right-hand side, which contradicts the shortest-height assumption. □

Lemma 6. *If there is a proof of* $\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A^{+1}$ *then there is also a proof of* $\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1}$.

Proof If $\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A^{+1}$ is provable then, depending on whether or not A^{+1} is asynchronous, either $\vdash (\Gamma^{-1})^\perp : R^{+1} \Uparrow A^{+1}$ or $\vdash (\Gamma^{-1})^\perp : R^{+1}, A^{+1} \Uparrow$ is also provable, which by Lemma 5 means that $\vdash (\Gamma^{-1})^\perp :\Uparrow 0$ is provable. It can be ruled out that $\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A'$ is the conclusion of an initial rule for any subformula A' of A^{+1} because the translation excludes the possibility that $R^{+1} = (A')^\perp$. Thus the focusing stage of $\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A'$ will end in a $R\Downarrow$ reaction, which means that the subproofs of $\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A^{+1}$ are of the form

$$
\dfrac{\begin{array}{c}\vdots \\ \vdash (\Theta^{-1})^\perp : R^{+1}, \Delta \Uparrow A'\end{array}}{\vdash (\Theta^{-1})^\perp : R^{+1}, \Delta \Downarrow A'}\,R\Downarrow
\qquad \text{or} \qquad
\dfrac{\begin{array}{c}\vdots \\ \vdash (\Theta^{-1})^\perp : \Delta \Uparrow A'\end{array}}{\vdash (\Theta^{-1})^\perp \Delta \Downarrow A'}\,R\Downarrow
$$

By a simple observation on LLF inference rules, $(\Theta^{-1})^\perp$ is a superset of $(\Gamma^{-1})^\perp$ and thus $\vdash (\Theta^{-1})^\perp :\Uparrow 0$ ($!\Theta \vdash_{LL} 0$) is also provable. Then by cut elimination and the completeness of LLF, $\vdash (\Theta^{-1})^\perp :\Uparrow \Delta, A'$ and therefore (since all elements of Δ are synchronous or atomic) $\vdash (\Theta^{-1})^\perp : \Delta \Uparrow A'$ are provable. Thus every subproof of $\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A^{+1}$ can be replaced with a subproof with R^{+1} removed, and hence we have a proof of $\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1}$.[2] $\square$

These lemmas allow us to show that focused proofs with non-intuitionistic sequents can still be matched with sequents of the appropriate restricted format needed for intuitionistic logic (see case 4 of Proposition 7).

In formulating the correspondence between LJF proofs and LLF proofs we also diverge slightly from LLF in using multisets for both the boxed and unboxed contexts, whereas LLF uses a list for the linear context. This relaxation allows any asynchronous formula on the left or right to be decomposed first. Ignoring the order of this list is justified by the permutabilities between asynchronous inference rules and is proved easily using the *inversion lemma* of [1].

Proposition 7. *Let* $(\Gamma^{-1})^\perp$ *be the multiset* $\{(D^{-1})^\perp \mid D \in \Gamma\}$*. There is a proof of* $\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}$ *if and only if there is a proof of* $\Gamma \vdash_I R$*. Furthermore, if* Γ *is consistent (that is, if there is no proof of* $\Gamma \vdash_I false$*) then the correspondence between proofs is bijective modulo the order of asynchronous decomposition.*

Proof We show representative cases of the mapping.

1. $((A \wedge^+ B)^{-1})^\perp = (A^{-1} \otimes B^{-1})^\perp = (A^{-1})^\perp \parr (B^{-1})^\perp$:

$$\frac{[\Gamma], \Theta, A, B \longrightarrow R}{[\Gamma], \Theta, A \wedge^+ B \longrightarrow R}\ \wedge^+ L \qquad \longleftrightarrow \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp, (A^{-1})^\perp, (B^{-1})^\perp}{\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp, (A^{-1})^\perp \parr (B^{-1})^\perp}\ [\parr]$$

2. $(A \wedge^- B)^{+1} = A^{+1} \,\&\, B^{+1}$:

$$\frac{[\Gamma] \longrightarrow A \quad [\Gamma] \longrightarrow B}{[\Gamma] \longrightarrow A \wedge^- B}\ \wedge^- R \qquad \longleftrightarrow \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Uparrow A^{+1} \quad \vdash (\Gamma^{-1})^\perp :\Uparrow B^{+1}}{\vdash (\Gamma^{-1})^\perp :\Uparrow A^{+1} \,\&\, B^{+1}}\ [\&]$$

3. $(A_1 \vee A_2)^{+1} = A_1^{+1} \oplus A_2^{+1}$:

$$\frac{[\Gamma] -_{A_i}\!\!\rightarrow}{[\Gamma] -_{A_1 \vee A_2}\!\!\rightarrow}\ \vee R \qquad \longleftrightarrow \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Downarrow A_i^{+1}}{\vdash (\Gamma^{-1})^\perp :\Downarrow A_1^{+1} \oplus A_2^{+1}}\ [\oplus]$$

4. $((A \supset B)^{-1})^\perp = (A^{+1} \multimap B^{-1})^\perp = A^{+1} \otimes (B^{-1})^\perp$:

$$\frac{[\Gamma] -_A\!\!\rightarrow \qquad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]}\ \supset L \qquad \longleftrightarrow \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1} \quad \vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow (B^{-1})^\perp}{\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A^{+1} \otimes (B^{-1})^\perp}\ [\otimes]$$

   The validity of this mapping is justified by Lemmas 5 and 6. That is, if there is a proof where the $\otimes$ rule splits the context differently due to the presence of 0, then there is also a proof where it is split as above.

5. The decision and reaction rules of LJF also emulated those of LLF *except* for the following case, which corresponds to the ? rule:

$$\frac{[C, \Gamma], \Theta \longrightarrow R}{[\Gamma], \Theta, C \longrightarrow R}\ []_l \qquad \longleftrightarrow \qquad \frac{\vdash (\Gamma^{-1})^\perp, (C^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp}{\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp, ?(C^{-1})^\perp}\ [?]$$

   The $[]_l$ rule includes the permeation of positive atoms into the boxed context, where they are then locked in place. Such positive atoms have the form $(!P)^\perp = ?P^\perp$ in the LLF proof. The reaction rules allow us to keep the focused sequent calculus compact, in contrast to LU and LC. Without these rules there would be, for example, four cases of the $\wedge^+ L$ rule: one for each combination of polarities.

[2] This proof could be made simpler if a cut rule can be given for LLF that leaves the conclusion focused. We shall develop such a set of cut rules for LJF in Section 6.

□

Theorem 8. *LJF is sound and complete with respect to intuitionistic logic.*

Proof All that remains is to show the soundness of LJF with respect to LJ. One can simply observe that, if the special notations of LJF sequents are removed, then the resulting inferences are valid in intuitionistic logic. Combined with Propositions 2, 4 and 7, the grand tour through linear logic is complete. □

6. Cut Elimination in LJF

One can argue that a consequence of the completeness of focusing proofs is that logical connectives should not be viewed as single, logical constant but rather as larger groupings of connectives, all of the same polarity. When one starts an introduction phase in a synchronous phase, one does not stop or interleave any other proof search phase: thus, that entire phase acts as a single connective. If one takes this shift in considering the "size" of logical connective, one should probably next study how cut-elimination works with these larger connectives. Exactly this kind of analysis of cut-elimination in a focused proof system was done in [27] for a particular presentation of linear logic. In this section, we prove cut-elimination for LJF in the more conventional sense by showing how the cut rule permutes over individual inference rules. In such a setting, we shall need a number of different cut-rules which will help account for how a cut moves through the small steps within the different phases of a focused proof. A presentation of a "big-step" cut elimination procedure for LJF is left for future work.

