Kripke Semantics and Proof Systems for Combining Intuitionistic Logic and Classical Logic

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Abstract

We combine intuitionistic logic and classical logic into a new, first-order logic called *Polarized Intuitionistic Logic.* This logic is based on a distinction between two dual polarities which we call *red* and *green* to distinguish them from other forms of polarization. The meaning of these polarities is defined model-theoretically by a Kripke-style semantics for the logic. Two proof systems are also formulated. The first system extends Gentzen's intuitionistic sequent calculus LJ. In addition, this system also bears essential similarities to Girard's *LC* proof system for classical logic. The second proof system is based on a semantic tableau and extends Dragalin's multiple-conclusion version of intuitionistic sequent calculus. We show that soundness and completeness hold for these notions of semantics and proofs, from which it follows that cut is admissible in the proof systems and that the propositional fragment of the logic is decidable.

1 Introduction

One of Gentzen's goals in designing the sequent calculus was to construct an *analytic* approach to proofs that could work for *both* classical and intuitionistic logic [Gen69]. While his natural deduction proof system did not allow him to prove the *Hauptsatz* uniformly for both of these logics, his design of the sequent calculus did allow the cut-elimination theorem to be proved for both logics using the same algorithm. This early attempt at providing a *unity of logic* also presented the first demonstration of the importance of structural rules in the presentation of proof systems: in particular, the rule of contraction is not allowed on the right of Gentzen's intuitionistic sequent calculus (LJ) while it is allowed on the right in his classical sequent calculus (LK). While his approach has provided us with a common framework for the proof theory of these two logics, it did not provide us with one logic that combines classical and intuitionistic logics. Translating between and combining these logics has been repeatedly considered over the past several decades.

An important property of intuitionistic logic is its ability to embed classical logic: for an overview of several such double-negation translations by Kolmogorov, Gödel, Gentzen, and others, see [FO10]. This ability suggests that intuitionistic logic already contains the potential to serve as a platform for combining intuitionistic and classical reasoning. The double negation translations not only embed classical logic within intuitionistic logic but also help to explain the differences between the two.

In Gentzen's original sequent calculi, contraction is not applicable to right-hand-side formula occurrences in intuitionistic sequents but it is available for such formula occurrences in classical sequents. One way to describe double negation translations is that they overcome this restriction in intuitionistic sequents by moving some right-hand-side formula occurrences in classical sequent proofs to negated left-hand-side formula occurrences in intuitionistic sequent proofs (where contraction is available). As has been shown by Lamarche [Lam08] and others, a different way to see the differences between sequent calculus proofs for both classical and intuitionistic logic is to use a *one-sided* sequent calculus but with a system of *polarization*—annotations such as *input* and *output*—that distinguishes those formula occurrences that are subject to structural rules from those that are not. One might argue that the polarized approach is nothing but double-negation in disguise. However, our goal is not to see classical logic as a fragment of intuitionistic logic but rather to build proof systems and semantics in which *all connectives*—classical *and* intuitionistic—may mix freely.

An attempt to achieve such mixtures with a double negation translation must address at least the following questions:

1. If intuitionistic connectives are mixed with classical connectives that have been translated via a double-negation, how does one distinguish the parts of a formula that represent classical formulas from the parts that are just intuitionistic?

2. Even more crucially, how does one obtain cut-elimination in such a mixed setting? In the context of sequents, a double-negation represents classical formulas on the left-hand side. However, the cut rule
$$\frac{\neg A, \Gamma \vdash B \quad A, \Gamma \vdash B}{\Gamma\Gamma' \vdash B}$$
is admissible in classical logic but *not* in intuitionistic logic. Is cut-elimination possible at all when intuitionistic and classical formulas can mix?

These questions point to the consideration of a logic in which classical connectives are added as primitives alongside intuitionistic connectives. Furthermore, it is well known that the "purely intuitionistic" connectives of implication and universal quantification exhibit characteristics that decisively distinguish them from the other connectives. In order to guarantee that these connectives do not collapse into their classical counterparts in this mixed setting, we shall rely on a polarization of connectives.

Assume for the moment that $\vee^i$ is the intuitionistic "or" that gives us the disjunction property, and that $\vee^c$ is the classical "or" that is subject to structural rules. If we are allowed to freely mix these connectives with the purely intuitionistic ones, questions arise that challenge our understanding of classical and intuitionistic logics. The two versions of disjunction would naturally give rise to two versions of *false*: assume that these are $\perp$ for $\vee^c$ and 0 for $\vee^i$. We know that $A \vee^i \neg A$ should not be provable. But what about $A \vee^c \neg A$? If negation is defined in terms of intuitionistic implication and 0 (i.e., $A \supset 0$) then the answer is *still no.* The constant 0, being associated with $\vee^i$, should not be subject to weakening. One might notice that this argument is essentially one of linear logic, and that the observations concerning structural rules do not necessarily apply in intuitionistic proof theory. In this paper, however, we shall provide a semantic explanation of the above phenomenon independently of linear logic. If we tried to explain the above non-provability in terms of a traditional double-negation translation in intuitionistic logic, we will find that neither $\neg\neg(A \vee \neg A)$ nor $\neg(A \wedge \neg A)$ are accurate translations, since they are intuitionistically provable.

To formulate a system in which the law of excluded middle can safely and transparently coexist with the disjunction property, we also require a classical notion of negation that exhibits the expected De Morgan dualities. In order to extend these dualities to a setting where arbitrary connectives may mix, there will be *dual connectives to intuitionistic implication and universal quantification.* The twin notions of negation also give rise to distinct levels of consistency: a characteristic that we explain by an enrichment of Kripke models. In particular, we shall admit *imaginary possible worlds* that may validate $\perp$ (but never 0). These models translate to Heyting algebras with an embedded boolean algebra, one that is different from the *skeleton* induced from Glivenko's transformation. We refer to this logic as *polarized intuitionistic logic* (PIL). Double negations will be crucially important in the semantic exposition of PIL, but the syntax and proof theory of PIL are independent of them.

This paper is organized as follows. Section 2 defines the syntax of formulas and their polarity assignments, without giving any meaning to these assignments. Section 3 defines the Kripke semantics for the *propositional* fragment of PIL. A translation to the Heyting algebra representation is also provided. The mixing of classical and intuitionistic quantifiers pose certain challenges. Thus we will present the first-order semantics separately in Section 5. Section 4 introduces the sequent

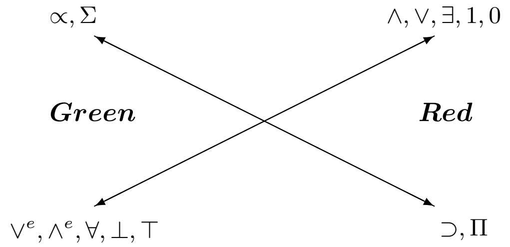

Figure 1: Classification of PIL Connectives

calculus *LP* and discusses its relationship to LJ and LC. Section 6 establishes the soundness and completeness of LP (with respect to the full first-order logic). The *admissibility of cut* follows from semantic completeness. In Section 7, we present another proof system for PIL based on semantic tableau. From the correctness of this system it is also shown that the propositional fragment of PIL is decidable. In Section 8, we discuss related works, which include various double-negation translations, dual-intuitionistic logic, linear logic, polarized linear logic, LU and LC, as well as some of our own work. We summarize in Section 9.

2 Syntax

The formulas of PIL are constructed from the following polarized connectives. The term "polarity" has been used in many contexts. Although our form of polarization is not entirely unrelated to these other uses, to avoid confusion and misconception we have chosen a pair of terms that are entirely neutral.

Red-Polarized: $\vee$, $\wedge$, $\exists$, 0, 1, $\supset$, Π.

Green-Polarized: $\wedge^e$, $\vee^e$, $\forall$, $\top$, $\bot$, $\propto$, Σ.

Throughout the paper we shall use the letter R to represent red-polarity formulas and E to represent green-polarity formulas, with frequent reminders of this convention.

The group of connectives $\supset$, Π, $\propto$ and Σ will also be referred to as the *purely intuitionistic* or *PI-connectives.* The set $\vee$, $\wedge$, $\exists$, $\vee^e$, $\wedge^e$ and $\forall$ are referred to as the *classically-oriented* or *LC-connectives.* The rationale for these designations will be clarified in subsequent sections. For each propositional LC-connective there is an associated logical constant: 0 for $\vee$, $\bot$ for $\vee^e$, $\top$ for $\wedge^e$ and 1 for $\wedge$. These symbols are obviously borrowed from linear logic. While we make no deliberate attempt to hide the influence of linear logic, in this paper we present PIL as an independent system. The classification of PIL connectives is also illustrated in Figure 1.

We assume that there are denumerably many parameters (aka terms), variables and predicate symbols. An atomic formula has the form $p(t_1, \ldots, t_n)$ for predicate symbol p and parameters and variables $t_1, \ldots, t_n$. We designate that *all atomic formulas are red-polarized.*[1]. For every atom a we also admit its *dual* $a^\perp$, which is green-polarized. *De Morgan negation* is extended to all formulas using the following dualities: $1/\bot$, $0/\top$, $\supset/\propto$, Σ/Π, $\vee/\wedge^e$, $\wedge/\vee^e$, $\exists/\forall$. It is a syntactic identity that $A^{\perp\perp} = A$ for all formulas A. We use the term *literal* to refer to atoms and their duals. An atom a is *not* considered a subformula of the literal $a^\perp$. All formulas are written in negation normal form: the negation $(\cdot)^\perp$ has only atomic scope. Intuitionistic implication is also dualized into the connectives $\supset$ (red) and $\propto$ (green). However, the dual of $A \supset B$ is $A \propto B^\perp$, and not $A^\perp \propto B^\perp$. We make this choice because we wish to think of $\propto$ as a form of (non-commutative) conjunction. The symbol $\propto$ was selected to convey an asymmetrical product. The red and green polarities are duals of each other: given A and $A^\perp$, one is red-polarized and the other is green-polarized.

Another form of negation in PIL is *intuitionistic negation,* which we write as $\sim A$, and is defined to be abbreviation for $A \supset 0$. Still other forms of negation can be defined in PIL, including $A \supset \bot$

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[1] This is an arbitrary choice for the sake of a smoother presentation - it is also possible to arbitrarily assign polarity to atoms.

(which will not always be equivalent to $A^{\perp}$).

The polarity of a formula is determined entirely by its top-level connective. Intuitionistic logic in the traditional sense uses only red-polarized formulas (including subformulas). PIL is an extension of intuitionistic logic that places *no restriction* on how formulas are composed using its connectives, constants, and literals. A sample mixed formula in PIL is $[A \wedge ((C \supset B) \vee^e C)] \supset (A \propto C)$.

3 Semantics: the Propositional Case

The two versions of falsehood—$\perp$and 0—suggest that there are two notions of inconsistency and, hence, two notions of consistency. In our semantics $0 \supset A$ will always be valid while $\perp \supset E$ will only be valid for *green* formulas E. An important aspect of PIL is that these two notions of inconsistency are allowed to coexist.

3.1 Kripke Models of Polarized Formulas

In a standard Kripke model for intuitionistic logic, the terminal nodes of the ordering relation represent classical worlds where intuitionistic implication collapses into a classical one and the excluded middle becomes valid. Because of the two notions of consistency, worlds *above* classical worlds will be needed. To account for the richer formulas of PIL, we allow possible worlds that can be inconsistent in terms of $\perp$ (but never in terms of 0). The use of inconsistent possible worlds is not without precedent [Vel76, ILH10] (though not in the context of two levels of consistency). While some authors referred to them as "exploded worlds," we prefer the term *imaginary world* in analogy to $\sqrt{-1}$ being an imaginary number. These worlds also distinguish PIL from some other efforts to semantically combine classical and intuitionistic logic [CH96].

We first define the semantic interpretation for the propositional fragment of PIL as it is self-contained and requires a simpler model structure and definition of "forcing" than first-order PIL. We define a *propositional Kripke hybrid model* as a structure $\langle \mathbf{W}, \preceq, \mathbf{C}, \models \rangle$ where $(\mathbf{W}, \preceq)$ is a non-empty Kripke frame of possible worlds $\mathbf{W}$ and $\preceq$ is a transitive and reflexive relation on $\mathbf{W}$. The set $\mathbf{C}$, the set of "classical worlds," is a subset of $\mathbf{W}$. The component $\models$ is a binary relation between elements of $\mathbf{W}$ and (red-polarized) atomic formulas.

Let $\triangle_{\mathbf{u}} = \{\mathbf{k} \in \mathbf{C} \mid \mathbf{u} \preceq \mathbf{k}\}$, i.e., the set of all classical worlds above $\mathbf{u}$. We say that a world $\mathbf{u}$ is *imaginary*, or $\perp$*-inconsistent*, if $\triangle_{\mathbf{u}}$ is empty. We also require the following conditions for all propositional models:

- $\models$ is *monotone*: that is, for any $\mathbf{u}, \mathbf{v} \in \mathbf{W}$ and atom a, if $\mathbf{u} \preceq \mathbf{v}$ then $\mathbf{u} \models a$ implies $\mathbf{v} \models a$
- $\triangle_{\mathbf{k}} = \{\mathbf{k}\}$ for all $\mathbf{k} \in \mathbf{C}$, i.e., there are no classical worlds properly above other classical worlds.

The satisfiability or *forcing* relation extends $\models$ from atoms to all propositional formulas using the following induction (on the structure of formulas). The key idea here is that a green formula is valid in a world $\mathbf{u}$ if it is valid in all classical worlds above $\mathbf{u}$. First, we define the red-polarity cases: here, $\mathbf{u}, \mathbf{v} \in \mathbf{W}$. When a is atomic, whether $\mathbf{u} \models a$ holds is given by the model. All other red-polarized cases are given below.

- $\mathbf{u} \models 1$ and $\mathbf{u} \not\models 0$
- $\mathbf{u} \models A \vee B$ iff $\mathbf{u} \models A$ or $\mathbf{u} \models B$
- $\mathbf{u} \models A \wedge B$ iff $\mathbf{u} \models A$ and $\mathbf{u} \models B$
- $\mathbf{u} \models A \supset B$ iff for all $\mathbf{v} \succeq \mathbf{u}$, $\mathbf{v} \models A$ implies $\mathbf{v} \models B$

The red connectives are thus interpreted in exactly the same way as in Kripke's semantics for intuitionistic logic. Next, we define the cases for green formulas, but only over classical worlds: here, $\mathbf{c} \in \mathbf{C}$ and $\mathbf{v} \in \mathbf{W}$.

- $\mathbf{c} \models a^{\perp}$ iff $\mathbf{c} \not\models a$, where a is an atom.

- $\mathbf{c} \models \top$ and $\mathbf{c} \not\models \bot$
- $\mathbf{c} \models A \propto B$ iff for some $\mathbf{v} \succeq \mathbf{c}$, $\mathbf{v} \models A$ and $\mathbf{v} \not\models B^{\perp}$
- $\mathbf{c} \models A \vee^{e} B$ iff $\mathbf{c} \models A$ or $\mathbf{c} \models B$
- $\mathbf{c} \models A \wedge^{e} B$ iff $\mathbf{c} \models A$ and $\mathbf{c} \models B$

The $\models$ relation is extended to all green formulas E in *any* world $\mathbf{u} \in \mathbf{W}$ by the condition

- $\mathbf{u} \models E$ if and only if for all $\mathbf{c} \in \triangle_{\mathbf{u}}$, $\mathbf{c} \models E$.

