An Errata for *McDowell and Miller's Cut-Elimination*

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1 The Issue

There appears to be a gap in McDowell and Miller's cut-elimination proof for $FO\lambda^{\Delta\mathbb{N}}$ [1]. In particular, the induction measure used for cut-elimination appears to be violated in the $def\mathcal{L}/\circ\mathcal{L}$ reduction case.

1.1 The Reduction

To review, the reduction case for $def\mathcal{L}/\circ\mathcal{L}$ applies to the following derivation Ξ,

$$\frac{\dfrac{\Pi_1}{\Delta_1 \longrightarrow B_1} \quad \cdots \quad \dfrac{\Pi_n}{\Delta_n \longrightarrow B_n} \quad \dfrac{\Pi}{B_1, \ldots, B_n, \Gamma \longrightarrow C}}{\Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow C}\ mc$$

where Π ends with a left rule other than $c\mathcal{L}$ acting on B_1 and Π_1 is

$$\frac{\left\{ \dfrac{\Pi_1^{\rho,\sigma,D}}{D\sigma, \Delta_1'\rho \longrightarrow B_1\rho} \right\}}{A, \Delta_1' \longrightarrow B_1}\ def\mathcal{L}$$

Then Ξ reduces to

$$\frac{\left\{ \dfrac{\dfrac{\Pi_1^{\rho,\sigma,D}}{D\sigma, \Delta_1'\rho \longrightarrow B_1\rho} \quad \left\{ \dfrac{\Pi_i\rho}{\Delta_i\rho \longrightarrow B_i\rho} \right\}_{i\in\{2..n\}} \quad \dfrac{\Pi\rho}{\ldots \longrightarrow C\rho}}{D\sigma, \Delta_1'\rho, \Delta_2\rho, \ldots, \Delta_n\rho, \Gamma\rho \longrightarrow C\rho}\ mc \right\}}{A, \Delta_1', \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow C}\ def\mathcal{L}$$

1.2 The Proof

In the proof of cut-elimination, we consider all derivations Ξ,

$$\frac{\dfrac{\Pi_1}{\Delta_1 \longrightarrow B_1} \quad \cdots \quad \dfrac{\Pi_n}{\Delta_n \longrightarrow B_n} \quad \dfrac{\Pi}{B_1, \ldots, B_n, \Gamma \longrightarrow C}}{\Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow C}\ mc$$

where the derivations $\Pi_1, \ldots, \Pi_n$ are reducible, and we want to show Ξ is reducible. The proof is by induction on $ht(\Pi)$ with subordinate inductions on n and on the reductions of $\Pi_1, \ldots, \Pi_n$. It is claimed that the order of the inductions on reductions is not important, thus we always assume we are working with Π_1.

To show that Ξ is reducible, we show that for every θ, there is a reducible reduct for $\Xi\theta$. To simplify things here, we will let θ be the identity and show that we cannot (easily) find a reducible reduct for Ξ.

Consider when Ξ falls into the $def\mathcal{L}/\circ\mathcal{L}$ case. Here we make an appeal to the induction hypothesis using $\Pi_1^{\rho,\sigma,D}, \Pi_2\rho, \ldots, \Pi_n\rho$, and Π. By a lemma, we know that $\Pi_2\rho, \ldots, \Pi_n\rho$ are all reducible. We also know that $\Pi_1^{\rho,\sigma,D}$ is a predecessor of Π_1, thus it appears we can appeal to the inner induction hypothesis. The problem, however, is that the reductions of $\Pi_2, \ldots, \Pi_n$ have been changed to the reductions of $\Pi_2\rho, \ldots, \Pi_n\rho$. We have no lemma which relates the reduction tree of Π_i with that of $\Pi_i\rho$ except to say that the existence of the former implies the existence of the latter. The appeal to the inner induction hypothesis would only be valid if the order of the inductions on the reductions of $\Pi_1, \ldots, \Pi_n$ was considered. But if such an order is considered important, then we can no longer assume we are working on Π_1 and the proof breaks down in the same place.

2 A Possible Fix

The problem above is caused by the lack of a close relationship between the reduction of Π and the reduction of $\Pi\rho$. To establish such a relationship, we define a notion of the height of a reduction.

Definition 1. *Let* Π *be a reducible derivation. We define the reduction height of* Π, $rh(\Pi)$, *as*

$$rh(\Pi) = \begin{cases} 0 & \textit{if } \Pi \textit{ has no predecessors} \\ 1 + lub\{rh(\Pi') : \Pi' \textit{ a predecessor of } \Pi\} & \textit{otherwise} \end{cases}$$

Given this definition we can establish the following key property.

Lemma 1. *If* Π *is reducible, then for all* ρ, $rh(\Pi) = rh(\Pi\rho)$.

Proof. The proof is by induction on the reduction of Π and case splitting based on the various cases in the original definition of reducibility. □

Notice that if Π' is a predecessor of Π then $rh(\Pi) > rh(\Pi')$, thus any induction on the reduction of Π can safely be replaced with an induction on $rh(\Pi)$. The fix for the cut-elimination proof is then to induct on $rh(\Pi_1), \ldots, rh(\Pi_n)$.

3 An Earlier Fix

Alwen Tiu noticed this gap as well and fixed it in his cut-elimination proof for LINC, a related logic. His fix was to generalize the reducibility lemma. See the proof given in his dissertation [2].

References

[1] Raymond McDowell and Dale Miller. Cut-elimination for a logic with definitions and induction. *Theoretical Computer Science*, 232:91–119, 2000.

[2] Alwen Tiu. *A Logical Framework for Reasoning about Logical Specifications.* PhD thesis, Pennsylvania State University, May 2004.