Given the different forms of sequents, there can be many cut rule for *LJF*. We propose the following seven such rules. Recall that P denotes a positive formula while C denotes a negative formula or positive atom,

$$\frac{[\Gamma], \Theta \longrightarrow P \qquad [\Gamma'], \Theta', P \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}}\ Cut^{+} \qquad \frac{[\Gamma], \Theta \longrightarrow C \qquad [C, \Gamma'], \Theta' \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}}\ Cut^{-}$$

$$\frac{[\Gamma] -_{A}\!\!\rightarrow \qquad [\Gamma'], \Theta', A \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}} \Downarrow Cut^{+} \qquad \frac{[\Gamma], \Theta \longrightarrow B \qquad [\Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'], \Theta \longrightarrow [R]} \Downarrow Cut^{-}$$

$$\frac{[\Gamma] \xrightarrow{B} [P] \qquad [\Gamma'], P \longrightarrow [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\ Cut_1^{\leftarrow} \qquad \frac{[\Gamma] \longrightarrow G \qquad [G, \Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\ Cut_2^{\leftarrow}$$

$$\frac{[\Gamma] -_{C}\!\!\rightarrow \qquad [C, \Gamma'] -_{R}\!\!\rightarrow}{[\Gamma\Gamma'] -_{R}\!\!\rightarrow}\ Cut^{\rightarrow}$$

The last three cut rules retain focus in the conclusion. Similar cut-rules are shown to be admissible in *LJQ′* [4] and are used to study term-reduction systems. The rules $\Downarrow Cut^{+}$ and $\Downarrow Cut^{-}$ include the "key cases": cases where the cut formula is principal in both premises.

Note the restrictions on the forms of $Cut^{\rightarrow}$ and $Cut_1^{\leftarrow}$. These rules preserve focus in the conclusion. Consider the following variations on the $Cut^{\rightarrow}$ and $Cut_1^{\leftarrow}$ rules:

$$\frac{[A, A \supset C] \longrightarrow C \qquad [C] -_{C}\!\!\rightarrow}{[A, A \supset C] -_{C}\!\!\rightarrow} \qquad \frac{[\Gamma] \xrightarrow{N} [N] \qquad [N, M] \longrightarrow [M]}{[M, \Gamma] \xrightarrow{N} [M]}$$

Here, C is a positive atom and N and M are distinct negative atoms. Neither rule is admissible as their conclusions are not provable. In the $Cut^{\rightarrow}$ rule, both premises must be focused,[3] and in the $Cut_1^{\leftarrow}$ rule, the cut formula must be positive.

[3] a similar restriction also appears in [4].

There are, however, a number of variations that are admissible. For example:

$$\frac{[\Gamma], \Theta \longrightarrow [P] \qquad [\Gamma'], \Theta', P \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}}$$

This rule easily reduces to the Cut^+ rule by applying the $[]_r$ rule to the left premise. To avoid involving even more variations of the cut rule during cut-elimination, the *height* measurement of the subproofs of cuts can be modified to discount certain reaction rules such as $[]_r$ in this situation.

For cut-elimination involving quantifiers, a first-order substitution lemma is required:

Lemma 9. *If* $[\Gamma], \Theta \longrightarrow \mathcal{R}$ *is provable in LJF then there is also a proof of* $[\Gamma[t/x]], \Theta[t/x] \longrightarrow \mathcal{R}[t/x]$.

The proof of this lemma offers no difficulty if we assume that the assignment of polarity to atomic formulas is stable under substitution (see Section 5). Such an assignment is possible if, for example, the polarity of an atomic formula is based on its predicate head. The completeness of focusing holds, however, even when first-order substitution can change the polarities of literals. If we allow this flexibility, the lemma still holds by completeness, but currently there is not a more direct procedure to describe how a proof can be transformed in general when a negative atom becomes positive.

Theorem 10. *The rules* Cut^+, Cut^-, $\Downarrow Cut^+$, $\Downarrow Cut^-$, $Cut^{\rightharpoonup}$, $Cut_1^{\leftharpoonup}$ *and* $Cut_2^{\leftharpoonup}$ *are admissible in LJF.*

Proof By constructing a mutually-recursive cut-elimination procedure. The inductive measure is the usual lexicographical ordering on the degree of the cut formula and the (adjusted) heights of the subproofs of the cut. We show two of the "key" cases; that is, cases where both cut formulas are principal *and* where the synchronous cut formula is under focus:

$$\cfrac{\cfrac{[\Gamma] -_{A}\!\rightarrow \quad [\Gamma] -_{B}\!\rightarrow}{[\Gamma] -_{A\wedge^+ B}\!\rightarrow}\ \wedge^+ R \qquad \cfrac{[\Gamma'], \Theta', A, B \longrightarrow \mathcal{R}}{[\Gamma'], \Theta', A \wedge^+ B \longrightarrow \mathcal{R}}\ \wedge^+ L}{[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}}\ \Downarrow Cut^+$$

is replaced by

$$\cfrac{[\Gamma\Gamma'] -_{B}\!\rightarrow \qquad \cfrac{[\Gamma] -_{A}\!\rightarrow \quad [\Gamma'], \Theta', A, B \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta', B \longrightarrow \mathcal{R}}\ \Downarrow Cut^+}{[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}}\ \Downarrow Cut^+$$

For a left-focusing example:

$$\cfrac{\cfrac{[\Gamma], \Theta, A \longrightarrow B}{[\Gamma], \Theta \longrightarrow A \supset B}\ \supset R \qquad \cfrac{[\Gamma'] -_{A}\!\rightarrow \quad [\Gamma'] \xrightarrow{B} [R]}{[\Gamma'] \xrightarrow{A \supset B} [R]}\ \supset L}{[\Gamma\Gamma'], \Theta \longrightarrow [R]}\ \Downarrow Cut^-$$

is replaced by

$$\cfrac{\cfrac{[\Gamma'] -_{A}\!\rightarrow \quad [\Gamma], \Theta, A \longrightarrow B}{[\Gamma\Gamma'], \Theta \longrightarrow B}\ \Downarrow Cut^+ \qquad [\Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'\Gamma'], \Theta \longrightarrow [R]}\ \Downarrow Cut^-$$

Since the boxed context of LJF never decreases from conclusion to premise, a simple induction on cut-free LJF proofs shows that if the above inference can be made cut free then there is also a proof with the conclusion $[\Gamma\Gamma'], \Theta \longrightarrow [R]$. One can, in fact, treat the boxed contexts as sets.

In each key case, the cut is replaced by zero or more cuts with cut formulas of lower degree.