That is, a green formula is considered valid in world $\mathbf{u}$ *if it is valid in all classical worlds at or above* $\mathbf{u}$. The $\models$ relation is well-defined: in particular, if $\mathbf{u}$ is classical, then the clauses above defining $\models$ for classical worlds coincide since $\triangle_{\mathbf{u}}$ is the singleton set $\{\mathbf{u}\}$. Furthermore, *if* $\triangle_{\mathbf{u}}$ *is empty, then all green formulas are satisfied vacuously in* $\mathbf{u}$.

The following lemma formalizes the essential characteristics of the possible worlds of a model.

Lemma 1 *In a propositional Kripke hybrid model, for every* $\mathbf{u}, \mathbf{v} \in \mathbf{W}$*, every* $\mathbf{c} \in \mathbf{C}$*, and every (propositional) formula* A*, the following hold.*

a. *if* $\mathbf{u} \preceq \mathbf{v}$*, then* $\mathbf{u} \models A$ *implies* $\mathbf{v} \models A$ *(monotonicity)*

b. $\mathbf{c} \models A$ *iff* $\mathbf{c} \not\models A^{\perp}$ *(excluded middle)*

c. $\mathbf{u} \models A$ *and* $\mathbf{u} \models A^{\perp}$ *for some formula* A *iff* $\triangle_{\mathbf{u}}$ *is empty (*$\mathbf{u}$ *is imaginary).*

The first two properties are proved by induction on formulas. Crucially, in the case of monotonicity for green formulas, assume that $\mathbf{u} \preceq \mathbf{v}$. Then it holds by transitivity that $\triangle_{\mathbf{v}} \subseteq \triangle_{\mathbf{u}}$, and thus $\mathbf{u} \models E$ implies $\mathbf{v} \models E$ for green formulas E. This argument includes the case where $\triangle_{\mathbf{v}}$ is empty. The third property follows from the first two. An equivalent version of the third property is that $\mathbf{u} \not\models E$ *for some green formula* E *iff* $\triangle_{\mathbf{u}}$ *is non-empty.*

We make some additional observations about these semantic definitions.

- At first examination, the use of "*for some* $\mathbf{v} \succeq \mathbf{u}$ ..." in the clause defining the semantics of $\propto$ may appear to break the monotonicity condition. In particular, if c and c' are two classical worlds such that $\mathbf{c} \preceq \mathbf{c}'$, why should $\mathbf{c} \models A \propto B$ imply $\mathbf{c}' \models A \propto B$ since they could use different witnesses for the existential quantification "*for some...*?" The reason is simple: since $\triangle_{\mathbf{c}}$ is a singleton, $\mathbf{c} = \mathbf{c}'$. Our interpretation of $\propto$ is a "dual" of intuitionistic implication in the *De Morgan* sense, unlike in some versions of *dual-intuitionistic logic* or *bi-intuitionistic logic* [Rau74, Gor00].
- Let $\mathbf{v}$ be a world above the classical world $\mathbf{c}$. The condition $\mathbf{v} \not\models A^{\perp}$ is in fact *stronger* than $\mathbf{v} \models A$. This property follows from (the contrapositive of) the monotonicity property and the excluded middle property of classical worlds. The condition $\mathbf{v} \not\models A$ also implies that either $\mathbf{v} = \mathbf{c}$ or A is a red formula.
- While 0 and $\bot$ are clearly distinct in this semantics, 1 and $\top$ are, in fact, equivalent: they are simply red and green-polarized versions of the same truth value. This equivalence does not affect cut-elimination for the full logic. In general, it is possible for green-polarized formulas to be logically equivalent to red-polarized ones.

We say that a model $\mathcal{M}$ satisfies A, or $\mathcal{M} \models A$, if $\mathbf{u} \models A$ for every $\mathbf{u} \in \mathbf{W}$. A formula is valid if it is satisfied in all models.

Our definition of classical satisfiability is similar to the notion of "covers" found in Beth models [Bet59]. In these models, $w \models A \vee B$ if there exists a cover above w in the sense that every maximal path through w intersects the cover, and for each world v in the cover, $v \models A$ or $v \models B$. All finite Beth models are classical models since the leafs cover all worlds. In PIL's *hybrid* models, the "cover" of w is fixed to be $\triangle_{w}$, which *could be empty.* A key difference between our semantics and those of Beth and Kripke is *polarization.* The cover is only used in the interpretation of the green connectives while the red ones are interpreted as in Kripke models. Non-classical models can remain finite. All hybrid models are Kripke models with a refined classification of possible worlds into three disjoint sets: classical, imaginary, and non-classical worlds with a non-empty cover. We prefer a Kripke-style semantics because of its simplicity and utility: for example, much of the difference between classical and intuitionistic logic can be explained clearly by a few small Kripke models. This characteristic is preserved in PIL.

3.2 Important Countermodels

We give two important examples of invalid formulas and their countermodels. Both examples involve formulas that are not found in traditional intuitionistic logic.

The formula $\sim a \vee^e \sim\sim a$ is not valid. Despite using the classical $\vee^e$, this version of the excluded middle does not hold in PIL precisely because there may be imaginary worlds above the classical worlds. A countermodel is

$$\begin{tikzcd}
s_1 : \{a, a^\perp\} & & s_2 : \{a^\perp\} \\
& k : \{a^\perp\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The notation is intended to indicate that the classical world k does not satisfy a (so it must satisfy $a^\perp$), but k also does not satisfy $a \supset 0$ since there is a world s_1 above k that satisfies a. The world k does not satisfy $\sim\sim a$ because no world above s_2 satisfies a. The same model shows that $a \vee^e \sim a$ is also not valid (s_2 is not needed here). Intuitionistic implication does not collapse into a classical one even when interpreted at the classical level. The excluded middle is valid in the form $a \vee^e a^\perp$.

The formula $(p \wedge^e q) \supset p$ is not valid. A countermodel is:

$$\begin{tikzcd}
k : \{p, q\} \\
s : \{\} \arrow[u]
\end{tikzcd}$$

Although every classical world *above* s satisfies p and q, s does not satisfy p. The same model shows that several other formulas, including $(p \vee^e q) \supset (p \vee q)$, are not valid. The key characteristic of this model is the gap between the classical worlds and the possible worlds beneath them. Such models show how the mixture of intuitionistic and classical reasoning must be restricted. They suggest the need of special proof-theoretic devices that are required for cut-elimination. For example, in designing a proof system involving sequents, restrictions must be made on the application of the following kind of introduction rule, which may at first appear harmless:

$$\frac{P, \Gamma \vdash R \quad Q, \Gamma \vdash R}{P \vee^e Q, \Gamma \vdash R} \vee^e L$$

This small model in fact explains an invariant of PIL proofs found in the following section, one that extends the *stoup* concept found in Girard's *LC* system [Gir91].

3.3 An Algebraic Perspective

Our refinement of Kripke models translates to Heyting algebras in a natural way. The purpose of this translation is to expose some additional properties of PIL, especially its relationship to double negation. Since every hybrid model is also a Kripke model, this translation will extend the standard one (see [Fit69]).

Every Kripke frame corresponds to a Heyting algebra. Specifically, from a hybrid model $\langle \mathbf{W}, \preceq, \mathbf{C}, \models \rangle$ we define the Heyting algebra $\mathcal{H} = \langle \mathcal{U}(\mathbf{W}), \sqsubseteq, \sqcup, \sqcap, \rightarrow, \mathbf{0} \rangle$ where $\mathcal{U}(\mathbf{W})$ is the set of all upwardly closed subsets of $\mathbf{W}$: a set S is *upwardly closed* if, whenever $a \in S$ and $a \preceq b$, $b \in S$. The ordering relation $\sqsubseteq$ of $\mathcal{H}$ is just the regular subset relation. Join and meet are defined by set union ($\sqcup$) and intersection ($\sqcap$) respectively. We have chosen to use different symbols from $\subseteq$, $\cup$, $\cap$ so as to distinguish the algebra's operations from other statements concerning sets.

The relative pseudo-complement $A \rightarrow B$ is the largest $x \in \mathcal{U}(\mathbf{W})$ such that $(A \sqcap x) \sqsubseteq B$. It can be shown that this element is in fact the largest upwardly closed subset (*interior*) of $(\mathbf{W} - A) \cup B$ where "$-$" represents set subtraction. The least element of $\mathcal{U}(\mathbf{W})$, denoted by $\mathbf{0}$, is the empty set. Formulas A are interpreted in this algebra by a mapping h defined as

$$h(A) = \{\mathbf{u} \in \mathbf{W} : \ \mathbf{u} \models A\}$$

Each $h(A)$ is upwardly closed by the monotonicity of $\models$. A formula A is valid in the algebra if $h(A) = \mathbf{W}$: that is, when $\mathbf{u} \models A$ for all $\mathbf{u} \in \mathbf{W}$.

All this is well known. But now notice that *the set of all imaginary worlds in* $\mathbf{W}$ *is an upwardly closed set.* This set corresponds to a unique element of $\mathcal{U}(\mathbf{W})$, which we can call $\perp\!\!\!\perp$:

$$\perp\!\!\!\perp = \{\mathbf{u} \in \mathbf{W} : \ \triangle_{\mathbf{u}} \text{ is empty}\} = \{\mathbf{u} \in \mathbf{W} : \ \mathbf{u} \models \perp\}$$

If there are no imaginary worlds in $\mathbf{W}$ then, indeed, $\perp\!\!\!\perp = \mathbf{0}$. Clearly $h(\perp) = \perp\!\!\!\perp$. It is now possible to characterize the interpretation of all formulas in terms of operations inside the algebra. Every green formula is the dual of a red formula. A green formula is interpreted by the pseudo-complement of its dual relative to $\perp\!\!\!\perp$. From this semantic perspective the most important addition to intuitionistic logic found in PIL is $\perp$.

Lemma 2 *The following holds for h:*

- *$h(A \vee B) = h(A) \sqcup h(B)$, $h(A \wedge B) = h(A) \sqcap h(B)$, and $h(A \supset B) = h(A) \rightarrow h(B)$.*
- *$h(R^{\perp}) \ = \ h(R) \rightarrow \perp\!\!\!\perp$ for all green formulas $R^{\perp}$.*

Proof The cases for $\vee$, $\wedge$ and $\supset$ are exactly the same as in intuitionistic logic. The first two cases are trivial; for the third case, see [Fit69, Chapter 1]. We shall prove the new property for green-polarized formulas $R^{\perp}$. Let $P = h(R) \rightarrow \perp\!\!\!\perp$. This means that for any X in $\mathcal{H}$, $h(R) \sqcap X \sqsubseteq \perp\!\!\!\perp$ if and only if $X \sqsubseteq P$. The forward direction holds because $h(R) \sqcap h(R^{\perp}) \sqsubseteq \perp\!\!\!\perp$ (by Lemma 1c) and thus $h(R^{\perp}) \sqsubseteq P$ since P is assumed to be largest. In the other direction, suppose $\mathbf{u} \in P$. Assume that $\mathbf{u} \notin h(R^{\perp})$, which means that $\mathbf{u} \not\models R^{\perp}$. Since $R^{\perp}$ is green-polarized, this implies that for some $\mathbf{k} \in \triangle_{\mathbf{u}}$, $\mathbf{k} \not\models R^{\perp}$. Since $\mathbf{k}$ is classical, this means that $\mathbf{k} \models R$, so $\mathbf{k} \in h(R)$. But $\mathbf{k} \in P$ because P is upwardly closed, so $\mathbf{k} \in h(R) \sqcap P$. This implies $\mathbf{k} \in \perp\!\!\!\perp$, which contradicts the assumption that $\mathbf{k}$ is classical. □

In particular, $h(A \vee^e A^{\perp}) = \mathbf{W}$ since $h(A \wedge A^{\perp}) \sqsubseteq \perp\!\!\!\perp$ and so $h(A \wedge A^{\perp}) \sqcap \mathbf{W} \sqsubseteq \perp\!\!\!\perp$, which means that $h(A \wedge A^{\perp}) \rightarrow \perp\!\!\!\perp = \mathbf{W}$. The lemma implies that $h(R^{\perp})$ is also equivalent to $h(R \supset \perp)$. Thus in PIL, $R^{\perp} \equiv R \supset \perp$ for all red formulas R, but the same is not true of green formulas: $E \supset \perp$ is *not* equivalent to $E^{\perp}$. In fact, we can show the important equivalence $(R \supset \perp) \supset \perp \ \equiv \ (R \vee^e \perp)$:

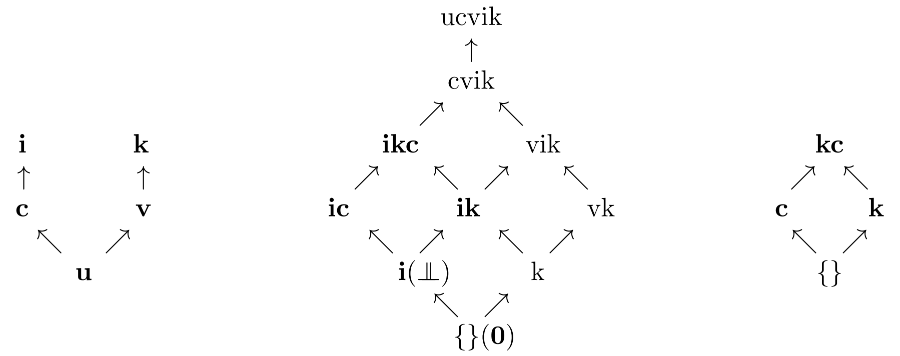
$$h(R \vee^e \perp) = h(R^{\perp} \wedge 1) \rightarrow \perp\!\!\!\perp = h(R^{\perp}) \rightarrow \perp\!\!\!\perp = (h(R) \rightarrow \perp\!\!\!\perp) \rightarrow \perp\!\!\!\perp = h((R \supset \perp) \supset \perp)$$

The double-negation "lifts" R to a green formula (but remember that R may contain more than just conjunction and disjunction). This form of "double-negation," however, is in terms of $\perp$, which is green-polarized as opposed to 0 or some arbitrary atom, which are red-polarized.

The equivalence $R^{\perp} \equiv R \supset \perp$ is the same as $E \equiv E^{\perp} \supset \perp$. However, the fact that the green connectives can be defined in terms of red connectives and $\perp$ does not mean that a system in which they are considered primitive is not useful: for example, most presentations of classical logic contain a full set of connectives despite the well known fact that every one of them can be defined in terms of the others. The lifting of the green connectives to first-class status clearly simplifies the combination of classical connectives with the intuitionistic ones.