Except for the greater number of cases, the details of the rest of the cut-elimination procedure do not differ significantly from other cut-elimination procedures. In the non-key cases, the introduction of the principal formula is simply permuted beneath the cut. This process continues until a formula must be selected for focus (i.e., the cut must now be permuted above a Rf or Lf rule):

$$\dfrac{\dfrac{[\Gamma] \longrightarrow [P]}{[\Gamma] \longrightarrow P}\,[]_r \qquad [\Gamma'], P \longrightarrow [R]}{[\Gamma\Gamma'] \longrightarrow [R]}\,Cut^+ \qquad \text{or} \qquad \dfrac{[\Gamma] \longrightarrow C \quad [C,\Gamma'] \longrightarrow [R]}{[\Gamma\Gamma'] \longrightarrow [R]}\,Cut^-$$

At least one of the premises must be the conclusion of a Rf or Lf rule. If the formula selected for focus is *not* the cut formula, then it is replaced by a cut with a focused conclusion ($Cut_1^{\leftarrow}$ or $Cut_2^{\leftarrow}$). For example, if the upper-left premise of the Cut^+ above is the conclusion of a Lf rule, then the cut is replaced by

$$\dfrac{\dfrac{[\Gamma] \xrightarrow{N} [P] \quad [\Gamma'], P \longrightarrow [R]}{[\Gamma\Gamma'] \xrightarrow{N} [R]}\,Cut_1^{\leftarrow}}{[\Gamma\Gamma'] \longrightarrow [R]}\,Lf$$

Selecting the cut formula for focus will lead to a key cut ($\Downarrow Cut^-$ or $\Downarrow Cut^+$). The case involving the Lf rule is of particular interest because it embeds a contraction:

$$\dfrac{[\Gamma] \longrightarrow C \quad \dfrac{[C,\Gamma'] \xrightarrow{C} [R]}{[C,\Gamma'] \longrightarrow [R]}\,Lf}{[\Gamma\Gamma'] \longrightarrow [R]}\,Cut^-$$

This case is replaced by the following:

$$\dfrac{[\Gamma] \longrightarrow C \qquad \dfrac{[\Gamma] \longrightarrow C \quad [C,\Gamma'] \xrightarrow{C} [R]}{[\Gamma\Gamma'] \xrightarrow{C} [R]}\,Cut_2^{\leftarrow}}{[\Gamma\Gamma\Gamma'] \longrightarrow [R]}\,\Downarrow Cut^-$$

Traditionally, cut-elimination with explicit contraction has required some form of Gentzen's *mix* rule [28] (also known as the *multicut* rule). The structure of focused proofs allows us to reduce the cuts more directly. Of the two replacement cuts, the upper $Cut_2^{\leftarrow}$ has a lower height measure than the original cut and the lower cut is a *key-case* cut, which, as noted, can be reduced to other cuts with cut formulas of smaller degree. In contrast to the focused case, consider cut-elimination in an unfocused sequent calculus where we can replace

$$\dfrac{\Gamma \vdash C \quad \dfrac{C, C, \Gamma' \vdash R}{C, \Gamma' \vdash R}\,Contract}{\Gamma\Gamma' \vdash R}\,Cut \qquad \text{with the two cuts} \qquad \dfrac{\Gamma \vdash C \quad \dfrac{\Gamma \vdash C \quad C, C, \Gamma' \vdash R}{C, \Gamma\Gamma' \vdash R}\,Cut.}{\Gamma\Gamma\Gamma' \vdash R}\,Cut$$

The upper cut also has a lower height measure, but there is nothing we can say about the second cut.

To summarize the mutually recursive cut-elimination procedure, the "principal" cuts (Cut^- and Cut^+) eventually transition to either focused cuts ($Cut_1^{\leftarrow}$ and $Cut_2^{\leftarrow}$) or key cuts ($\Downarrow Cut^-$ and $\Downarrow Cut^+$). The key cuts are reduced to other key cuts of smaller degree until either none exists or the focused cut formula becomes asynchronous, which will cause a reversion back to a principal cut (via a reaction rule). The focused cuts are also permuted above introduction rules (on the focused formula) until a polarity switch reverts them back to principal cuts. The $Cut^{\rightarrow}$ rule is also used in the reduction of the two other focused cuts.

See the report [21] for the full details of the cut-elimination proof. □

7. Embedding Intuitionistic Systems in LJF

The *LJF* proof system can be used to "host" other focusing proof system for intuitionistic logic. For example, Figure 7 contains the purely negative fragment of *LJF* and only allows focusing on the left: this restriction describes only *uniform proofs* [2] for intuitionistic logic and is essentially equivalent to LJT [3].

$$\frac{N, \Gamma \xrightarrow{N} [R]}{N, \Gamma \longrightarrow [R]}\ Lf \qquad \frac{\Gamma \longrightarrow [M]}{\Gamma \longrightarrow M}\ []_r \text{ atomic } M \qquad \frac{}{\Gamma \xrightarrow{N} [N]}\ I_l \text{, atomic } N$$

$$\frac{\Gamma \xrightarrow{N_i} [R]}{\Gamma \xrightarrow{N_1 \wedge N_2} [R]}\ \wedge L^- \qquad \frac{\Gamma \longrightarrow N \quad \Gamma \longrightarrow M}{\Gamma \longrightarrow N \wedge M}\ \wedge R^- \qquad \frac{\Gamma \xrightarrow{N[t/x]} [R]}{\Gamma \xrightarrow{\forall x N} [R]}\ \forall L$$

$$\frac{\Gamma \longrightarrow N}{\Gamma \longrightarrow \forall y N}\ \forall R \qquad \frac{\Gamma \longrightarrow N \qquad \Gamma \xrightarrow{M} [R]}{\Gamma \xrightarrow{N \supset M} [R]}\ \supset L \qquad \frac{N, \Gamma \longrightarrow M}{\Gamma \longrightarrow N \supset M}\ \supset R$$

Figure 7: The Negative Fragment of LJF

Various other proof systems can be embedded into *LJF* by mapping intuitionistic formulas to intuitionistic formulas in such a way that focusing features in *LJF* are stopped by the insertion of *delay* operators. In particular, if we define $\partial^-(B) = true \supset B$ and $\partial^+(B) = true \wedge^+ B$, then B, $\partial^-(B)$, and $\partial^+(B)$ are all logically equivalent but $\partial^-(B)$ is always negative and $\partial^+(B)$ is always positive.

The following translation uses these devices to embed *LJQ′* [4] in LJF. The translation needs to stop asynchronous decomposition on both the left and right-hand side. It also needs to stop left-side focusing. However, the only left-synchronous formulas in *LJQ′* are of the form $A \supset B$, thus it suffices to stop focusing on B. The translation uses l and r labels to indicate left and right-side translations.

- atom B: $B^l = B^r = B$
- $false^l = \partial^-(false)$, $false^r = false$
- $(A \wedge B)^l = \partial^-(A^l \wedge^+ B^l)$, $(A \wedge B)^r = A^r \wedge^+ B^r$
- $(A \vee B)^l = \partial^-(A^l \vee B^l)$, $(A \vee B)^r = A^r \vee B^r$
- $(A \supset B)^l = A^r \supset \partial^+(B^l)$, $(A \supset B)^r = \partial^+(A^l \supset B^r)$

Since all formulas A^l are negative and all A^r are positive, we can embed the *LJQ′* sequent $\Gamma \Rightarrow G$ as $[\Gamma^l] \longrightarrow [G^r]$. *LJQ′* focusing sequents $\Gamma \to G$ naturally becomes $[\Gamma^l] -_{G^r}\!\!\rightarrow$. The correspondence between *LJQ′* rules and LJF derivations is sampled below:

1.

$$\frac{\Gamma \to R}{\Gamma \Rightarrow R}\ Der \qquad \longrightarrow \qquad \frac{[\Gamma^l] -_{R^r}\!\!\rightarrow}{[\Gamma^l] \longrightarrow [R^r]}\ Rf$$

2. $(A \wedge B)^r = A^r \wedge^+ B^r$:

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\ R\wedge' \qquad \longrightarrow \qquad \frac{[\Gamma^l] -_{A^r}\!\!\rightarrow \quad [\Gamma^l] -_{B^r}\!\!\rightarrow}{[\Gamma^l] -_{A^r \wedge^+ B^r}\!\!\rightarrow}\ \wedge^+ R$$

3. $(A \supset B)^r = \partial^+(A^l \supset B^r) = true \wedge^+ (A^l \supset B^r)$:

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \to A \supset B}\ R\supset' \qquad \longrightarrow \qquad \frac{\dfrac{}{[\Gamma^l] -_{true}\!\!\rightarrow}\ t \qquad \dfrac{\dfrac{\dfrac{[\Gamma^l, A^l] \longrightarrow B^r}{[\Gamma^l], A^l \longrightarrow B^r}\ []_l}{[\Gamma^l] \longrightarrow A^l \supset B^r}\ \supset R}{[\Gamma^l] -_{A^l \supset B^r}\!\!\rightarrow}\ R_r}{[\Gamma^l] -_{\partial^+(A^l \supset B^r)}\!\!\rightarrow}\ \wedge^+ R$$

4. All left rules begin with Lf ($[D_2]$) and $(A \supset B)^l = A^r \supset \partial^+(B^l) = A^r \supset (true \wedge^+ B^l)$:

$$\frac{\Gamma \to A \quad \Gamma, B \Rightarrow R}{\Gamma, A \supset B \Rightarrow R}\; L \supset'$$

becomes

$$\dfrac{\dfrac{[\Gamma^l, A^r \supset \partial^+(B^l)] -_{A^r}\!\!\rightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \supset B)^l, B^l] \longrightarrow [R^r]}{[\Gamma^l, A^r \supset \partial^+(B^l), B^l], true \longrightarrow [R^r]}\; trueL}{[\Gamma^l, A^r \supset \partial^+(B^l)], true, B^l \longrightarrow [R^r]}\; []_l}{[\Gamma^l, A^r \supset \partial^+(B^l)], true \wedge^+ B^l \longrightarrow [R^r]}\; \wedge^+ L}{[\Gamma^l, A^r \supset \partial^+(B^l)] \xrightarrow{true \wedge^+ B^l} [R^r]}\; R_l}{[\Gamma^l, A^r \supset \partial^+(B^l)] \xrightarrow{A^r \supset \partial^+(B^l)} [R^r]}\; \supset L}{[\Gamma^l, A^r \supset \partial^+(B^l)] \longrightarrow [R^r]}\; Lf$$

The correctness of the embedding is also proved by the above construction.

Proposition 11. *The system LJQ′ can be embedded inside LJF in the following sense:*

1. $\Gamma \Rightarrow G$ *is provable in LJQ′ if and only if* $[\Gamma^l] \longrightarrow [G^r]$ *is provable in LJF.*
2. $\Gamma \to G$ *is provable in LJQ′ if and only if* $[\Gamma^l] -_{G^r}\!\!\rightarrow$ *if provable in LJF*

A similar embedding can be designed to embed LJT, which must stop focusing on the right hand side. Forgoing other details[4], the mapping for the always-interesting $\supset L$ rule is be

$$\frac{\Gamma \Rightarrow A \quad \Gamma \xrightarrow{B} [R]}{\Gamma \xrightarrow{A \supset B} [R]}\; \supset L \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \supset B)^l] \longrightarrow A^r}{[\Gamma^l, (A \supset B)^l], true \longrightarrow A^r}\; trueL}{[\Gamma^l, (A \supset B)^l] \longrightarrow true \supset A^r}\; \supset R}{[\Gamma^l, (A \supset B)^l] -_{\partial^-(A^r)}\!\!\rightarrow}\; R_r \qquad [\Gamma^l, (A \supset B)^l] \xrightarrow{B^l} [R^r]}{[\Gamma^l, (A \supset B)^l] \xrightarrow{\partial^-(A^r) \supset B^l} [R^r]}\; \supset L$$

Naturally, it is possible to embed arbitrary LJ proofs as LJF proofs. The embedding is given in Table 2. The labels l/r are reused for convenience. This embedding echos the 0/1 translation of $\vdash_I$ proofs into linear logic.

F	F^l (left)	F^r (right)
atom C	C	C
false	$\partial^-(\mathit{false})$	*false*
true	$\partial^-(\mathit{true})$	*true*
$A \wedge B$	$\partial^+(A^l) \wedge^- \partial^+(B^l)$	$\partial^+(A^r \wedge^- B^r)$
$A \vee B$	$\partial^-(A^l \vee B^l)$	$\partial^-(A^r) \vee \partial^-(B^r)$
$A \supset B$	$\partial^-(A^r) \supset \partial^+(B^l)$	$\partial^+(A^l \supset B^r)$
$\exists x A$	$\partial^-(\exists x A^l)$	$\exists x \partial^-(A^r)$
$\forall x A$	$\forall x \partial^+(A^l)$	$\partial^+(\forall x A^r)$

Table 2: Embedding of LJ proofs as LJF proofs

[4]Several versions of LJT exist in literature, including versions that include full sets of connectives [5].

Proposition 12. *$\Gamma \vdash_I G$ is provable in LJ if and only if $[\Gamma^l] \longrightarrow [G^r]$ is provable in LJF.*

Proof All forms A^l are negative or atomic and all forms A^r are positive or atomic. A mapping between proofs can be constructed which implicitly provides an induction on the height of proofs.

- embedding of $\wedge R$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma] \longrightarrow [A^r]}{[\Gamma] \longrightarrow A^r}\,[]_r \quad \dfrac{[\Gamma] \longrightarrow [B^r]}{[\Gamma] \longrightarrow B^r}\,[]_r}{[\Gamma] \longrightarrow A^r \wedge^- B^r}\,\wedge^- R}{[\Gamma] -_{A^r \wedge^- B^r}\rightarrow}\,R_r \qquad \dfrac{}{[\Gamma] -_{true}\rightarrow}\,trueR}{[\Gamma] -_{(A^r \wedge^- B^r) \wedge^+ true}\rightarrow}\,\wedge^+ R}{[\Gamma] \longrightarrow [\partial^+(A^r \wedge^- B^r)]}\,Rf$$

- embedding of $\supset L$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \supset B)^l] \longrightarrow A^r}{[\Gamma^l, (A \supset B)^l], true \longrightarrow A^r}\,trueL}{[\Gamma^l, (A \supset B)^l] \longrightarrow true \supset A^r}\,\supset R}{[\Gamma^l, (A \supset B)^l] -_{\partial^-(A^r)}\rightarrow}\,R_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \supset B)^l, B^l] \longrightarrow [R^r]}{[\Gamma^l, \partial^-(A^r) \supset \partial^+(B^l)], B^l \longrightarrow [R^r]}\,[]_l}{[\Gamma^l, \partial^-(A^r) \supset \partial^+(B^l)], true, B^l \longrightarrow [R^r]}\,trueL}{[\Gamma^l, \partial^-(A^r) \supset \partial^+(B^l)], true \wedge^+ B^l \longrightarrow [R^r]}\,\wedge^+ L}{[\Gamma^l, \partial^-(A^r) \supset \partial^+(B^l)] \overset{true \wedge^+ B^l}{\longrightarrow} [R^r]}\,R_l}{\dfrac{[\Gamma^l, (A \supset B)^l] \overset{\partial^-(A^r) \supset \partial^+(B^l)}{\longrightarrow} [R^r]}{[\Gamma^l, (A \supset B)^l)] \longrightarrow [R^r]}\,Lf}\,\supset L$$

  This embedding is indeed a combination of that of LJQ' and LJT.

Other cases are similar. □

Conjunction can also be embedded with $\wedge^+$. In that case, we would have $(A \wedge B)^l = \partial^-(A^l \wedge^+ B^l)$ and $(A \wedge B)^r = \partial^-(A^r) \wedge^+ \partial^-(B^r)$.