Note that whether $\perp$ can be considered a *"unit"* is dependent on the interpretation of *"equivalence:"* classical equivalence should not be defined in terms of intuitionistic implication. For all green-polarity formulas, $E \vee^e \perp$ is always equivalent to E, in the sense that $h(E \vee^e \perp) = h(E)$. From the properties of h we also see that $A \supset E$ is equivalent to the classical implication $A^{\perp} \vee^e E$ (for green-polarized E), since

$$h(A \supset E) = h(A) \rightarrow (h(E^{\perp}) \rightarrow \perp\!\!\!\perp) = (h(A) \sqcap h(E^{\perp})) \rightarrow \perp\!\!\!\perp = h(A \wedge E^{\perp}) \rightarrow \perp\!\!\!\perp = h(A^{\perp} \vee^e E)$$



Kripke Frame with $\mathbf{C} = \{\mathbf{c}, \mathbf{k}\}$ Heyting Algebra with $⫫$ Boolean Algebra $2^{\mathbf{C}}$

Figure 2: The algebraic interpretation of a sample Kripke-hybrid model

The same is not true for $A \supset R$: properly intuitionistic implication will require a red formula at the head.

Since $R \not\equiv (R \supset \bot) \supset \bot$, the operation $(.) \to ⫫$ is not an involutive operation in the algebra. To find such an operation, we will first identify a proper boolean algebra embedded within the Heyting algebra. It is well known that the powerset of any set forms a boolean algebra. Let $\mathcal{B} = \langle \mathbf{2^C}, \cup, \cap, \{\}\rangle$ be the boolean algebra formed from the subsets of $\mathbf{C}$. This algebra is embedded within $\mathcal{H}$ as:

$$\mathcal{BH} \;=\; \{⫫ \cup K : \; K \subseteq \mathbf{C}\}$$

For any subset K of classical worlds, $⫫ \cup K$ is an upwardly closed set since $⫫$ is upwardly closed and $\triangle_{\mathbf{k}} = \{\mathbf{k}\}$ for all $\mathbf{k} \in \mathbf{C}$ in propositional models. Since K and $⫫$ are disjoint, the implied mapping from $\mathcal{B}$ to $\mathcal{BH}$ is clearly one-to-one. The least element of the embedded algebra[2] is $⫫$. It is easy to see that $\mathcal{BH}$ is closed under $\cup$ and $\cap$ (and thus under $\sqcup$ and $\sqcap$). However, it is *not* closed under $\to$. That is to say, intuitionistic implication requires "escaping" the boolean algebra into the larger Heyting algebra. The diagrams in Figure 2 illustrate a sample frame of a hybrid model where **c** and **k** are considered classical, the corresponding Heyting algebra, and the boolean algebra in its embedded and independent forms. Every choice of $\mathbf{C}$ corresponds to some boolean algebra that is already embedded within the Heyting algebra. In the example of Figure 2, if **u** is considered classical, which means that all the other worlds are imaginary, then the embedded boolean algebra consists of only the top two nodes of the Heyting algebra.

Contrast the algebra $\mathcal{BH}$ with the boolean algebra identified by Glivenko [Gli29], with which many double-negation translations correspond. Glivenko showed that the *skeleton* of a pseudo-complemented lattice, consisting of all points of the form $x \to 0$ (equivalently all points of the form $(x \to 0) \to 0$), can form a boolean lattice. However, unlike $\mathcal{BH}$ it is not a *sublattice* as it does not preserve joins: joins inside the skeleton boolean lattice must be redefined as $((a \sqcup b) \to 0) \to 0$. With this double-negation, intuitionistic implication cannot be embedded inside a classical disjunction without losing its strength. This form of double-negation confines intuitionistic implication inside the skeleton *from whence there is no escape.*

We now define a mapping of formulas into $\mathcal{BH}$ based on the double negation $(A \supset \bot) \supset \bot$. Let $\diamond = ⫫ \cup \mathbf{C}$, the top element of $\mathcal{BH}$ (not to be confused with 1). For $K \subseteq \mathbf{C}$, let $\overline{K}$ represent the complement of K in the boolean algebra $\mathcal{B}$. Extend this operation to $\mathcal{BH}$ so that $\overline{K \cup ⫫} = \overline{K} \cup ⫫$. In the boolean algebra $\mathcal{B}$, $\overline{K}$ is just $\mathbf{C} - K$. Define the secondary mapping h' from formulas into $\mathcal{BH}$ by:

$$h'(A) = h(A \vee^e \bot) \sqcap \diamond$$

To further analyze the properties of h', we can divide it into separate cases for red formulas R and green formulas E:

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[2] It is not technically a "subalgebra" since it does not preserve the least element of $\mathcal{H}$; it is a sublattice.

- $h'(E) \ = h(E \vee^e \bot) \sqcap \Diamond = h(E) \sqcap \Diamond \ = (h(E) \cap \mathbf{C}) \cup \perp\!\!\!\perp$
- $h'(R) \ = h(R \vee^e \bot) \sqcap \Diamond \ = (h(R \vee^e \bot) \cap \mathbf{C}) \cup \perp\!\!\!\perp \ = (h(R) \cap \mathbf{C}) \cup \perp\!\!\!\perp.$

To see the equalities above, note that $h(E)$ always contains $\perp\!\!\!\perp$ (all green formulas are valid in imaginary worlds). Also classical worlds do not satisfy $\bot$, so $h(E \vee^e \bot) = h(E)$ and $h(R \vee^e \bot) \cap \mathbf{C} = h(R) \cap \mathbf{C}$.

Thus for every formula A, $h'(A)$ is *always inside the embedded algebra* $\mathcal{BH}$ and in fact $h'(A)$ can be defined as $(h(A) \cap \mathbf{C}) \cup \perp\!\!\!\perp$.

Now by the property that $\mathbf{c} \models A$ iff $\mathbf{c} \not\models A^{\perp}$ for all $\mathbf{c} \in \mathbf{C}$, we see that

- $\overline{h'(A)} = \overline{(h(A) \cap \mathbf{C})} \cup \perp\!\!\!\perp = (h(A^{\perp}) \cap \mathbf{C}) \cup \perp\!\!\!\perp = h'(A^{\perp})$
- $\overline{\overline{h'(A)}} = h'(A)$

We have our involutive operator that interprets $A^{\perp}$. The following properties can also be established:

- $h'(1) = h'(\top) = \Diamond$
- $h'(\bot) = h'(0) = \perp\!\!\!\perp$
- $h'(A \wedge B) = h'(A \wedge^e B) = h'(A) \sqcap h'(B)$
- $h'(A \vee B) = h'(A \vee^e B) = h'(A) \sqcup h'(B)$

Inside the boolean algebra $\vee^e$ and $\vee$ are equivalent, as are $\wedge^e$ and $\wedge$. 0 and $\bot$ also become equivalent. However, the homomorphic properties for h' do not extend to $\rightarrow$. That is, although intuitionistic implication can be mapped by h' into the embedded boolean algebra, it *does not hold* that $h'(A \supset B) = h'(A) \rightarrow h'(B)$. To demonstrate this, consider the Kripke frame of Figure 2, but as part of a model where $\mathbf{C} = \{\mathbf{u}\}$ and with $\mathbf{i} \models a$ and $\mathbf{k} \models b$. Then $\perp\!\!\!\perp = \{cvik\}$ and $h'(a \supset b) = \perp\!\!\!\perp$ but $h'(a) \rightarrow h'(b) = \mathbf{W}$. Neither h' nor h is a perfect homomorphism: h fails with respect to De Morgan negation and h' fails with respect to intuitionistic implication. Each compensates for a deficit in the other. A perfect homomorphism in either case would represent a collapse into classical logic.

The satisfiability of a green formula E in a hybrid model is entirely determined by its satisfiability in the classical worlds. Thus it follows that

- $h'(E) = \Diamond$ if and only if $h(E) = \mathbf{W}$

The interpretation of green formulas by h can be replaced by that of h'. However, the fact that $h(R)$ and $h'(R)$ are distinct for red formulas R allows us to combine classical and intuitionistic logics in a way that does not destroy the latter. This pair of *pseudo-homomorphisms* will correspond to two distinct modes of derivation in the proof theory of PIL: any correct proof system for PIL must implement this distinction.

4 Sequent Calculus

We present the sequent calculus LP for PIL in Figure 3. Although a semantics for first-order quantifiers is not presented until later in Section 5, we find it convenient to include these quantifiers in LP at this point. In all rules, Γ and Θ are multisets of formulas, E is a *green* formula, R is a *red* formula, and a is any atom. Although our use of polarization eliminates the *need* for two-sided sequents, we nevertheless choose this style in presenting LP because it improves readability and offers a better correspondence to the semantics. We use the symbols $\vdash_{\circ}$ and $\vdash_{\bullet}$ to represent two modes of proof. In terms of the algebraic interpretation, $\vdash_{\circ}$ corresponds to the pseudo-homomorphism h and $\vdash_{\bullet}$ corresponds to h'. We write $\vdash_{*}$ to denote either $\vdash_{\circ}$ or $\vdash_{\bullet}$. The interpretation of a sequent $\Gamma \vdash_{\circ} A$ is the formula $\bigwedge \Gamma \supset A$ where $\bigwedge \Gamma$ is the $\wedge$-conjunction over formulas in Γ, with an empty

Structural Rules and Identity

$$\frac{\Gamma \vdash_{\bullet} E}{\Gamma \vdash_{\circ} E}\ \textit{Signal} \qquad \frac{A^{\perp}, \Gamma \vdash_{\bullet} \Theta}{\Gamma \vdash_{\bullet} A, \Theta}\ \textit{Store} \qquad \frac{A^{\perp}, \Gamma \vdash_{\circ} A}{A^{\perp}, \Gamma \vdash_{\bullet}}\ \textit{Load} \qquad \frac{}{a, \Gamma \vdash_{\circ} a}\ I$$

Right-Red Introduction Rules

$$\frac{\Gamma \vdash_{\circ} A \quad \Gamma \vdash_{\circ} B}{\Gamma \vdash_{\circ} A \wedge B}\ \wedge R \qquad \frac{\Gamma \vdash_{\circ} A_i}{\Gamma \vdash_{\circ} A_1 \vee A_2}\ \vee R \qquad \frac{A, \Gamma \vdash_{\circ} B}{\Gamma \vdash_{\circ} A \supset B}\ \supset R$$

Left-Red Introduction Rules

$$\frac{A, B, \Gamma \vdash_{\circ} R}{A \wedge B, \Gamma \vdash_{\circ} R}\ \wedge L \qquad \frac{A, \Gamma \vdash_{\circ} R \quad B, \Gamma \vdash_{\circ} R}{A \vee B, \Gamma \vdash_{\circ} R}\ \vee L \qquad \frac{A \supset B, \Gamma \vdash_{\circ} A \quad B, \Gamma \vdash_{\circ} R}{A \supset B, \Gamma \vdash_{\circ} R}\ \supset L$$

Right-Green Introduction Rules

$$\frac{\Gamma \vdash_{\bullet} A \quad \Gamma \vdash_{\bullet} B}{\Gamma \vdash_{\bullet} A \wedge^{e} B}\ \wedge^{e} R \qquad \frac{\Gamma \vdash_{\bullet} A, B}{\Gamma \vdash_{\bullet} A \vee^{e} B}\ \vee^{e} R \qquad \frac{\Gamma \vdash_{\circ} A \quad \Gamma \vdash_{\bullet} B}{\Gamma \vdash_{\bullet} A \propto B}\ \propto R$$

Rules for Quantifiers

$$\frac{\Gamma \vdash_{\circ} A[t/x]}{\Gamma \vdash_{\circ} \exists x.A}\ \exists R \qquad \frac{\Gamma \vdash_{\circ} A}{\Gamma \vdash_{\circ} \Pi y.A}\ \Pi R \qquad \frac{A, \Gamma \vdash_{\circ} R}{\exists y.A, \Gamma \vdash_{\circ} R}\ \exists L \qquad \frac{A[t/x], \Pi x.A, \Gamma \vdash_{\circ} R}{\Pi x.A, \Gamma \vdash_{\circ} R}\ \Pi L$$

$$\frac{\Gamma \vdash_{\bullet} A[t/x]}{\Gamma \vdash_{\bullet} \Sigma x.A}\ \Sigma R \qquad \frac{\Gamma \vdash_{\bullet} A}{\Gamma \vdash_{\bullet} \forall y.A}\ \forall R$$

In all rules, y is not free in Γ and R.

Rules for Constants

$$\frac{}{\Gamma \vdash_{\circ} 1}\ 1R \qquad \frac{\Gamma \vdash_{\circ} R}{1, \Gamma \vdash_{\circ} R}\ 1L \qquad \frac{}{0, \Gamma \vdash_{\circ} R}\ 0L \qquad \frac{\Gamma \vdash_{\bullet}}{\Gamma \vdash_{\bullet} \perp}\ \perp R \qquad \frac{}{\Gamma \vdash_{\bullet} \top}\ \top R$$

Figure 3: The LP proof system

Γ representing 1; the interpretation of $\Gamma \vdash_{\bullet} \Theta$ is $\bigwedge \Gamma \supset \bigvee \Theta$ where $\bigvee \Theta$ is the $\vee^{e}$-disjunction over Θ, with an empty Θ representing $\perp$.

An important requirement of LP is that the *end-sequents* of proofs have the form $\Gamma \vdash_{\circ} A$, where A can be of any polarity. The theorems of PIL are those formulas A such that $\vdash_{\circ} A$ is provable.

The introduction rules of LP are classified both by the polarity of the principal formula and by the side of the sequent where it occurs. Thus a classification such as **left-red** refers to *red formulas on the left-hand side.* The rules of LP can be divided into three groups: the right-red and left-red rules, the right-green rules, and the structural rules. The rules for quantifiers and constants follow the same pattern of classification as the propositional cases. Below, we provide some further explanation of each group of rules.

Red Introduction Rules: The LJ Fragment. These rules correspond to those of Gentzen's LJ sequent calculus, with some common variations in the handling of contraction and weakening and in the $\wedge L$ rule. These variations are well known to be equivalent to Gentzen's original presentation (see the system "G3i" in [TS96]). LP is thus clearly an extension of LJ. Rules of this group use the $\vdash_{\circ}$ mode exclusively. A formula is *purely red* if all of its subformulas are red. If a sequent with $\vdash_{\circ}$ is also composed exclusively of purely red formulas then only red-introduction rules will ever be applicable in any attempted derivation of the sequent. Thus LJ is contained in LP in a very strong sense: *an LP proof of an intuitionistic sequent* is *an LJ proof.*

It would be wrong, however to exclusively associate $\vdash_{\bullet}$ with classical deduction and $\vdash_{\circ}$ with intuitionistic deduction. Classical deduction can also use the $\vdash_{\circ}$ mode.

We have chosen to use multisets in this presentation of LP, although it is also possible to use sets. Using multisets allow us to avoid some unnecessary contractions such as in the left-red rules for $\vee$ and $\wedge$.

Structural Rules. A green-polarized formula can occur on the right-hand side of $\vdash_\circ$ because, for example, such a formula can be embedded inside the scope of a red connective. When this *polarity switch* occurs, the sequent must be preceded from above by a *Signal* rule in proofs. The structural rules *Signal*, *Store* and *Load* are critical because they allow classical and intuitionistic logics to coexist as more than disjoint fragments. These rules allow the green and red connectives to mix without restriction. The *Store* rule is a generalized form of *reductio ad absurdum.* The *Load* rule allows even classical proofs to use the $\vdash_\circ$ mode, after a contraction.