The above proposition, along with cut elimination, also forms the basis of a completeness proof for LJF independently of its translation to linear logic.

The system λRCC presents interesting choices. In particular, it may not always be the best choice to focus maximally. Forward chaining may generate a new formula or "clause" that may need to be used multiple times. In a $\supset L$ rule on formulas $E \supset D$ where E is a positive atom, one may not wish to decompose the formula D immediately. This is accomplished in the linear translation with a !. It can also be accomplished by using formulas $E \supset \partial^+(D)$ in case D is negative, and $E \supset \partial^+(\partial^-(D))$ in case D is positive. Note that unlike the l/r translations for LJQ and LJ, these simple devices do not hereditarily alter the structure of D.

It is possible to view focusing proof systems as describing new sets of "big connectives", which are collections of either all asynchronous or all synchronous "small connectives" (*i.e.*, *true*, *false*, $\wedge^+$, $\vee$, $\exists$, $\wedge^-$, $\supset$, and $\forall$). The embedding of various proof system into LJF using the delays $\partial^+(\cdot)$ and $\partial^-(\cdot)$ can be seen as describing how the big connectives of LJF can be systematically broken into the smaller connectives described by the other focusing proof systems.

8. Embedding Classical Logic in LJF

We can use *LJF* to formulate a focused sequent calculus for classical logic that reveals the latter's constructive content in the style of LC. While it is possible to derive such a system again using linear logic, classical logic can also be embedded within intuitionistic logic using the well-known *double-negation* translations of Gödel [29], Gentzen, and Kolmogorov [30]. The translation of Kolmogorov liberally places double negations ($\sim\sim$) in formulas in a manner similar to the liberal use of ! in the 0/1 translation of intuitionistic to linear logic. The explicit use of double-negation has the effect of throttling focused proofs, similar to the role of !. The *"negative"* translations of Gödel and Gentzen use fewer double negations. In fact, negation is only used in translating the positives. A formula $\forall x \forall y A$ will still have the form $\forall x \forall y A'$ after translation. However, $\exists x \exists y A$ will have the form $\sim\forall x \sim\sim \forall y \sim A'$. The Gödel-Gentzen translation can only give us the asynchronous half of focusing but not the crucial synchronous half. Girard's *polarized* version of the double negation translation for LC approaches the problem of capturing duality in a more subtle way. The proof system LC is not, in fact, a focusing proof system. The focusing proof system for classical logic, *LKF* that we present below is essentially a focused version of LC. The system *LKF* differs also from LC in the minor sense that it uses polarized versions of all the propositional connectives (compared to the polarization of only $\wedge$ in *LJF*).

We first separate classical from intuitionistic polarity since these are different notions (see Section 4).

Definition 13. Atoms are arbitrarily classified as either positive or negative. The literal $\neg A$ has the opposite polarity of the atom A. *Positive formulas* are among positive literals, $\mathcal{T}$, $\mathcal{F}$, $A \wedge^+ B$, $A \vee^+ B$, $A \supset^+ B$ and $\exists x A$. *Negative formulas* are among negative literals, $\neg\mathcal{T}$, $\neg\mathcal{F}$, $A \wedge^- B$, $A \vee^- B$, $A \supset^- B$ and $\forall x A$. Negation $\neg A$ is defined by the de Morgan dualities for $\neg A/A$, $\wedge^+/\vee^-$, $\wedge^-/\vee^+$ and $\forall/\exists$. Negative implication $A \supset^- B$ is defined as $\neg A \vee^- B$ and $A \supset^+ B$ is defined as $\neg A \vee^+ B$. We often assume that formulas are in negation normal form; that is, formulas that do not contain implications and where negations have atomic scope.

The constants $\mathcal{T}$, $\mathcal{F}$, $\neg\mathcal{T}$ and $\neg\mathcal{F}$ are best described, respectively, as 1, 0, $\perp$ and $\top$ in linear logic. Just as we have dual versions of each connective, we also have dual versions of each identity. But this is not linear logic as the formulas are polarized *in the extreme.* The distinction between the positive and negative versions of each connective affects only the structure of proofs and not provability.

$\mathcal{A}^{\approx}$	$\mathcal{B}^{\approx}$	$(\mathcal{A} \wedge^+ \mathcal{B})^{\approx}$	$(\mathcal{A} \wedge^- \mathcal{B})^{\approx}$	$(\mathcal{A} \vee^+ \mathcal{B})^{\approx}$	$(\mathcal{A} \vee^- \mathcal{B})^{\approx}$	$(\neg\mathcal{A})^{\approx}$
A	B	$A \wedge^+ B$	$\sim(\sim A \vee \sim B)$	$A \vee B$	$\sim(\sim A \wedge^+ \sim B)$	$\sim A$
A	$\sim B$	$A \wedge^+ \sim B$	$\sim(\sim A \vee B)$	$A \vee \sim B$	$\sim(\sim A \wedge^+ B)$	$\cdot$
$\sim A$	B	$\sim A \wedge^+ B$	$\sim(A \vee \sim B)$	$\sim A \vee B$	$\sim(A \wedge^+ \sim B)$	A
$\sim A$	$\sim B$	$\sim A \wedge^+ \sim B$	$\sim(A \vee B)$	$\sim A \vee \sim B$	$\sim(A \wedge^+ B)$	$\cdot$

$\mathcal{A}^{\approx}$	$\mathcal{B}^{\approx}$	$(\mathcal{A} \supset^+ \mathcal{B})^{\approx}$	$(\mathcal{A} \supset^- \mathcal{B})^{\approx}$	$(\forall x\mathcal{A})^{\approx}$	$(\exists x\mathcal{A})^{\approx}$
A	B	$\sim A \vee B$	$\sim(A \wedge^+ \sim B)$	$\sim(\exists x \sim A)$	$\exists x A$
A	$\sim B$	$\sim A \vee \sim B$	$\sim(A \wedge^+ B)$	$\cdot$	$\cdot$
$\sim A$	B	$A \vee B$	$\sim(\sim A \wedge^+ \sim B)$	$\sim(\exists x A)$	$\exists x \sim A$
$\sim A$	$\sim B$	$A \vee \sim B$	$\sim(\sim A \wedge^+ B)$	$\cdot$	$\cdot$

Table 3: Polarized embedding of classical logic. The $(\cdot)^{\approx}$ translation on compound formulas is given above (there, A, B represent formulas not preceded by $\sim$). The logical constants are translated as $\mathcal{T}^{\approx} = \mathit{true}$, $\mathcal{F}^{\approx} = \mathit{false}$, $(\neg\mathcal{T})^{\approx} = \sim \mathit{true}$, $(\neg\mathcal{F})^{\approx} = \sim \mathit{false}$. A positive (classical) atom P is translated via $P^{\approx} = P$ and a negative (classical) atom N via $N^{\approx} = \sim N$: in both cases, when atoms P and N are considered as intuitionistic atoms, they are assigned positive polarity.

Let $\sim A$ represent the intuitionistic formula $A \supset \phi$ where ϕ is *some unspecified positive atom.* The "$\approx$" embedding of classical logic is found in Table 3. Note that the classical $\wedge^-$ is not defined in terms of the intuitionistic $\wedge^-$. The embeddings are selected to enforce the dualities $\wedge^-/\vee^+$ and $\wedge^+/\vee^-$. Variations on this embedding are also possible. The cases all follow the pattern P or $\sim P$ where P is a positive intuitionistic formula: negative intuitionistic atoms are not used in the embedding.

The $\approx$ embedding induces the *LKF* sequent calculus in Figure 8 from the image of *LJF* proofs, analogous to how *LJF* was derived from LLF. LKF sequents of the form $\vdash [\Theta], \Gamma$ are unfocused while those of the form $\mapsto [\Theta], A$ focus on the *stoup* formula A.

Translating an LKF endsequent $\vdash [], \Gamma$ into LJF involves the following steps:

1. Divide Γ into the form Δ', Ψ', where Δ' contains all positive formulas of Γ and Ψ' contains all negative formulas of Γ.

Decision, Reaction, Initial

$$\frac{\vdash [\Theta, C], \Gamma}{\vdash [\Theta], \Gamma, C}\ [] \qquad \frac{\mapsto [P, \Theta], P}{\vdash [P, \Theta]}\ Focus \qquad \frac{\vdash [\Theta], N}{\mapsto [\Theta], N}\ Release \qquad \frac{}{\mapsto [\neg P, \Theta], P}\ Id\ (\text{literal } P)$$

Asynchronous Connectives

$$\frac{}{\vdash [\Theta], \Gamma, \neg\mathcal{F}}\ absurd \qquad \frac{\vdash [\Theta], \Gamma}{\vdash [\Theta], \Gamma, \neg\mathcal{T}}\ trivial \qquad \frac{\vdash [\Theta], \Gamma, A \quad \vdash [\Theta], \Gamma, B}{\vdash [\Theta], \Gamma, A \wedge^- B}\ \wedge^-$$

$$\frac{\vdash [\Theta], \Gamma, A, B}{\vdash [\Theta], \Gamma, A \vee^- B}\ \vee^- \qquad \frac{\vdash [\Theta], \Gamma, B, \neg A}{\vdash [\Theta], \Gamma, A \supset^- B}\ \supset^- \qquad \frac{\vdash [\Theta], \Gamma, A}{\vdash [\Theta], \Gamma, \forall x A}\ \forall$$

Synchronous Connectives

$$\frac{}{\mapsto [\Theta], \mathcal{T}}\ \mathcal{T} \qquad \frac{\mapsto [\Theta], A \quad \mapsto [\Theta], B}{\mapsto [\Theta], A \wedge^+ B}\ \wedge^+ \qquad \frac{\mapsto [\Theta], A_i}{\mapsto [\Theta], A_1 \vee^+ A_2}\ \vee^+$$

$$\frac{\mapsto [\Theta], A[t/x]}{\mapsto [\Theta], \exists x A}\ \exists \qquad \frac{\mapsto [\Theta], \neg A}{\mapsto [\Theta], A \supset^+ B}\ \supset^+ \qquad \frac{\mapsto [\Theta], B}{\mapsto [\Theta], A \supset^+ B}\ \supset^+$$

Figure 8: The Classical Sequent Calculus LKF. Here, P is positive, N is negative, C is a positive formula or a negative literal, Θ consists of positive formulas and negative literals, and x is not free in Θ, Γ. Endsequents have the form $\vdash [], \Gamma$.

2. Translate Δ' and Ψ' into $\Delta'^{\approx}$ and $\Psi'^{\approx}$. We have that $\Delta'^{\approx} = \{P_1, \ldots, P_j\}$ for positive intuitionistic formulas $P_1 \ldots P_j$ and that $\Psi'^{\approx} = \{\sim Q_1, \ldots, \sim Q_k\}$ for positive intuitionistic formulas $Q_1 \ldots Q_k$.
3. Form the LJF sequent $[\Delta], \Psi \longrightarrow [\phi]$, where $\Delta = \{\sim P_1, \ldots \sim P_j\}$ and $\Psi = \{Q_1, \ldots Q_k\}$.

Proposition 14. *Let Δ, Δ', Ψ and Ψ' be as defined above.*

1. $\vdash [\Delta'], \Psi'$ *is provable if and only if* $[\Delta], \Psi \longrightarrow [\phi]$ *is provable.*
2. $\mapsto [\Delta'], P$ *is provable if and only if* $[\Delta] -_{P^{\approx}}\!\!\rightarrow$ *is provable.*

Proof We prove this proposition by providing the following mapping between LKF proofs and LJF proofs. Some representative cases of this mapping are presented.

1. an example of an asynchronous rule:

$$\frac{[\Delta], \Psi, A, B \longrightarrow [\phi]}{[\Delta], \Psi, A \wedge^+ B \longrightarrow [\phi]}\ \wedge^+ L \qquad \longleftrightarrow \qquad \frac{\vdash [\Theta], \Gamma, A, B}{\vdash [\Theta], \Gamma, A \vee^- B}\ \vee^-$$

2. an example of a synchronous rule:

$$\frac{[\Delta] -_{A_i}\!\!\rightarrow}{[\Delta] -_{A_1 \vee A_2}\!\!\rightarrow}\ \vee R \qquad \longleftrightarrow \qquad \frac{\mapsto [\Theta], A_i}{\mapsto [\Theta], A_1 \vee^+ A_2}\ \vee^+$$

3. a reaction rule, which terminates the focus:

$$\dfrac{\dfrac{\dfrac{[\Delta], P \longrightarrow [\phi]}{[\Delta], P \longrightarrow \phi}\ []_r}{[\Delta] \longrightarrow P \supset \phi}\ \supset R}{[\Delta] -_{\sim P}\!\!\rightarrow}\ R_r \qquad \longleftrightarrow \qquad \frac{\vdash [\Theta], N}{\mapsto [\Theta], N}\ Release$$

where N is classically negative and P is intuitionistically positive.

$\square$

The correctness of LKF with respect to LK can be proved by reduction to the correctness of the Gödel-Gentzen translation, which is (equivalent to) the following:

$$\begin{array}{l}
g(A) \;=\; \sim\sim A \;\text{ for atom } A\\
g(\neg A) \;=\; \sim g(A)\\
g(A \wedge B) \;=\; g(A) \wedge g(B)\\
g(A \vee B) \;=\; \sim(\sim g(A) \wedge \sim g(B))\\
g(\forall x A) \;=\; \forall x(g(A))\\
g(\exists x A) \;=\; \sim \forall x \sim g(A)
\end{array}$$

Implication is a derived connective and is not considered here. This translation has the property that if $\vdash_C F$ is provable then $\vdash_I g(F)$ is provable, where $\vdash_C$ represents classical provability.

The following notations are used to relate classical formulas with LKF formulas and with LJF formulas (the latter being annotated formulas).

Definition 15. Let $F^\bullet$ be some annotation of the classical formula F in which all occurrences of $\wedge$ are replaced by $\wedge^-$ or $\wedge^+$, all occurrences of $\vee$ are replaced by $\vee^-$ or $\vee^+$, all occurrences of *false* are replaced by $\mathcal{F}$ or $\neg\mathcal{T}$, and all occurrences of *true* are replaced by $\neg\mathcal{F}$ or $\mathcal{T}$. (Such replacement does not need to be uniform but can differ for differ occurrences.) Given an intuitionistic LJF formula G, let G° represent G with all occurrences of $\wedge^-$ and $\wedge^+$ replaced by $\wedge$.

Also, we assign all classical atoms positive polarity. Negative atoms are not strictly needed because of classical negation. The following lemma relates our translation with the Gödel-Gentzen one.