Right-Green Introduction Rules. We have chosen to allow introduction rules for the green-polarity connectives to occur only on the right-hand side. The reason for this choice is twofold. The first is one of economy: the missing "left-green" rules are not needed because the *Load* rule, which embodies a contraction, can be used to move a green formula to the right-hand side of the sequent. The second reason is that such a choice will force even classical proofs to have an intuitionistic structure, in the style of LC (see below). Specifically, when a green formula on the left is moved to the right, it becomes red-polarized and subject to introduction rules in the $\vdash_\circ$ mode. Left-green introduction rules will be used in an alternative proof system in Section 7. Of the right-green introduction rules, $\propto R$ is the only rule outside of the structural rules that involves a switch between the $\vdash_\bullet$ and $\vdash_\circ$ modes.

A critical invariant of LP is that *no green introduction rule is possible in the* $\vdash_\circ$ *mode.* This invariant defines the limitations of combining classical and intuitionistic reasoning, as explained by PIL's semantics. It is crucial to cut-elimination.

Contraction and weakening do not appear as explicit rules in LP except for the embedded contraction in *Load* and embedded weakening in I. Contraction and weakening are admissible by the following lemma, which is proved by induction on the structure of proofs. We also include the usual first-order substitution rule in this lemma.

Lemma 3 *If* $A, A, \Gamma \vdash_* \Theta$ *is provable, then* $A, \Gamma \vdash_* \Theta$ *is provable. If* $\Gamma \vdash_* \Theta$ *is provable, then* $A, \Gamma \vdash_* \Theta$ *is provable. If* x *is not free in* Γ *and* $\Gamma \vdash_* A$ *is provable then* $\Gamma \vdash_* A[t/x]$ *is provable for any term* t.

The following lemma, which follows from the admissibility of weakening, further relates the $\vdash_\circ$ and $\vdash_\bullet$ modes of proof.

Lemma 4 *If* $\Gamma \vdash_\circ A$ *is provable then* $\Gamma \vdash_\bullet A$ *is also provable. Furthermore, the sequent* $\vdash_\bullet A$ *is provable if and only if* $A^\perp \vdash_\circ A$ *is provable.*

Another important property, proved by induction on formulas, is that the initial rule can be applied to all formulas, not just atomic formulas.

Lemma 5 $A, \Gamma \vdash_\circ A$ *is provable for all formulas* A.

The two modes $\vdash_\circ$ and $\vdash_\bullet$ naturally define two forms of *equivalence.* A is equivalent to B in PIL if $\vdash_\circ (A \supset B) \wedge (B \supset A)$ is provable. On the other hand, we also have a classical notion of provable equivalence, which holds if $A \vdash_\bullet B$ and $B \vdash_\bullet A$ are both provable. We can refer to this latter form as *green-equivalence.*

In the context of the sequent calculus, we say that a set of formulas Γ is 0*-consistent* if $\Gamma' \vdash_\circ 0$ is not provable for any finite subset Γ' of Γ, and that the set is $\perp$*-consistent* if $\Gamma' \vdash_\circ \perp$ is not provable for any finite subset Γ' of Γ. If a set is $\perp$-consistent, then it is also 0-consistent, but *the reverse does not hold.* The set $\{a, a^\perp\}$ is not $\perp$-consistent, but it is 0-consistent. In intuitionistic logic only red-polarized formulas are used, so one cannot even speak of $\perp$-inconsistency. In classical logic, 0 is equivalent to $\perp$, so again the problem is nullified. The discrepancy appears because we allow the classical and intuitionistic polarities to mix freely in PIL.

It must be emphasized that it would be wrong to simply associate $\vdash_{\bullet}$ with green formulas and $\vdash_{\circ}$ with red formulas. Red formulas may also occur to the right-side of $\vdash_{\bullet}$. When $\vee$ occurs on the right-side of $\vdash_{\bullet}$, the availability of contraction (via *Store* and *Load*) renders it equivalent to $\vee^e$.

In PIL, the *law of excluded middle* is provable in the form $\vdash_{\circ} A \vee^e A^{\perp}$ as well as $\vdash_{\bullet} A \vee A^{\perp}$. On the other hand, the *disjunction and existence properties* are also retained in the forms $\vdash_{\circ} A \vee B$ and $\vdash_{\circ} \exists x.A$. As further examples, *Peirce's formula* $((p \supset q) \supset p) \supset p$ becomes provable by replacing the outermost and innermost $\supset$ with classical implication (definable as $A^{\perp} \vee^e B$), while keeping the middle one intuitionistic. We give a detailed proof below of this version of Peirce's formula as an illustration of how to use the LP system. The formula in negation normal form that we shall prove is $((p^{\perp} \vee^e q) \propto p^{\perp}) \vee^e p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p^{\perp}, p, q^{\perp} \vdash_{\circ} p}\,I}{p^{\perp}, p, q^{\perp} \vdash_{\bullet}}\,\mathit{Load}}{p^{\perp} \vdash_{\bullet} p^{\perp}, q}\,\mathit{Store} \times 2}{p^{\perp} \vdash_{\bullet} p^{\perp} \vee^e q}\,{\vee^e}R}{p^{\perp} \vdash_{\circ} p^{\perp} \vee^e q}\,\mathit{Signal} \qquad \dfrac{\dfrac{\dfrac{}{p, p^{\perp} \vdash_{\circ} p}\,I}{p, p^{\perp} \vdash_{\bullet}}\,\mathit{Load}}{p^{\perp} \vdash_{\bullet} p^{\perp}}\,\mathit{Store}}{p^{\perp} \vdash_{\bullet} (p^{\perp} \vee^e q) \propto p^{\perp}}\,{\propto}R}{\vdash_{\bullet} (p^{\perp} \vee^e q) \propto p^{\perp}, p}\,\mathit{Store}}{\vdash_{\bullet} ((p^{\perp} \vee^e q) \propto p^{\perp}) \vee^e p}\,{\vee^e}R}{\vdash_{\circ} ((p^{\perp} \vee^e q) \propto p^{\perp}) \vee^e q}\,\mathit{Signal}$$

The following version of *Markov's Principle* is also directly provable.

$$[(\Pi x. \sim P(x) \vee \sim (P(x)^{\perp})) \wedge (\sim\sim \exists x.P(x))] \supset \Sigma x.P(x)$$

Of course, had we written a purely classical version of the formula, using De Morgan negation instead of $\sim$, the principle would become trivialized. This particular mixture of green and red polarities is more meaningful, however, because its proof must invoke the Π clause, which states the "decidability" of P in yet another form of the excluded middle that is possible in PIL. For any valid term t one of $P(t)$ and $P(t)^{\perp}$ is red-polarized and the other is green-polarized. Assuming that $P(t)$ is red-polarized, the meaning of $\sim P(t) \vee \sim (P(t)^{\perp})$ is *either $P(t)$ is never true or there will always be future classical worlds in which it is true* (see Section 5 for formal semantics). All the admissible rules of intuitionistic logic are inherited by PIL, which is clear from the strong embedding of LJ inside LP.

4.1 The LC Fragment

The *purely classical* fragment of PIL consists of all connectives except $\supset$, $\propto$, Π and Σ. Classical end-sequents can be of the form $\vdash_{\circ} A_1 \vee^e \ldots \vee^e A_n \vee^e \perp$ or $\vdash_{\bullet} A_1, \ldots, A_n$. In this fragment, $\vee$ and $\vee^e$, as well as $\wedge$ and $\wedge^e$, are provably equivalent. One can show, for example, that $A \vee^e B \vdash_{\bullet} A \vee B$ is provable for each possible polarity combination of A and B (the equivalence is also semantically provable). The constants 0 and $\perp$ are also classically equivalent. However, even in the $\vdash_{\bullet}$ mode, that is to say, even when contraction is available, intuitionistic implication and universal quantification do *not* collapse into their classical counterparts (classical implication should be defined from disjunction). This phenomenon has been well explained in our semantics.

The completeness of this fragment with respect to classical logic can be proved by showing the admissibility of the rules of LK. However, the classical fragment of LP bears closer resemblance to *LC* [Gir91] than to LK. LC defines the polarities "positive" and "negative." In terms of PIL, $\vee$, $\wedge$, $\exists$, 1 and 0 would be considered positive while their duals are negative. Thus if we stay exclusively within the classically-oriented fragment, positive formulas are red-polarized and negative ones are green-polarized. However, the polarity of an LC formula is also dependent on the polarity of its subformulas. When A and B are both positive, $A \vee B$ in LC is interpreted as $A \vee B$ in PIL; otherwise, it is $A \vee^e B$ (and dually for $\wedge$). LC sequents with a non-empty "stoup" correspond to the $\vdash_\circ$ mode while those without a stoup correspond to $\vdash_\bullet$. LC introduction rules on the stoup formula correspond to right-red introduction rules in LP; the introduction rules for "negative" connectives *in the presence of a stoup* correspond to left-red rules while those without a stoup correspond to right-green rules. The following are representative examples of LC rules and their equivalents in LP. Here, P is positive and N is negative. The formula to the right of the semicolon is the stoup.

$$\frac{\vdash \Gamma, N, P; S}{\vdash \Gamma, N \vee P; S} \quad \longmapsto \quad \frac{\Gamma, P, N \vdash_\circ S}{\Gamma, P \wedge N \vdash_\circ S}\ \wedge L$$

$$\frac{\vdash \Gamma, N, P;}{\vdash \Gamma, N \vee P;} \quad \longmapsto \quad \frac{\Gamma \vdash_\bullet N, P}{\Gamma \vdash_\bullet N \vee^e P}\ \vee^e R$$

$$\frac{\vdash \Gamma; P \quad \vdash \Delta, N;}{\vdash \Gamma\Delta; P \wedge N} \quad \longmapsto \quad \frac{\Gamma\Delta \vdash_\circ P \quad \dfrac{\Gamma\Delta \vdash_\bullet N}{\Gamma\Delta \vdash_\circ N}\ \mathit{Signal}}{\Gamma\Delta \vdash_\circ P \wedge N}\ \wedge R$$

The structural rules of LC, including weakening and contraction outside of the stoup, are admissible in LP. The splitting of the context in the LC $\wedge$-rule does not change classical provability (we could have done the same in LP). Except for different styles in the management of sequents, LC is a fragment of LP. An important difference is that the original LC does not contain intuitionistic implication. While it allows classical logic to share the structure of intuitionistic *proofs,* it does not allow the two logics to mix at the level of *formulas.* In our version of LC the classical connectives can also join formulas containing purely intuitionistic connectives.

The original LC sequent calculus contains an invariant that no positive introduction rule can be applied outside of the stoup. This invariant is subsumed by the LP invariant that no green introduction rule is possible in the $\vdash_\circ$ mode[3].

4.2 Cut-Elimination

The admissible cut rule of LP in terms of end-sequents appears as follows:

$$\frac{\Gamma \vdash_\circ A \quad A, \Gamma' \vdash_\circ B}{\Gamma\Gamma' \vdash_\circ B}\ \mathit{Cut}$$

With the two modes of sequents in PIL, several other cuts are also admissible, including:

$$\frac{\Gamma \vdash_\bullet A, \Theta \quad A, \Gamma' \vdash_\bullet \Theta'}{\Gamma\Gamma' \vdash_\bullet \Theta\Theta'}\ \mathit{cut} \qquad \frac{\Gamma \vdash_\circ A \quad A, \Gamma' \vdash_\bullet \Theta'}{\Gamma\Gamma' \vdash_\bullet \Theta'}\ \mathit{cut} \qquad \frac{\Gamma \vdash_\circ A \quad \Gamma' \vdash_\circ A^\perp}{\Gamma\Gamma' \vdash_\bullet}\ \mathit{cut}$$

Still other forms can be written, but *Cut* is the principal one with a conclusion in the $\vdash_\circ$ mode. The admissibility of other forms of cut may also depend on the polarity of the cut formula.

Cut-elimination can be proved in the usual way, although here we formally derive the result semantically in Section 6. It is worthwhile, however, to also point out the most important aspects of the syntactic proof. Such proofs are tedious because they involve many cases, but ones that require special attention are those where the cut formula is subject to contraction. The LP invariant that *no green introduction rule is possible in the* $\vdash_\circ$ *mode,* which we motivated semantically in Section 3, is needed syntactically because some *but not all* formulas are subject to contraction in proofs.

The syntactic proof is by a simultaneous induction on the several variants of *cut.* The inductive measure is the usual lexicographical ordering on the size of the cut formula and the height of subproofs. The size of all literals is one, so A and $A^\perp$ are always of the same size. Also needed is a strengthening of Lemma 3, which not only admits weakening and contraction on the left side but also preserves the height of proofs. Cuts are separated into *key cases,* where the cut formula is principal in both premises, and *parametric* cases, where at least one cut formula is not immediately principal. Parametric cuts are shown to be always permutable to cuts of smaller height measures.

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[3] Because it so happens that all red-polarized classical connectives are positive, the LC version of the invariant accidentally resembles *focusing.* Completely focusing LP is much more difficult (see Section 8 for a brief discussion).

Cut-elimination in LP diverges from that of LJ when a formula is contracted as a result of *Load*. Here the polarities and special structural rules of LP become relevant. We show one of the most representative cases. Suppose the cut to be reduced is of the form

$$\dfrac{\dfrac{\dfrac{\dfrac{A^{\perp}, B^{\perp}, \Gamma \vdash_{\bullet}}{\Gamma \vdash_{\bullet} A, B}\,\mathit{Store} \times 2}{\Gamma \vdash_{\bullet} A \vee^{e} B}\,\vee^{e} R}{\Gamma \vdash_{\circ} A \vee^{e} B}\,\mathit{Signal} \qquad \dfrac{\dfrac{A \vee^{e} B, \Gamma' \vdash_{\circ} A^{\perp} \quad A \vee^{e} B, \Gamma' \vdash_{\circ} B^{\perp}}{A \vee^{e} B, \Gamma' \vdash_{\circ} A^{\perp} \wedge B^{\perp}}\,\wedge R}{A \vee^{e} B, \Gamma' \vdash_{\bullet}}\,\mathit{Load}}{\Gamma\Gamma' \vdash_{\bullet}}\,\mathit{cut}$$

Crucially, a *key-case* cut on such a green-polarized formula can only occur in this form, with the right-side subproof ending in the $\vdash_{\bullet}$ mode, because of the invariant concerning the $\vdash_{\circ}$ mode. We can show that, without loss of generality, the sequent $\Gamma \vdash_{\bullet} A, B$ can be assumed to be preceded from above by two *Store* rules. This cut is permuted into the following form:

$$\dfrac{\dfrac{\Gamma \vdash_{\circ} A \vee^{e} B \quad A \vee^{e} B, \Gamma' \vdash_{\circ} B^{\perp}}{\Gamma\Gamma' \vdash_{\circ} B^{\perp}}\,\mathit{Cut} \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash_{\circ} A \vee^{e} B \quad A \vee^{e} B, \Gamma' \vdash_{\circ} A^{\perp}}{\Gamma\Gamma' \vdash_{\circ} A^{\perp}}\,\mathit{Cut} \qquad A^{\perp}, B^{\perp}, \Gamma \vdash_{\bullet}}{B^{\perp}, \Gamma\Gamma' \vdash_{\bullet}}\,\mathit{cut}}{\Gamma\Gamma' \vdash_{\bullet}}\,\mathit{cut}$$

We have implicitly used the strengthened form of Lemma 3 here to avoid writing copies of Γ and Γ'. The original *cut* is reduced to four cuts: the lower two cuts involve smaller cut formulas while the upper two involve smaller height measures.

The other cases of the syntactic cut-elimination proof for LP are either variations of the above case or are the same as those of a typical cut-elimination proof for an intuitionistic sequent calculus based on LJ.

Although syntactic cut-elimination is expected of any Gentzen-style system, the mostly mechanical procedure does not reveal all the subtleties that allow the process to succeed. One way to understand Gentzen's analysis of cut-elimination for intuitionistic logic is that the following cut must *not* be admissible:

$$\dfrac{P, \Gamma \vdash Q \quad {\sim}P, \Gamma' \vdash Q}{\Gamma\Gamma' \vdash Q}\,\mathit{wrong\text{-}cut}$$

Allowing this cut is equivalent to admitting the law of excluded middle as a global axiom. Gentzen's solution to excluding this cut is to force the conclusion of the cut to be written in the following form: $\Gamma\Gamma' \vdash Q, Q$. Then, by restricting proofs to single formulas on the right, this conclusion obviously has no cut-free proof.

The single-conclusion invariant of LJ has been extended to a more generic form in LP, which is worth repeating: *green-polarity introduction rules are not available in the* $\vdash_{\circ}$ *mode.* The "*wrong cut*" in the context of LP can also have the following form:

$$\dfrac{\Gamma \vdash_{\bullet} P \quad P, \Gamma' \vdash_{\circ} Q}{\Gamma\Gamma' \vdash_{\circ} Q}\,\mathit{bad\text{-}cut}$$

Here, P and Q are both red-polarized. The *non-admissibility* of this cut is easily shown semantically: the validity of P in classical worlds does not imply its validity in all worlds. During the permutation of cuts, an attempted cut of the above form is never encountered. Now suppose we had violated the $\vdash_{\circ}$ invariant and allowed the following introduction rule:

$$\dfrac{A, A \wedge^{e} B, \Gamma \vdash_{\circ} R}{A \wedge^{e} B, \Gamma \vdash_{\circ} R}\,\mathit{naive\text{-}}\wedge^{e} L$$

Then given the following cut

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash_{\bullet} A \quad \Gamma \vdash_{\bullet} B}{\Gamma \vdash_{\bullet} A \wedge^{e} B}\,\wedge^{e} R}{\Gamma \vdash_{\circ} A \wedge^{e} B}\,\mathit{Signal} \qquad \dfrac{A, A \wedge^{e} B, \Gamma' \vdash_{\circ} R}{A \wedge^{e} B, \Gamma' \vdash_{\circ} R}\,\mathit{naive\text{-}}\wedge^{e} L}{\Gamma\Gamma' \vdash_{\circ} R}\,\mathit{Cut}$$

we would be led to attempt the following, invalid reduction:

$$\dfrac{\Gamma \vdash_{\bullet} A \qquad \dfrac{\Gamma \vdash_{\circ} A \wedge^{e} B \quad A, A \wedge^{e} B, \Gamma' \vdash_{\circ} R}{A, \Gamma\Gamma' \vdash_{\circ} R}\, \mathit{Cut}}{\Gamma\Gamma' \vdash_{\circ} R}\, \textit{bad-cut}$$

This kind of subtlety can easily be missed in a mechanical, inductive proof. The ability to reason semantically about the proof system complements the inductive procedure. We return to semantics in the following sections.

5 The Semantics of First-Order PIL

The interpretation of the propositional fragment is self-contained and relatively clear. The propositional fragment already illustrates many of the essential characteristics of the full semantics. The introduction rules for the quantifiers in LP are fairly standard: indeed proof-theoretically the first-order quantifiers are rather mundane. The same is not true in model theory, much of which is centered on the interpretation of predicate logic. The extension of PIL semantics to the first-order case also cannot merely follow the example of first-order intuitionistic model theory.

First-order PIL poses the following challenge. In order to preserve the excluded middle property at classical worlds $\mathbf{c}$, what should be the *domain* $\mathcal{D}(\mathbf{c})$ of these worlds? If the classical domains are universal (i.e., contain all possible terms), then the intuitionistic Π would collapse into the classical $\forall$ when interpreted in the classical worlds, a fact that is inconsistent with the interpretation of $\supset$. It is also not clear how to specify a proof system for this interpretation that preserves cut-elimination (the LP invariant will have to be compromised). With a restricted domain, each classical world can no longer claim to be *maximally* $\perp$*-consistent* in the sense that *"*$\mathbf{c} \models A$ *if and only if* $\mathbf{c} \not\models A^{\perp}$*."* For some new parameter (term) t_n, adding a proposition $p(t_n)$ may result in a $\perp$-consistent extension. Each classical world can at most claim to be maximally consistent with respect to its domain. This entails that there could be more than a single "layer" of classical worlds, and that the restriction $\triangle_{\mathbf{c}} = \{\mathbf{c}\}$ can no longer be observed. There could now be classical worlds above other classical worlds as well as non-classical worlds in between them. Most importantly, there can be infinite chains of classical worlds without bound. In this context there is a danger of loosing monotonicity for the green connectives, most notably in the case of the classical universal quantifier $\forall$. The simultaneous preservation of monotonicity and an acceptable form of the excluded middle property becomes a challenge.

A first-order Kripke hybrid model is a structure $\langle \mathbf{W}, \preceq, \mathbf{C}, \models, \mathcal{D} \rangle$ where $\mathbf{W}$, $\preceq$, $\mathbf{C}$, and $\models$ are as in the propositional case. The component $\mathcal{D}$ is a domain function mapping elements of $\mathbf{W}$ to non-empty sets of parameters. $\mathcal{D}(\mathbf{u})$ is called the domain of $\mathbf{u}$. For convenience we do not distinguish predicates and parameters in the syntax from their semantic interpretation, since this correspondence is quite standard. Let $Pm(A)$ represent the set of all parameters appearing in formula A. Let $\mathcal{L}(D)$ be the set of all formulas with parameters in the set D. The following conditions are also required of any model:

1. For $\mathbf{u}, \mathbf{v} \in \mathbf{W}$ and atomic formula a, if $\mathbf{u} \preceq \mathbf{v}$ then $\mathbf{u} \models a$ implies $\mathbf{v} \models a$.
2. if $\mathbf{u} \preceq \mathbf{v}$ then $\mathcal{D}(\mathbf{u}) \subseteq \mathcal{D}(\mathbf{v})$.
3. if $\mathbf{u} \models p(t_1, \ldots, t_n)$ then $t_1, \ldots, t_n \in \mathcal{D}(\mathbf{u})$

The definition of $\models$ for the first-order case not only defines the cases for the quantifiers, but must modify the green propositional cases as well. The red-polarity cases are not modified and remain identical to their definition in traditional Kripke models for intuitionistic logic. For convenience however, we list all the cases. All cases of $\models$ are now defined with the understanding that they contain the implicit stipulation that $\mathbf{u} \models A$ *only if all parameters of* A *are in* $\mathcal{D}(\mathbf{u})$.

We assume $\mathbf{c}, \mathbf{k} \in \mathbf{C}$ but that $\mathbf{v}$ and $\mathbf{u}$ represent arbitrary worlds. Again, $\triangle_{\mathbf{c}}$ is $\{\mathbf{k} \in \mathbf{C} : \mathbf{k} \succeq \mathbf{c}\}$.

- $\mathbf{u} \models \Pi x.A$ iff for all $\mathbf{v} \succeq \mathbf{u}$, and for all $t \in \mathcal{D}(\mathbf{v})$, $\mathbf{v} \models A[t/x]$.

- $\mathbf{u} \models \exists x.A$ iff for some $s \in \mathcal{D}(\mathbf{u})$, $\mathbf{u} \models A[s/x]$.
- $\mathbf{u} \models 1$ and $\mathbf{u} \not\models 0$
- $\mathbf{u} \models A \vee B$ iff $\mathbf{u} \models A$ or $\mathbf{u} \models B$
- $\mathbf{u} \models A \wedge B$ iff $\mathbf{u} \models A$ and $\mathbf{u} \models B$
- $\mathbf{u} \models A \supset B$ iff for all $\mathbf{v} \succeq \mathbf{u}$, $\mathbf{v} \models A$ implies $\mathbf{v} \models B$
- $\mathbf{c} \models \forall x.A$ iff for all $\mathbf{k} \in \triangle_{\mathbf{c}}$ and all $t \in \mathcal{D}(\mathbf{k})$, $\mathbf{k} \not\models A[t/x]^{\perp}$.
- $\mathbf{c} \models \Sigma x.A$ iff for all $\mathbf{k} \in \triangle_{\mathbf{c}}$, there exist some $\mathbf{v} \succeq \mathbf{k}$ and some $s \in \mathcal{D}(\mathbf{v})$ such that $\mathbf{v} \not\models A[s/x]^{\perp}$.
- $\mathbf{c} \models \top$ and $\mathbf{c} \not\models \bot$
- $\mathbf{c} \models a^{\perp}$ iff for all $\mathbf{k} \in \triangle_{\mathbf{c}}$, $\mathbf{k} \not\models a$, for all green literals $a^{\perp}$.
- $\mathbf{c} \models A \propto B$ iff for all $\mathbf{k} \in \triangle_{\mathbf{c}}$, there exists some $\mathbf{v} \succeq \mathbf{k}$ such that $\mathbf{v} \models A$ and $\mathbf{v} \not\models B^{\perp}$.
- $\mathbf{c} \models A \vee^{e} B$ iff for all $\mathbf{k} \in \triangle_{\mathbf{c}}$, $\mathbf{k} \not\models A^{\perp}$ or $\mathbf{k} \not\models B^{\perp}$.
- $\mathbf{c} \models A \wedge^{e} B$ iff for all $\mathbf{k} \in \triangle_{\mathbf{c}}$, $\mathbf{k} \not\models A^{\perp}$ and $\mathbf{k} \not\models B^{\perp}$.

The critical *"lifting"* rule for green-polarized E at all worlds is retained, and remains consistent with the definition of $\models$ in classical worlds:

- $\mathbf{u} \models E$ iff for all $\mathbf{k} \in \triangle_{\mathbf{u}}$, $\mathbf{k} \models E$.

Note that *if* we can assume the property $\triangle_{\mathbf{c}} = \{\mathbf{c}\}$, as in the propositional fragment, then the definitions for the green-polarity cases will collapse into forms that are exact duals of the red cases. Then it follows easily that $\mathbf{c} \models A$ if and only if $\mathbf{c} \not\models A^{\perp}$. Therefore, the definition of $\models$ in the propositional cases would become equivalent to their original forms in Section 3. The first-order semantics is thus entirely consistent with the propositional one and a completeness proof for the first-order semantics would be valid for all of PIL. In fact, if we can only assume that there can be no infinite chains of classical worlds, then these new definitions would also not be needed.

The core properties of Lemma 1 are now modified. Monotonicity is sustained, but the excluded middle property for classical worlds takes on a weakened *but adequate* form:

Lemma 6 *In a first-order Kripke hybrid model, for every* $\mathbf{u}, \mathbf{v} \in \mathbf{W}$ *and every* $\mathbf{c}, \mathbf{k} \in \mathbf{C}$*:*

a. *If* $\mathbf{u} \preceq \mathbf{v}$*, then* $\mathbf{u} \models A$ *implies* $\mathbf{v} \models A$ *(monotonicity)*

b. *If* $\mathbf{c} \models A$ *then* $\mathbf{c} \not\models A^{\perp}$

c. *If* $\mathbf{c} \not\models E$ *then for some* $\mathbf{k} \in \triangle_{\mathbf{c}}$*,* $\mathbf{k} \models E^{\perp}$*, for all green formulas* E*.*

The three properties are easily verified from the definitions. Monotonicity is built in to the definitions of the green connectives by using "*for all* $\mathbf{k} \in \triangle_{\mathbf{c}}$, *. . .,*" which precedes all clauses.

Although the following arguments are implied by the completeness proof to follow, we provide them here separately as illustrations of reasoning within this semantics.

- $\mathbf{c} \models A \vee^{e} A^{\perp}$

  This property shows that the definitions of $\models$ in the green polarity cases, although not precisely "duals" of their red counterparts, nevertheless preserve the classical disjunction. To see this, assume that for some classical $\mathbf{c}$, $\mathbf{c} \not\models A \vee^{e} A^{\perp}$, then by the third property of the lemma above there is some classical $\mathbf{k} \succeq \mathbf{c}$ such that $\mathbf{k} \models A^{\perp} \wedge A$. But this implies that $\mathbf{k} \models A$ and $\mathbf{k} \models A^{\perp}$, which contradicts the second property of the lemma.

- It is no longer the case that $\mathbf{c} \models A \vee A^{\perp}$, which holds in propositional models. However, this failure does not contradict the *classical* equivalence of $\vee$ and $\vee^{e}$. Recall that in LP, the provability of the sequent $\vdash_{\bullet} A \vee A^{\perp}$ should actually be interpreted as the provability of $\vdash_{\circ} (A \vee A^{\perp}) \vee^{e} \perp$. Thus we should be asking if $\mathbf{c} \models (A \vee A^{\perp}) \vee^{e} \perp$. This holds as follows: suppose $\mathbf{c} \not\models (A \vee A^{\perp}) \vee^{e} \perp$, then there is some classical $\mathbf{k} \succeq \mathbf{c}$ such that $\mathbf{k} \models A^{\perp} \wedge^{e} A$. But this means that *for all* $\mathbf{k}' \in \triangle_{\mathbf{k}}$, $\mathbf{k}' \not\models A$ and $\mathbf{k}' \not\models A^{\perp}$. However, $\mathbf{k} \in \triangle_{\mathbf{k}}$ and either A or $A^{\perp}$ must be green-polarized: without loss of generality, assume it is A. By the third property of the lemma, there is *some* $\mathbf{k}' \in \triangle_{\mathbf{k}}$ such that $\mathbf{k}' \models A^{\perp}$: a contradiction.
- Although the new definition of $\models$ for $\forall$ resembles intuitionistic universal quantification, it is in fact still classical. Assume that $\mathbf{c} \models \forall x.(A \vee^{e} B)$ where x does not occur free in A. The $\forall$ quantifier allows A to "escape its scope," for it also holds that $\mathbf{c} \models A \vee^{e} \forall x.B$. To see this, suppose $\mathbf{c} \not\models A \vee^{e} \forall x.B$, then for some $\mathbf{k} \in \triangle_{\mathbf{c}}$, $\mathbf{k} \models A^{\perp} \wedge \exists x.B^{\perp}$. But by monotonicity we also have that $\mathbf{k} \not\models A^{\perp} \wedge B^{\perp}[t/x]$ for any $t \in \mathcal{D}(\mathbf{k})$: a contradiction follows. The same argument fails for Π, the dual of which is Σ: from $\mathbf{k} \models \Sigma x.B^{\perp}$ we get that for some $\mathbf{v} \succeq \mathbf{k}$, $\mathbf{v} \not\models B[s/x]$ for some $s \in \mathcal{D}(\mathbf{v})$, which leads to no contradiction.