Lemma 16. *For all classical formulas F:*

1. $g(F) \;\vdash_I F^{\bullet\approx\circ}$ *if $F^\bullet$ is negative.*
2. $g(F) \;\vdash_I \sim\sim F^{\bullet\approx\circ}$ *if $F^\bullet$ is positive.*

Here, $F^{\bullet\approx\circ}$ is an unannotated intuitionistic formula that results from decorating F using $\bullet$, then translating using $\approx$ to an LJF formula, and then erasing the $+/-$ decoration on $\wedge$ using $\circ$.

Proof By induction on F. Each entry of the $\approx$ translation table needs to be verified separately. We provide two representative cases.

1. For an example of case 1, consider $F = \mathcal{A} \wedge \mathcal{B}$ such that $\mathcal{A}^\bullet$ is positive and $\mathcal{B}^\bullet$ is negative. Let $F^\bullet = (\mathcal{A}^\bullet \wedge^- \mathcal{B}^\bullet)$. We have that $F^{\bullet\approx\circ} = \sim(\sim A^\circ \vee B^\circ)$ where $\mathcal{A}^{\bullet\approx} = A$ and $\mathcal{B}^{\bullet\approx} = \sim B$. Observe the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{g(\mathcal{A}) \vdash_I \sim\sim A^\circ}{g(\mathcal{A}) \wedge g(\mathcal{B}) \vdash_I \sim\sim A^\circ}\wedge L \qquad \sim\sim A^\circ, \sim A^\circ \vdash_I \phi}{g(\mathcal{A}) \wedge g(\mathcal{B}), \sim A^\circ \vdash \phi}\,Cut \qquad \dfrac{\dfrac{g(\mathcal{B}) \vdash_I \sim B^\circ}{g(\mathcal{A}) \wedge g(\mathcal{B}) \vdash_I \sim B^\circ}\wedge L \qquad \sim B^\circ, B^\circ \vdash_I \phi}{g(\mathcal{A}) \wedge g(\mathcal{B}), B^\circ \vdash_I \phi}\,Cut}{g(\mathcal{A}) \wedge g(\mathcal{B}), (\sim A^\circ \vee B^\circ) \vdash_I \phi}\vee L}{g(\mathcal{A}) \wedge g(\mathcal{B}) \vdash_I \sim(\sim A^\circ \vee B^\circ)}\supset R$$

   The premises of the derivation follow from inductive hypotheses: case 2 for the left premise and case 1 for the right premise. Note that if $\wedge^+$ was used instead of $\wedge^-$ in the $\bullet$ decoration, the translation would match up more directly with the Gödel-Gentzen translation.
2. For an example of case 2, consider $F = \exists x\mathcal{A}$ where $\mathcal{A}^\bullet$ is positive and $\mathcal{A}^{\bullet\approx} = A$, so $F^{\bullet\approx\circ} = \exists x A^\circ$.

$g(F) = \sim\forall x \sim g(\mathcal{A})$. We have

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dots g(\mathcal{A}) \vdash_I \sim\sim A^\circ}{\dots g(\mathcal{A}) \vdash_I \exists x \sim\sim A^\circ}\,\exists R \qquad \dots \phi \vdash_I \phi}{\dots g(\mathcal{A}), \sim\exists x \sim\sim A^\circ \vdash_I \phi}\,\supset L}{\dots \sim\exists x \sim\sim A^\circ \vdash_I \sim g(\mathcal{A})}\,\supset R}{\dots \sim\exists x \sim\sim A^\circ \vdash_I \forall x \sim g(\mathcal{A})}\,\forall R \qquad \dots \phi \vdash_I \phi}{\sim\forall x \sim g(\mathcal{A}), \sim\exists x \sim\sim A^\circ \vdash_I \phi}\,\supset L}{\sim\forall x \sim g(\mathcal{A}) \vdash_I \sim\sim\exists x \sim\sim A^\circ}\,\supset R \qquad \sim\sim\exists x \sim\sim A^\circ \vdash_I \sim\sim\sim\sim\exists x A^\circ}{\sim\forall x \sim g(\mathcal{A}) \vdash_I \sim\sim\sim\sim\exists x A^\circ}\,Cut \qquad \sim\sim\sim\sim\exists x A^\circ \vdash_I \sim\sim\exists x A^\circ}{\sim\forall x \sim g(\mathcal{A}) \vdash_I \sim\sim\exists x A^\circ}\,Cut
$$

It is easily shown that $\exists x \sim\sim B \vdash_I \sim\sim\exists x B$ is provable. The remaining premise follows from inductive hypothesis (case 2).

The other cases follow these patterns. □

Theorem 17. *LKF is sound and complete with respect to classical logic.*

Proof Completeness is proved by the correctness of the Gödel-Gentzen translation, lemma 16, proposition 14 and the completeness of LJF. The soundness is trivial. □

We have constructed this embedding of classical logic as a further demonstration of the abilities of *LJF* as a hosting framework. The embedding also revealed interesting relationships between classical and intuitionistic polarity. It is also possible to derive LKF from linear logic: one needs to define each connective to be either wholly positive or negative. For example, the translation of $(A \vee^- B)^p$ is $A^p \parr B^p$ if A^p and B^p are both negative; is $A^p \parr\, ?B^p$ if only A^p is negative; is $?A^p \parr B^p$ if only B^p is negative; and is $?A^p \parr ?B^p$, if A^p and B^p are both positive. This translation is called the *"polaro"* translation in [12], where it was used to formulate LK_p^η, the first focused proof system for classical logic. Like the $\approx$ translation, the polaro translation is a derivative of the LC/LU analysis of polarity. With the same special treatment of positive and negative atoms, LKF is derivable from LLF using essentially the polaro translation in the same manner that *LJF* is derived.

LK_p^η was extended to $LK_{pol}^{\eta;\rho}$ in [13]. These systems were formulated independently of Andreoli's results. The authors of [12] opted not to present LK_p^η as a sequent calculus because they feared that it will have the cumbersome size of LU. Such cumbersomeness can, in fact, be avoided by adopting LLF-style *reaction* rules.

Given our goals, the choice in adopting Andreoli's system is justified in that LKF and LJF have the form of compact sequent calculi ready for further application and implementation. More significantly perhaps, LK_p^η and $LK_{pol}^{\eta;\rho}$ define focusing for classical logic by mapping to polarized forms of linear logic (LLP and LL_{pol}). LLF is defined for full classical linear logic. LKF can be embedded within LLF in the same way that LC is embedded within LU. Both LLF are LJF are well suited for hosting other systems.

We have shown how LKF is derivable from LJF, which is in turn derivable from LLF. There is also an unmistakable relationship between LKF and MALL, the fragment of linear logic without ! and ? (but we may consider MALL with the quantifiers). In particular, one can consider making the following modifications to LKF.

1. change the *Focus* rule to eliminate the embedded contraction:

$$\dfrac{\mapsto [\Theta], P}{\vdash [P, \Theta]}\ Focus$$

2. eliminate the context from the identity and $\mathcal{T}$ rules:

$$\dfrac{}{\mapsto [\neg P], P}\ Id,\ \text{(literal } P\text{)} \qquad \dfrac{}{\mapsto [], \mathcal{T}}\ \mathcal{T}$$

3. split the context in the $\wedge^+$ rule:

$$\frac{\mapsto [\Theta_1], A \quad \mapsto [\Theta_2], B}{\mapsto [\Theta_1\Theta_2], A \wedge^+ B} \wedge^+$$

The resulting system is isomorphic to LLF restricted to MALL. Every MALL proof is trivially a classical proof. One might argue that we could have arrived at LKF through a much simpler route, but such modifications alone do not provide completeness results. We have come nearly full circle in our analysis of focused sequent calculi: begining with full linear logic and ending with MALL, a focused version of which can be seen as a restriction to LKF. The MALL fragment of LLF and LKF share the simplest structure among the focused sequent calculi we have considered. They are not affected by the asymmetry of intuitionistic logic nor do they exhibit the peculiar behavior of unrestricted uses of ! and ? with respect to synchrony and asynchrony.