Now consider the sentence

$$(\perp \supset 0) \ \wedge \ \forall x.p(x) \ \wedge \ \Sigma y.p(y)^{\perp}$$

This sentence is consistent in the given semantics. However, it can only have infinite models, with no upper-bound to classical worlds. The clause $\perp \supset 0$, which is consistent, excludes models with any imaginary worlds (without this clause the two green-polarized formulas can be satisfied by a model with a single imaginary world). This suggests a model with the following structure:

$$\begin{array}{ccll}
\textit{Domain} & & \models & \\
\vdots & & \vdots & \\
t_1, t_2, t_3 & \mathbf{c_3} & \models & p(t_1), p(t_2), p(t_3) \\
t_1, t_2, t_3 & \mathbf{v_2} & \models & p(t_1), p(t_2) \\
t_1, t_2 & \mathbf{c_2} & \models & p(t_1), p(t_2) \\
t_1, t_2 & \mathbf{v_1} & \models & p(t_1) \\
t_1 & \mathbf{c_1} & \models & p(t_1)
\end{array}$$

There must be some classical world $\mathbf{c_1}$ in any model of this sentence where $p(s)$ is forced for every parameter s in $\mathcal{D}(\mathbf{c_1})$. In order to satisfy $\Sigma y.p(y)^{\perp}$, there also needs to be a new world $\mathbf{v_1}$ above $\mathbf{c_1}$ such that $\mathbf{v_1} \not\models p(t)$ for some $t \in \mathcal{D}(\mathbf{v})$. To satisfy monotonicity with respect to the classical world, the domain of $\mathbf{v_1}$ must be expanded with a $t \notin \mathcal{D}(\mathbf{c_1})$. However, there must be another classical world $\mathbf{c_2}$ above $\mathbf{v_1}$ since $\mathbf{v_1}$ cannot be imaginary. To satisfy the monotonicity of $\forall x.p(x)$, $p(t)$ must now be satisfied in $\mathbf{c_2}$, which necessitates another world $\mathbf{v_2}$ with yet another new parameter, resulting in an infinite chain of alternating classical and non-classical worlds. Under the restriction $\triangle_{\mathbf{c}} = \{\mathbf{c}\}$, there is no model for this sentence. The generalized definition of $\models$ is required to distinguish the classical quantifiers from the purely intuitionistic ones.

From the definitions of $\models$ for $\vee^{e}$ and $\forall$, one might observe that the set of classical worlds in a hybrid model in fact forms a Kripke model on their own. In this submodel, these classical connectives are interpreted *almost* as if they were intuitionistic ones (thinking of classical implication as $A^{\perp} \vee^{e} B$). However, there is an important distinction: instead of "$\mathbf{k} \models A$" we say "$\mathbf{k} \not\models A^{\perp}$," which can be considered as another form of double negation.

When restricted to intuitionistic logic (purely red-polarized formulas), the interpretation of the connectives and quantifiers preserves intuitionistic validity. Every hybrid model can be considered a regular Kripke model by simply ignoring the definition of $\models$ on green formulas.

6 Soundness and Completeness

The soundness and completeness of LP with respect to the Kripke-style semantics are proved with respect to cut-free proofs. We use the notation $\mathcal{M} \models (\Gamma \vdash_{\circ} A)$ to mean $\mathcal{M} \models \bigwedge \Gamma \supset A$, where $\bigwedge \Gamma$

is the $\wedge$-conjunction of formulas in Γ. Sequents $\Gamma \vdash_\bullet A_1, \ldots, A_n$ are included in this definition as $\Gamma \vdash_\circ A_1 \vee^e \ldots \vee^e A_n \vee^e \perp$ (so $\Gamma \vdash_\bullet$ is treated as $\Gamma \vdash_\circ \perp$).

We prove the results for the full first-order semantics, which subsumes the propositional case.

Theorem 7 *(Soundness) If $\Gamma \vdash_\circ A$ is provable, then $M \models (\Gamma \vdash_\circ A)$ for every Kripke hybrid model M.*

The soundness direction is proved by induction on the structure of proofs. We show two cases. The most interesting case is that of the structural rule *Store*. Without loss of generality, assume that the context Θ contains a single formula B (which could be $\perp$). Assume that for all models $\mathcal{M}$ and all u in $\mathcal{M}$, $u \models (A^\perp \wedge \bigwedge \Gamma)$ implies $u \models (B \vee^e \perp)$. Now to prove the conclusion, assume $u \models \bigwedge \Gamma$. For *all* classical worlds $c_u \succeq u$ in the model, we need to show that $c_u \models A \vee^e B$. Now, for all $c \in \triangle_{c_u}$, it is thus assumed that $c \not\models (A^\perp \wedge \bigwedge \Gamma)$ or $c \models B \vee^e \perp$ (since c is itself a world in $\mathcal{M}$). If $c \models B \vee^e \perp$, then by definition, for all $k \in \triangle_c$, $k \not\models B^\perp$ (since $k \models \perp^\perp = 1$). If $c \not\models (A^\perp \wedge \bigwedge \Gamma)$, then $c \not\models A^\perp$, since $c \models \bigwedge \Gamma$ by monotonicity. In either case we have shown that that for all $c \in \triangle_{c_u}$, $c \not\models A^\perp$ or $c \not\models B^\perp$ and thus by definition $c_u \models A \vee^e B$. In the case of ΣR, the assumption is that $u \models \bigwedge \Gamma$ implies $c_u \models A[t/x]$ for all $c_u \in \triangle_u$, which implies $c_u \not\models A[t/x]^\perp$ by Lemma 6. Then either $t \in \mathcal{D}(c_u)$ or the substitution is vacuous. If it is not vacuous, then t is the existential witness and since $c_u \succeq c_u$, $c_u \models \Sigma x.A$. If the substitution is vacuous, then since $\mathcal{D}(c_u)$ is always non-empty, we can select any element of $\mathcal{D}(c_u)$ as the witness. The other cases of the proof are either similar or straightforward.

The organization and presentation of our completeness proof follows that of Fitting [Fit69], which is in turn based on Kripke's proof [Kri65] and relies on showing the existence of Hintikka-type saturations. This style of proof shows the existence of a countermodel given an unprovable formula. Another style of proof follows the strategy of Henkin in creating maximally consistent sets, which in turn creates a canonical countermodel that falsifies every unprovable formula. We wish to have a completeness proof that suggests a procedure for constructing countermodels. While our proof here is designed to invalidate a given formula, we also rely on the existence of Henkin-style maximally consistent saturations in forming the classical worlds. Additionally, our proof is given directly for a primarily single-conclusion system as opposed to a multiple-conclusion one (i.e., the Beth-Fitting tableau system). First, we modify LP as given by using sets (on the left-hand side) as opposed to multisets in the representation of sequents. We also assume that the principal formula is always persistent in the left-side context in the premise of each inference rule: this assumption is valid by the admissibility of weakening and contraction (Lemma 3).

We define an *antisequent* to be a pair of enumerable sets of formulas Γ and Δ, which we write as $\Gamma \not\vdash_\circ \Delta$. An antisequent $\Gamma \not\vdash_\circ \Delta$ is defined to be *consistent* if for all finite subsets Γ' of Γ and for all formulas $A \in \Delta$, $\Gamma' \vdash_\circ A$ is not provable in LP; otherwise, it is *inconsistent.* Antisequents associated with the initial rules of LP, such as $a, \Gamma \not\vdash_\circ a, \Delta$ and $0, \Gamma \not\vdash_\circ \Delta$, are clearly inconsistent. If $\perp$ appears on the right-hand side of a consistent antisequent, then the left-hand context is $\perp$-consistent in the sense of Section 4. Intuitively, an antisequent represents a (possibly infinite) multiple conclusion sequent, or a set of signed formulas in a tableau with formulas in Γ signed T and formulas in Δ signed F.

We define a *Hintikka-Henkin pair* as a pair of sets of *consistent* antisequents of the form $(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ is a non-empty set of the form $\{S_1, S_2, \ldots, S_i, \ldots\}$ and $\mathcal{K}$ is a (possibly empty) subset of $\mathcal{H}$, and is enumerable in the form $\{K^1, K^2, \ldots, K^j, \ldots\}$. Given an antisequent $S_i = \Gamma_i \not\vdash_\circ \Delta_i$, let $Pm(S_i)$ now also represent the set of all parameters occurring in Γ_i or Δ_i. For each S_i in $\mathcal{H}$ there is also an associated non-empty set of parameters D^{S_i} such that $Pm(S_i) \subseteq D^{S_i}$. Every antisequent S_i in each set of the pair must satisfy the following properties. Here, E is again a *green*-polarity formula:

1. if $A \wedge B \in \Gamma_i$, then $A \in \Gamma_i$ and $B \in \Gamma_i$
2. if $A \vee B \in \Gamma_i$, then $A \in \Gamma_i$ or $B \in \Gamma_i$
3. if $A \supset B \in \Gamma_i$, then either $A \in \Delta_i$ or $B \in \Gamma_i$.

4. if $A \wedge B \in \Delta_i$ then either $A \in \Delta_i$ or $B \in \Delta_i$.
5. if $A \vee B \in \Delta_i$ then $A \in \Delta_i$ and $B \in \Delta_i$.
6. if $A \supset B \in \Delta_i$ then there exists an $S_j = \Gamma_j \nvdash_\circ \Delta_j$ in $\mathcal{H}$ such that $D^{S_i} \subseteq D^{S_j},\ A, \Gamma_i \subseteq \Gamma_j$ and $B \in \Delta_j$.
7. if $\exists x.A \in \Gamma_i$, then $A[t/x] \in \Gamma_i$ for some $t \in D^{S_i}$.
8. if $\Pi x.A \in \Gamma_i$, then $A[t/x] \in \Gamma$ for all $t \in D^{S_i}$.
9. if $\exists x.A \in \Delta_i$, then $A[t/x] \in \Delta_i$ for all $t \in D^{S_i}$.
10. if $\Pi x.A \in \Delta_i$, then there is an $S_j = \Gamma_j \nvdash_\circ \Delta_j$ in $\mathcal{H}$ such that $D^{S_i} \subseteq D^{S_j}$, $\Gamma_i \subseteq \Gamma_j$ and for some $t \in D^{S_j}$, $A[t/x] \in \Delta_j$.
11. if $E \in \Delta_i$ then there exists a $K^j = \Gamma^j \nvdash_\circ \Delta^j$ in $\mathcal{K}$ such that $D^{S_i} \subseteq D^{K^j}$ and $E^\perp, \Gamma_i \subseteq \Gamma^j$.
12. $\perp \in \Delta^i$ for each $K^i = \Gamma^i \nvdash_\circ \Delta^i$ in $\mathcal{K}$.
13. given $K^i = \Gamma^i \nvdash_\circ \Delta^i$ in $\mathcal{K}$, for any formula A with parameters in D^{K^i}, either $A \in \Gamma^i$, or $A, \Gamma^i \nvdash_\circ \perp$ is inconsistent.
14. given $K^i = \Gamma^i \nvdash_\circ \Delta^i$ in $\mathcal{K}$, for any formula A with parameters in D^{K^i}, either $A \in \Delta^i$ or $\Gamma^i \nvdash_\circ A$ is inconsistent.

Note that the rules for the red connectives $\wedge$, $\vee$ and $\supset$ are the same for the $\mathcal{K}$ antisequents. These rules also cover derivations such as

$$\dfrac{\dfrac{\dfrac{\dfrac{A, B, A \wedge B, \Gamma \vdash_\bullet}{A \wedge B, \Gamma \vdash_\bullet A^\perp, B^\perp}\ \textit{Store} \times 2}{A \wedge B, \Gamma \vdash_\bullet A^\perp \vee^e B^\perp}\ \vee^e R}{A \wedge B, \Gamma \vdash_\circ A^\perp \vee^e B^\perp}\ \textit{Signal}}{A \wedge B, \Gamma \vdash_\bullet}\ \textit{Load} \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{A \supset B, \Gamma \vdash_\circ A \qquad \dfrac{A \supset B, B, \Gamma \vdash_\bullet}{A \supset B, \Gamma \vdash_\bullet B^\perp}\ \textit{Store}}{A \supset B, \Gamma \vdash_\bullet A \propto B^\perp}\ \propto R}{A \supset B, \Gamma \vdash_\circ A \propto B^\perp}\ \textit{Signal}}{A \supset B, \Gamma \vdash_\bullet}\ \textit{Load}$$

Introduction rules for the green connectives are thus also accounted for by these properties. Completeness of provability is preserved by aggressively applying *Store* (this holds by the admissibility of weakening). While the first ten clauses correspond to introduction rules of LP, the last few describe the saturation of structural rules. A significant distinction in LP is that all introduction rules satisfy the subformula property, while the structural rules only satisfy this property *up to duality.* For example, by clause 14, if $E \in \Gamma^i$, then $E^\perp \in \Delta_i$ since if the conclusion of *Load* is consistent, then so must be its premise.

The following critical lemma shows that a collection satisfying the above properties forms a model that is consistent with its antisequents.

Lemma 8 *Given a Hintikka-Henkin pair $(\mathcal{H}, \mathcal{K})$, for all $S_i = \Gamma_i \nvdash_\circ \Delta_i$ in $\mathcal{H}$, let $S_a \preceq S_b$ if $\Gamma_a \subseteq \Gamma_b$ and $D^{S_a} \subseteq D^{S_b}$. Let $\mathcal{D}(S_i) = D^{S_i}$. Let $S_i \models a$ for atomic formulas a if $a \in \Gamma_i$. Extend $\models$ as defined for hybrid models, with $\mathcal{K}$ used as the classical worlds. Then $\langle \mathcal{H}, \preceq, \mathcal{K}, \models, \mathcal{D} \rangle$ is a hybrid model, and*

1. *if $A \in \Delta_i$ then $S_i \not\models A$ for all formulas A.*
2. *if $A \in \Gamma_i$ then $S_i \models A$ for all formulas A.*

Proof That this structure is a model is easily verified since transitivity and monotonicity are consequences of the definition of $\preceq$ by $\subseteq$. The two remaining properties of the lemma are proved by a simultaneous induction on formulas. The intuitionistic red-polarity cases are as in Fitting's proof. For green formula E, if $E \in \Gamma_i$ then either $S_i \models E$ vacuously, or for each $K^j \in \mathcal{K}$ such that $K^j \succeq S_i$, $E \in \Gamma^j$ (with $K^j = \Gamma^j \nvdash \Delta^j$). Then by the *Load* rule and properties 12 and 14 of Hintikka-Henkin pairs, $E^\perp \in \Delta^j$. We need to show that $K^j \models E$; then by the conditions of forcing it will follow that $S_i \models E$. Proceed for each case of E:

- If E is a green literal $d^{\perp}$, where d is a red atom, then $d^{\perp}$ and d cannot both be in Γ^j because $\perp \in \Delta^j$. Thus by the definition of the extension of the $\models$ relation, $K^j \not\models d$ and this also holds for all $K^l \succeq K^j$ since $d^{\perp} \in \Gamma^l$. Thus $K^j \models d^{\perp}$.

- If E is $A \vee^e B$, then it is also in all Γ^l for all $K^l \succeq K^j$, and thus $A^{\perp} \wedge B^{\perp} \in \Delta^l$ (by the *Load* rule), which means that either $A^{\perp}$ or $B^{\perp}$ is in Δ^l. Then by inductive hypothesis it follows that $K^l \not\models A^{\perp}$ or $K^l \not\models B^{\perp}$, thus $K^j \models A \vee^e B$.

- If E is $A \wedge^e B$, then it is also in all Γ^l for all $K^l \succeq K^j$. Thus $(A^{\perp} \vee B^{\perp}) \in \Delta^l$ and so both $A^{\perp}, B^{\perp} \in \Delta^l$ and by inductive hypothesis we have $K^l \not\models A^{\perp}$ and $K^l \not\models B^{\perp}$, and thus by definition $K^j \models A \wedge^e B$.

- If E is $A \propto B$, then $(A \supset B^{\perp}) \in \Delta^l$ for all $K^l \succeq K^j$ since $A \propto B \in \Gamma^l$, and there is some S_m in $\mathcal{H}$ such that $A, \Gamma^l \subseteq \Gamma_m$ and $B^{\perp} \in \Delta_m$, and so by inductive hypothesis $S_m \models A$ and $S_m \not\models B^{\perp}$, and by definition of $\models$, $K^j \models A \propto B$.

- If E is $\Sigma x.A$ then $\Pi x.A^{\perp} \in \Delta^l$ for all $K^l \succeq K^j$. But then there exists a $S_m \succeq K^l$ and $t \in D^{S_m}$ such that $A[t/x]^{\perp} \in \Delta_m$ and so again by inductive hypothesis $S_m \not\models A[t/x]^{\perp}$ and so by the definition of $\models$, $K^j \models \Sigma x.a$.

- if E is $\forall x.A$ then $\exists x.A^{\perp} \in \Delta^l$ for all $K^l \succeq K^j$. But then $A[t/x]^{\perp} \in \Delta^l$ for all $t \in D^{K^l}$ thus by inductive hypothesis, $K^l \not\models A[t/x]^{\perp}$ for all $t \in D^{K^l}$ and so $K^j \models \forall x.A$.

- The cases of $\perp$ and $\top$ are trivial.

For the other half of the mutual induction, if $E \in \Delta_i$, then by property 11 there is some K^j in $\mathcal{K}$ such that $E^{\perp}, \Gamma_i \subseteq \Gamma^j$, and similar inductive arguments on the red $E^{\perp}$ can then be applied to show $K^j \models E^{\perp}$, which implies $K^j \not\models E$. This half of the argument mirrors the cases of $R \in \Gamma_i$ for red formulas R.

- In the case $E^{\perp}$ is an atomic formula d, then $K^j \models d$ and thus $K^j \not\models d^{\perp}$, both by definition.

- If $E^{\perp}$ is $A \supset B$, then by monotonicity of $\subseteq$ for each $S_m \succeq K^j$, $A \supset B \in \Gamma_m$, and thus either $A \in \Delta_m$ or $B \in \Gamma_m$. By inductive hypothesis $S_m \not\models A$ or $S_m \models B$, and so by definition of $\models$, $K^j \models A \supset B$.

- If $E^{\perp}$ is $\Pi x.A$, then it is also found in Γ_m for all $S_m \succeq K^j$, and thus $A[t/x] \in \Gamma_m$ for each $t \in D^{S_m}$.

The other cases are similar. □

As an added consequence of the lemma, if $S_i \in \mathcal{K}$ we also have that if $A \in \Gamma_i$ then $S_i \not\models A^{\perp}$ by virtue of the *Load* rule.

The completeness proof continues by showing that a Hintikka-Henkin pair exists for every unprovable sequent. Given an antisequent $S = \Gamma \nvdash \Delta$, we identify *generators* that will form new antisequents as follows. An *s-generator antisequent* of an antisequent S is of the form $\Gamma \nvdash_{\circ} A \supset B$ if $(A \supset B) \in \Delta$. A *k-generator antisequent* of S is $\Gamma \nvdash_{\circ} E$ if $E \in \Delta$ for some green formula E. Finally, we also define a *π-generator antisequent* of S as $\Gamma \nvdash_{\circ} \Pi x.A$, if $\Pi x.A \in \Delta$.

The generators are kernels of two types of closures: the s- and π-generators will form Hintikka-style downward saturations relative to some set of parameters P, labeled S^*_P, while the k-generators will form a Henkin-style maximally consistent set with existential witnesses, also relative to some parameters P, labeled S^{**}_P. Each such maximal antisequent, generated from a k-generator, will represent a classical world. Both types of saturations will satisfy the requirements of a Hintikka-Henkin pair.

Some intricacy is required to define the domain of parameters D^s associated with each saturated antisequent S in a monotonic manner.

To form S^*_P from a generator $G_0 = \Gamma \nvdash_{\circ} B$, first let $t_1, t_2, \ldots$ be a denumerable set of parameters disjoint from $Pm(G_0)$. Let D^s_0 be some set of parameters that contains $Pm(G_0)$ and let $\mathbf{P} = D^s_0 \cup \{t_1, t_2, \ldots\}$. Fix an enumeration of all the subformulas of formulas in Γ, B, with parameters in $\mathbf{P}$, as $A_1, A_2, \ldots$. We generate a sequence of antisequents $G_0, \ldots, G_n, \ldots$, as well as a sequence of sets of parameters $D_0^s, \ldots, D_n^s, \ldots$, as follows. For each G_n already formed, define a secondary, finite sequence $R_n^0, \ldots, R_n^n$. Let $R_n^0 = G_n = \Gamma_n^0 \nvdash_\circ \Delta_n^0$. For each $i > 0$, consider the subformula A_i. A_i can be in at most one of Γ_n^{i-1} and Δ_n^{i-1} because of the assumption of consistency. Define R_n^i as follows:

- if A_i is in neither Γ_n^{i-1} nor in Δ_n^{i-1}, let $R_n^i = R_n^{i-1}$
- The cases of $A_i \in \Delta$ where A_i is of the form $A \supset B$, $\Pi x.A$ and green formula E are ignored, as these cases represent s-, π- and k-generators. For each such A_i, let $R_n^i = R_n^{i-1}$.
- if $A_i \in \Gamma_n^{i-1}$ and is of the form $A \supset B$, then let $R_n^i = \Gamma_n^{i-1} \nvdash_\circ \Delta_n^{i-1}, A$ if it is consistent, else let $R_n^i = B, \Gamma_n^{i-1} \nvdash_\circ \Delta_n^{i-1}$ (one of these must be consistent).
- if $A_i \in \Delta_n^{i-1}$ and is of the form $A \vee B$, then let $R_n^i = \Gamma_n^{i-1} \nvdash_\circ \Delta_n^{i-1}, A, B$; the case of $A \wedge B \in \Gamma_n^{i-1}$ is similarly defined.
- if $A_i \in \Delta_n^{i-1}$ and is of the form $A \wedge B$, then let $R_n^i = \Gamma_n^{i-1} \nvdash_\circ \Delta_n^{i-1}, A$ if it is consistent, else let $R_n^i = \Gamma_n^{i-1} \nvdash_\circ \Delta_n^{i-1}, B$ (one of these must be consistent). The case of $A \vee B \in \Gamma_n^{i-1}$ is similarly defined.
- if A_i is $\exists x.A \in \Gamma_n^{i-1}$, let t_j be the first unused parameter in $t_1, t_2, \ldots$, and mark it as *used*. Let $R_n^i = A[t_j/x], \Gamma_n^{i-1} \nvdash_\circ \Delta_n^{i-1}$.
- if A_i is $\exists x.A \in \Delta_n^{i-1}$, let $\Delta_n^i = \Delta_n^{i-1} \cup \{A[t/x] : \ t \textit{ used or } t \in D_n^s\}$ and let $R_n^i = \Gamma_n^{i-1} \nvdash_\circ \Delta_n^i$. The case of $A_i = \Pi x.A \in \Gamma_n^{i-1}$ is treated symmetrically to this case.

Let $G_{n+1} = R_n^n$ and let D_{n+1}^s be D_n^s plus all parameters that were marked as used. Let $S_{\mathbf{P}}^* = \bigcup G_n$ and let $D^s = \bigcup D_n^s$.

The style of enumeration, using the secondary sequence, assures that each Π-formula in Γ_n^i and each $\exists$-formula in Δ_n^i is instantiated with all the new parameters that may be introduced for $\exists$-formulas in Γ_n^i. We do not have cases for green formulas, for these are only considered in classical worlds, which requires a different style of saturation. Since the initial set G_0 contains the original antisequent, all subformulas will be considered.

The $S_{\mathbf{P}}^*$ antisequents are not downward-saturated for any green formula, as that is left to the next form of closure.

The formation of K_P^{**} from a k-generator $K_0 = \Gamma \nvdash_\circ E$ is in several stages. Here, K is assumed to be of the form $E^\perp, \Gamma \nvdash_\circ \perp$, which is consistent if and only if K_0 is consistent. We first define a multi-stage operation that extends *any* given antisequent $K = \Gamma \nvdash_\circ \Delta$, with $\perp \in \Delta$, to a maximally consistent antisequent, then we add the existential witnesses. First let D_0^k be some set of parameters that contains $Pm(K)$.

1. Extend Γ to a maximally consistent set: fix an enumeration of all formulas of $\mathcal{L}(D_0^k)$ as $A_1, A_2, \ldots$. Let $\Gamma_0 = \Gamma$. For each A_i, if $A_i, \Gamma_{i-1} \nvdash_\circ \Delta$ is consistent, let $\Gamma_i = A_i, \Gamma_{i-1}$; else let $\Gamma_i = \Gamma_{i-1}$. Let $\Gamma^M = \bigcup \Gamma_n$. Define $K_{D_0^K}^1 = \Gamma^M \nvdash_\circ \Delta$. This set has the following maximality property, which holds by the admissibility of weakening: for all formulas A_k in $\mathcal{L}(D_0^k)$, if $A_k, \Gamma^M \nvdash_\circ \Delta$ is consistent, then $A_k \in \Gamma^M$.
2. Extend $K_{D_0^k}^1$ to a set $K_{D_0^k}^{12}$ as follows. Let $\Delta_0 = \Delta$. For each formula A_i (as defined above), if $\Gamma^M \nvdash_\circ \Delta_{i-1}, A_i$ is consistent, then let $\Delta_i = A_i, \Delta_{i-1}$; else, let $\Delta_i = \Delta_{i-1}$. Let $\Delta^M = \bigcup \Delta_m$. Define $K_{D_0^k}^{12} = \Gamma^M \nvdash_\circ \Delta^M$.

   Certain key characteristics of K^{12} are easily verified. Every formula A_i is in at most one side of the antisequent. Furthermore, by virtue of the *Load* and *Store* rules, if $A \in \Gamma^M$ then $A^\perp \in \Delta^M$, since $\perp \in \Delta^M$, and that if a green formula $E \in \Delta^M$ then $E^\perp \in \Gamma^M$. These properties are among those required by a Hintikka-Henkin pair.

3. The final stage is to introduce existential witnesses for $\exists$ in Γ^M (but *not* for Σ, as that forms a π-generator). Each new term added will require the antisequent to be further extended over the larger domain via the first two stages. We do not need to be concerned with the $\forall$ quantifier as its dual will be on the opposite side of the antisequent. Let $G_0 = K^{12}_{D^k_0} = \Gamma^0 \nvdash_\circ \Delta^0$. Let $t_1, t_2, \ldots$ be a denumerable set of parameters and let $\mathbf{P} = D^k_0 \cup \{t_1, t_2, \ldots\}$, as in the definition of S^*_P. Enumerate all formulas of $\mathcal{L}(\mathbf{P})$ as $A_1, A_2, \ldots$. Each successive G_i is formed from G_{i-1} as follows: find the first $A_k = \exists x.A \in \Gamma^{i-1}$ such that $A[s/x] \notin \Gamma^{i-1}$ for any s. Let t_n be the first new (unused) parameter in $t_1, t_2, \ldots$ that's not in D^k_{i-1}, and mark it as used. Let $D^k_i = D^k_{i-1} \cup \{t_n\}$. Let $G_i = (A[t_n/x], \Gamma^{i-1} \nvdash_\circ \Delta^{i-1})^{12}_{D^k_i}$. If there is no such A_k, let $G_i = G_{i-1}$ and $D^k_i = D^k_{i-1}$. Finally, let $K^{**}_{\mathbf{P}} = \bigcup G_n$ and let $D^k = \bigcup D^k_n$.

We now show how a model can be formed starting from any consistent antisequent.

Given an unprovable sequent $\vdash_\circ A^4$, let S_0 be the antisequent $\nvdash_\circ A$ and let $T_0 = Pm(S_0)$. If $Pm(S_0)$ is empty then let $T_0 = \{g\}$ for some reserved parameter g, so that it is always non-empty. Assume that some denumerable subset of the set of all parameters is disjoint from T_0, and is enumerated as follows:

$$\begin{array}{llllll} T_1: & t^1_1, & t^2_1, & t^3_1, & \ldots \\ T_2: & t^1_2, & t^2_2, & t^3_2, & \ldots \\ T_3: & t^1_3, & t^2_3, & t^3_3, & \ldots \\ & & \ldots\ldots & & \end{array}$$

Let $\mathbf{P_n} = T_0 \cup T_1 \cup T_2 \cup \ldots \cup T_n$.

Let $S_1 = (\nvdash_\circ A)^*_{\mathbf{P_1}}$, with $D^{S_1}_0 = T_0$. Take the first generator in S_1 and proceed to form S_2 as follows:

- for s-generator $\Gamma \nvdash_\circ A \supset B$, let $S_2 = (A, \Gamma \nvdash_\circ B)^*_{\mathbf{P_2}}$, with $D^{S_2}_0 = D^{S_1}$.
- for k-generator $\Gamma \nvdash_\circ E$, let $S_2 = (E^\perp, \Gamma \nvdash_\circ \perp)^{**}_{\mathbf{P_2}}$, with $D^{S_2}_0 = D^{S_1}$.
- for π-generator $\Gamma \nvdash_\circ \Pi x.A$, let $S_2 = (\Gamma \nvdash_\circ A[t^1_2/x])^*_{\mathbf{P_2}}$, with $D^{S_2}_0 = D^{S_1} \cup \{t^1_2\}$.
- If there are no (more) generators found in S_1, let $S_2 = S_1$ and $D^{S_2} = D^{S_1}$.

The result of the above step is a sequence S_1, S_2. Now take the next generator in S_1 and form S_3 relative to $\mathbf{P_3}$, and the first generator in S_2 and form S_4 relative to $\mathbf{P_4}$, as above. Now we have a sequence S_1, S_2, S_3, S_4. After each stage n we have a sequence $S_1, S_2, \ldots, S_{2^n}$ Proceed to stage $n+1$ by taking the next available generators in each S_i and create either a $S^*_{P_{2^n+i}}$ or a $K^{**}_{P_{2^n+i}}$ as above. This will double the sequence to $S_1, S_2, \ldots S_{2^{n+1}}$, and so on. This defines an enumerable sequence: *our model will remain countable.*

The Hintikka-Henkin pair $(\mathcal{H}, \mathcal{K})$ is formed with $\mathcal{H}$ being the set of all S_i and $\mathcal{K}$ being the set of all $K^{**}_{\mathbf{P_n}}$ that were generated from k-generators. For each S_i, there is an associated D^{S_i} as defined.

Note that a k-generator of a K^{**}_P closure in the sequence will only produce the same antisequent, since the left-hand side of the closure is already maximally consistent and because we have fixed the enumeration of subformulas. A s-generator will either lead to the same set, or to a left-hand side that's $\perp$-inconsistent, representing an imaginary world. That is, classical worlds are only created above other classical worlds when there is a need to extend the domain.