The possible variations of the cut rule in LKF are organized into three principal forms:

$$\frac{\vdash [\Theta], \Gamma, C \quad \vdash [\Theta'], \Gamma', \neg C}{\vdash [\Theta\Theta'], \Gamma\Gamma'} \; Cut_p, \; prime\ cut$$

$$\frac{\mapsto [\Theta], B \quad \vdash [\Theta'], \Gamma', \neg B}{\vdash [\Theta\Theta'], \Gamma'} \; Cut_k, \; key\ cut$$

$$\frac{\mapsto [\Theta, P], B \quad \vdash [\Theta'], \neg P}{\mapsto [\Theta\Theta'], B} \; Cut_f, \; focused\ cut$$

The key cut includes the “key cases” of cut elimination and the focused cut retains focus in the conclusion. In the focused cut, P cannot be a negative literal.

Theorem 18. *The rules Cut_p, Cut_k and Cut_f are admissible for LKF.*

Proof A mutually-recursive cut-elimination procedure can be given for LKF that is similar to that for LJF, except there are fewer cases since there are fewer cut rules to consider.

The structure of the procedure is to first permute the cut above asynchronous decomposition rules until a key case is reached; i.e., when the cut formula is principal (main) in both premises. For the Cut_p rule, this means we will eventually reach the situation

$$\frac{\vdash [\Theta], C \quad \vdash [\Theta'], \neg C}{\vdash [\Theta\Theta']} \; Cut_p$$

Exactly one of C and $\neg C$ is positive and only the positive formula can be selected for focus. Assume without loss of generality that C is positive, the other case being symmetrical. We then have:

$$\frac{\dfrac{\dfrac{\mapsto [\Theta, C], P}{\vdash [\Theta, C]} \; Focus}{\vdash [\Theta], C} \; [] \qquad \vdash [\Theta'], \neg C}{\vdash [\Theta\Theta']} \; Cut_p$$

Two cases are possible:

1. $P \in \Theta$ and $P \neq C$. In this case the cut is transformed into a focused cut at a lower height measure:

$$\frac{\dfrac{\mapsto [\Theta, C], P \quad \vdash [\Theta'], \neg C}{\mapsto [\Theta\Theta'], P} \; Cut_f}{\vdash [\Theta\Theta']} \; Focus$$

2. $P = C$. This case is reduced as follows:

$$\dfrac{\dfrac{\mapsto [\Theta, C], C \quad \vdash [\Theta'], \neg C}{\mapsto [\Theta\Theta'], C}\,Cut_f \qquad \vdash [\Theta'], \neg C}{\vdash [\Theta\Theta'\Theta']}\,Cut_k$$

Just as with LJF cut-elimination, the lower Cut_k is one of the key cases, which is reducible to other cuts with cut formulas of smaller degree. Genzten's *mix* rule is not required. Also as in LJF, contraction inside the boxed context is implicitly admissible.

As an example of the key case, the inference rules

$$\dfrac{\dfrac{\mapsto [\Theta], A \quad \mapsto [\Theta], B}{\mapsto [\Theta], A \wedge^+ B}\,\wedge^+ \qquad \dfrac{\vdash [\Theta'], \Gamma', \neg A, \neg B}{\vdash [\Theta'], \Gamma', \neg A \vee^- \neg B}\,\vee^-}{\vdash [\Theta\Theta'], \Gamma'}\,Cut_k$$

become the inference rules

$$\dfrac{\mapsto [\Theta], B \qquad \dfrac{\mapsto [\Theta], A \quad \vdash [\Theta'], \Gamma', \neg A, \neg B}{\vdash [\Theta\Theta'], \Gamma', \neg B}\,Cut_k}{\vdash [\Theta\Theta\Theta'], \Gamma'}\,Cut_k.$$

To illustrate the rewriting of a focused cut, consider the case where the focus formula is negative: here, a reaction rule transforms the cut into a prime cut. Otherwise, the positive formula under focus is principal. Consider the situation that the focus formula is a literal B:

$$\dfrac{\dfrac{}{\mapsto [\Theta, P], B}\,Id \qquad \vdash [\Theta'], \neg P}{\mapsto [\Theta\Theta'], B}\,Cut_f$$

It cannot be the case that $B = \neg P$ because *both* B *and* P *are positive.* Thus $B \in \Theta$ and the conclusion also follows by identity. If B is not a literal, the cut is permuted to its premises. For example, the inference rules

$$\dfrac{\dfrac{\mapsto [\Theta, P], A}{\mapsto [\Theta, P], A \vee^+ B}\,\vee^+ \qquad \vdash [\Theta'], \neg P}{\mapsto [\Theta\Theta'], A \vee^+ B}\,Cut_f$$

become

$$\dfrac{\dfrac{\mapsto [\Theta, P], A \quad \vdash [\Theta'], \neg P}{\mapsto [\Theta\Theta'], A}\,Cut_f}{\mapsto [\Theta\Theta'], A \vee^+ B}\,\vee^+$$

A more detailed proof can be found in the report [21]. □

Focusing can eliminate much of the non-determinism in cut-elimination. In fact, the only significant point of non-determinism that remains in our procedure is in prime cuts where neither cut formula is principal in the subproofs, and where the asynchronous context in both premises is non-empty. In this case it is possible to permute the cut above either premise first. For example:

$$\dfrac{\dfrac{\vdash [\Theta], \Gamma, A, C \quad \vdash [\Theta], \Gamma, B, C}{\vdash [\Theta], \Gamma, A \wedge^- B, C}\,\wedge^- \qquad \dfrac{\vdash [\Theta'], \Gamma', E, F, \neg C}{\vdash [\Theta'], \Gamma', E \vee^- F, \neg C}\,\vee^-}{[\Theta\Theta'], \Gamma\Gamma', A \wedge^- B, E \vee^- F}\,Cut_p$$

The cut can first be permuted above either asynchronous rule. In such cases, we can adopt some convention to eliminate the non-determinism. One possibility is to permute upwards the premise with the positive cut formula.

9. Conclusion and Future Work

We have studied focused proof construction in intuitionistic logic. The key to this endeavor is the definition of polarity for intuitionistic logic. The *LJF* proof system captures focusing using this notion of polarity. We illustrate how systems such as LJ, LJT, LJQ, and λRCC can be captured within *LJF* by assigning polarity to atoms and by adding to intuitionistic logic formulas annotations on conjunctions and delaying operators. We also use *LJF* to derive and justify the *LKF* focusing proof system for classical logic.

It remains to examine the impact of these focusing calculi on typed λ-calculi, logic programming, and theorem proving. Given the connections observed between LJT/LJQ and call-by-name/value, the *LJF* system could provide a framework for λ-term evaluations that combine the eager and lazy evaluation strategies. In the area of theorem proving, there is a number of completeness theorems for various restrictions to resolution: it would be interesting to see if any of these are captured by an appropriate mapping into *LKF*. In the area of logic programming, the connections between polarity and forward and backward chaining has been noted not only in this paper but also in numerous published paper. Miller and Nigam [25] have used the *LJF* proof system to provide a declarative means for insisting that if a fact is proved (and "tabled") then it is not reproved. Clearly, there should be many other opportunities for applying focusing proof systems.

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