To finish the completeness proof we just need to verify that $(\mathcal{H}, \mathcal{K})$ forms a Hintikka-Henkin pair. All of the cases are direct consequences of our definitions. For example, let $S_i = S^*_P = \Gamma \nvdash_\circ \Delta$. If $A \supset B \in \Gamma$, then at some point of the double-sequence enumeration (depending on the parameters in the formula) $A \supset B$ would be encountered. By the $\supset L$ rule, since the antisequent is consistent, either $\Gamma \nvdash_\circ A$ is consistent, or $B, \Gamma \nvdash_\circ \Delta$ is also consistent, and thus either $A \in \Delta$ or $B \in \Gamma$. If $A \supset B \in \Delta$, then $\Gamma \nvdash_\circ A \supset B$ will be an s-generator of S, which will then generate $(A, \Gamma \nvdash_\circ B)^*_{\mathbf{P_n}}$. Since by definition each $\mathbf{P_n}$ includes all parameters of $\mathbf{P_{n-1}}, \mathbf{P_{n-2}}, \ldots, \mathbf{P_1}$, the domain of this closure will contain the domain of S. Other cases are similar.

By Lemma 8, $S_1 \not\models A$. And thus we have a countermodel for A and completeness follows:

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[4] generalizable to any unprovable sequent $\Gamma \vdash_\circ A$

Theorem 9 *(Completeness) If a formula is satisfied in all Kripke hybrid models, then it is provable.*

The soundness and completeness of the semantics yield a trivial proof of the admissibility of cut: if $\mathcal{M} \models (\Gamma \vdash_\circ A)$ and $\mathcal{M} \models (A, \Gamma' \vdash_\circ B)$, then it follows directly that $\mathcal{M} \models (\Gamma\Gamma' \vdash_\circ B)$.

Corollary 10 *The Cut rule is admissible in LP.*

Several other versions of cut can also be shown to be admissible and countermodels can be given for those that are not.

The relationship between the details of the countermodel construction and Lemma 8 is a subtle one. For example, it would *not* be correct to assume that if an antisequent $\Gamma \not\vdash_\circ \Delta$ has a $\bot$-consistent Γ, then this antisequent will be beneath some classical world. As a simple example of the "procedure" suggested by the completeness proof, consider the sequent $\vdash_\circ d$ where d is a (red-polarized) propositional atom. From the initial antisequent $\not\vdash_\circ d$, the procedure creates only one world, using the S^*-type saturation, that does not force d. In fact the unextended $\models$ relation maps this world to the empty set. Since $\bot$ was never added to the right-hand side, there is no k-generator. Therefore, $\mathcal{K}$ is empty and this world is considered imaginary and forces $\bot$. This interpretation is technically consistent. Even if we only considered models that contained solely imaginary worlds, they would still be enough to interpret traditional intuitionistic logic. An important difference between the S^*-type closure and the K^{**}-closure is that the former only enumerates *subformulas* in S, while the maximally consistent closure enumerates through all formulas *and their duals.* We do not create classical worlds unless there is the *need* to create one, as indicated by the existence of a k-generator antisequent.

The Propositional Restriction

In the propositional case we can show that models can stay finite. The sequence $S_0, S_1, \ldots$ can be finite. First we note that in LP, all introduction rules exhibit the subformula property. The premises of the structural rules consist of either subformulas of the conclusion *or their duals.* The S^* closures are already formed from an enumeration of all subformulas of formulas in S. Given an initial antisequent $\not\vdash_\circ A$, it is also valid to restrict the enumeration of formulas in the definitions of and K^{**} to *all subformulas of A and their duals $A^\perp$.* In the propositional case, this enumeration is clearly finite. Since we are using sets instead of multisets, it is immediate that the number of possible antisequents consisting of subformulas of A and their duals is bounded. Furthermore, property 13 of Hintikka-Henkin pairs enforces the restriction $\triangle_{\mathbf{c}} = \{\mathbf{c}\}$ for all classical worlds. That is, an s-generator of a K^{**}-type antisequent can only create the same closure or an antisequent with a $\bot$-inconsistent left-hand side, i.e., an imaginary world.

7 Semantic Tableaux and Propositional Decidability

The antisequents used in the completeness proof of Section 6 are suggestive of a semantic tableau. However, the Henkin-type K^{**} closures used in the proof prevent the tableau from becoming a reasonable proof system directly. In particular, it does not suggest a decision procedure for the propositional fragment of PIL. Such a procedure can be shown to exist from the subformula property (up to duality) of cut-free LP proofs directly. However, we shall take the opportunity to define an alternative proof system for PIL. In this system, the distinction between $\vdash_\circ$ and $\vdash_\bullet$ will be replaced by a distinction between an empty right-hand side (corresponding to $\vdash_\bullet$) and a non-empty one (corresponding to $\vdash_\circ$). While LP is seen as an extension of Gentzen's LJ, this proof system will be an extension of the multiple-conclusion sequent calculus of Dragalin [Dra88]. The proof system, called *LPM* is shown in Figure 4.

An LPM sequent $\Gamma \vdash \Delta$ consists of a pair of finite *sets,* not multisets. The syntax A, Γ does not preclude the possibility that $A \in \Gamma$. The right-green introduction rules of LP have been replaced by the generalized left-red rules of LPM, which are applicable regardless of whether the right-hand

Right-Red Rules

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee R \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge R \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B, \Delta}\ \supset R \qquad \frac{}{\Gamma \vdash 1, \Delta}\ 1R$$

Left-Red Rules

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \vee L \qquad \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge L \qquad \frac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}\ \supset L \qquad \frac{}{0, \Gamma \vdash \Delta}\ 0L$$

Left-Green Rules

$$\frac{A, \Gamma \vdash \quad B, \Gamma \vdash}{A \vee^e B, \Gamma \vdash}\ \vee^e L \qquad \frac{A, B, \Gamma \vdash}{A \wedge^e B, \Gamma \vdash}\ \wedge^e L \qquad \frac{A, \Gamma \vdash B^\perp}{A \propto B, \Gamma \vdash}\ \propto L \qquad \frac{}{\perp, \Gamma \vdash}\ \perp L$$

The Lift Rule and Identity

$$\frac{E^\perp, \Gamma \vdash}{\Gamma \vdash E, \Delta}\ \mathit{Lift} \qquad \frac{}{a, \Gamma \vdash a, \Delta}\ I_r \qquad \frac{}{a, a^\perp, \Gamma \vdash}\ I_\ell$$

E is a green formula and a is an atomic formula

Figure 4: The Proof System LPM

side (Δ) is empty. The role of the *Load* rule in LP has been replaced by the left-green introduction rules of LPM. The *Lift* rule replaces *Signal* and *Store.* The intuitionistic fragment of this proof system corresponds to that of Dragalin with some discrepancies. The syntax of Dragalin's system is in the style of a sequent calculus in contrast to traditional tableaux using signed formulas: this we have adopted. However, Dragalin's system used lists instead of sets. More importantly, a sequent with an empty right-hand side in Dragalin's calculus should be interpreted here as $\Gamma \vdash 0$, since we reserve the empty right-side to represent $\perp$. The classical proof system embedded here resembles less that of LC: in particular there is no longer a rule to *load* the stoup. Since the left-red rules are applicable with an empty right-hand side, LPM contains all the rules of a one-sided classical sequent calculus, albeit it is the left-hand side that's used. The additional identity rule I_ℓ allows classical proofs to use only the left-hand side after an initial *Lift.* Once a classical sequent $\Gamma \vdash$ is reached in a bottom-up proof construction, the non-classical mode can only be returned to when a subformula containing $\supset$ or $\propto$ is encountered. When the right-hand side of a sequent is empty, clearly $\vee$ and $\vee^e$ become equivalent, as are $\wedge$ and $\wedge^e$, as well as $\perp$ and 0.

An LPM sequent with a non-empty right-hand side, $\Gamma \vdash A_1, \ldots, A_n$, has the same meaning as the LP sequent $\Gamma \vdash_\circ A_1 \vee \ldots \vee A_n$. The LPM sequent $\Gamma \vdash$, however, is equivalent to $\Gamma \vdash_\bullet$ in LP. The *end-sequents* of LPM are required to have a non-empty right-hand side. It may be more intuitive to regard an LPM sequent $\Gamma \vdash \Delta$ as the *antisequent* $\Gamma \not\vdash \Delta$.

Contraction is naturally available by the use of sets. Weakening is admissible on the left-side of the sequents; on the right-side, it is admissible *only if the right-side is already non-empty.* This requirement is necessary because here we are using the right-side context to serve two distinct purposes. With weakening and contraction we will be able to assume in the completeness proof that, without loss of generality, the principal formula of an introduction rule is always persistent in the premises except in $\supset R$ and *Lift*, where the right-side context Δ is deleted. Introduction rules for the quantifiers can be added in a predictable way, but here we shall focus on the propositional fragment[5].

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[5] The quantifier rules should be the following:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Pi y.A, \Delta}\ \Pi R \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x.A, \Delta}\ \exists R \qquad \frac{A[t/x], \Gamma \vdash \Delta}{\Pi x.A, \Gamma \vdash \Delta}\ \Pi L \qquad \frac{A, \Gamma \vdash \Delta}{\exists y.A, \Gamma \vdash \Delta}\ \exists L \qquad \frac{A[t/x], \Gamma \vdash}{\forall x.A, \Gamma \vdash}\ \forall L \qquad \frac{\Gamma \vdash A^\perp}{\Sigma y.A, \Gamma \vdash}\ \Sigma L$$

As usual, y is not free in Γ, Δ.

We now independently prove the propositional completeness of LPM. An LPM (anti)sequent is defined to be *consistent* if it is *not* provable. Clearly, if the conclusion of any inference rule is consistent, then at least one of its premises must also be consistent. The lone sequent found in the initial rules I_r, I_ℓ, $0L$, $\bot L$ and $1R$ are obviously inconsistent. Following the tradition of tableau, we call sequents of these forms *closed.* A branch of an LPM proof tree is closed if it terminates in a closed (anti)sequent, otherwise it is *open.* Every LPM "proof" is in fact a refutation that the concluding sequent is not provable.

Four rules of LPM stand out from the others: $\supset R$, $\supset L$, $\propto L$, and *Lift*. We can now define the following closure procedure:

Given a sequent $S = \Gamma \vdash \Delta$, we identify the following four types of formula occurrences found in the sequent as "generators." Each generator generates a new possible world.

1. $A \supset B \in \Delta$
2. $E \in \Delta$ for any green formula E
3. $A \propto B \in \Gamma$ and Δ is empty
4. $A \supset B \in \Gamma$, $B \notin \Gamma$ and Δ is empty.

We say that a sequent S' is *safe* with respect to S if both of the following conditions hold:

1. *S is the conclusion of some instance of an inference rule, S' is a premise of the inference rule, and the principal formula of the inference rule is not a generator formula.*
2. *The principal formula of the inference rule persists in S'*

We define a *saturated branch* of LPM as a consistent sequence of (anti)sequents as follows. Let S_0 be some given, consistent end-sequent $\Gamma_0 \vdash \Delta_0$. S_1 is formed from S_0 using a secondary sequence as follows. Let $T_0^0 = S_0$. Each successive T_0^j is a premise of an inference rule that is consistent and *safe* with respect to T_0^{j-1}. Each step preserves all formulas in the previous sequent. For example, if $T_0^{j-1} = C \supset D, \Gamma \vdash A \vee B, \Delta$, then T_0^j can be $(C \supset D, \Gamma \vdash A, B, \Delta)$ or $(C \supset D, D, \Gamma \vdash A \vee B, \Delta)$. By the subformula property (up to duality), each sequence $T_0^0, T_0^1, T_0^2, \ldots$ must terminate in some finite, saturated closure T_0^*. Let $S_1 = T_0^*$.

Let $F_1, \ldots, F_n$ be the generator formulas found in $S_1 = \Gamma_1 \vdash \Delta_1$. For each generator F_i, form S_{i+1} as follows:

- if F_i is of the form $A \supset B \in \Delta_1$ or $A \propto B^\perp \in \Gamma_1$ with Δ_1 empty, let $S_{i+1} = T_i^*$, where $T_i^0 = A, \Gamma_1 \vdash B$. Here, T_i^* is defined using the same procedure as T_0^*
- if F_i is a green polarity formula $E \in \Delta_1$, let $S_{i+1} = T_i^*$, where $T_i^0 = E^\perp, \Gamma_1 \vdash$. Furthermore, mark S_{i+1} as a *k-candidate.*
- if F_i is $A \supset B \in \Gamma_1$, $B \notin \Gamma_1$ and Δ_1 is empty, let $S_{i+1} = T_i^*$, where $T_i^0 = \Gamma_1 \vdash A$.

This defines the sequence $S_0, S_1, S_2, \ldots S_k$. Now take $S_2 = \Gamma_2 \vdash \Delta_2$, enumerate all generator formulas in S_2, then form S_{k+1}, S_{k+2}, etc ... Repeat the process exhaustively. Since we are using sets, the number of sequents consisting of subsets of a finite set of subformulas and their duals is clearly finite.

Given a consistent saturated branch $S_1, S_2, \ldots$, define a (propositional) hybrid model as follows. Let $\mathbf{W}$ be the set of all S_i. If $S_i = \Gamma_i \vdash \Delta_i$ and $S_j = \Gamma_j \vdash \Delta_j$, then let $S_i \preceq S_j$ if $\Gamma_i \subseteq \Gamma_j$. Let $\mathbf{C}$, the set of classical worlds, consist of all k-candidate sequents k that are *maximal* in the sense that if $k \preceq s$, then either $s = k$ or s is not a k-candidate. All k-candidates have the form $\Gamma \vdash$. Thus we have the property that $\triangle_{\mathbf{k}} = \{\mathbf{k}\}$ for all $\mathbf{k} \in \mathbf{C}$. This definition by maximality is of course only valid in the propositional case, where infinite models are not considered. Now let $S_i \models a$ if the atomic formula a is in Γ_i. This forms a model $\langle \mathbf{W}, \preceq, \mathbf{C}, \models \rangle$. We now prove a version of Lemma 8 for LPM:

Lemma 11 *In the model $\langle \mathbf{W}, \preceq, \mathbf{C}, \models \rangle$ formed from the consistent sequence $S_1, S_2, \ldots$, the following holds for each $S_i = \Gamma_i \vdash \Delta_i$:*

1. *if* $A \in \Gamma_i$ *then* $S_i \models A$
2. *if* $A \in \Delta_i$ *then* $S_i \not\models A$

Proof The proof is again by a simultaneous induction on both properties. The first six properties, and property 11, of Hintikka-Henkin pairs used in the first-order completeness proof still apply here. There are also additional properties such as: if $A \wedge^e B \in \Gamma$ and $\Gamma \vdash \quad \in \mathbf{C}$, then $A, B \in \Gamma$. We show some of the cases, including those that differ significantly from Lemma 8

- Assume that $A \propto B \in \Gamma_i$. Then it is also in Γ^j for all $K^j = \Gamma^j \vdash$ in $\mathbf{C}$ such that $K^j \succeq S_i$. Then $A \propto B$ becomes a generator for K^j so there exists an $S_m \succeq K^j$ such that $A \in \Gamma_m$ and $B^\perp \in \Delta_m$. Thus by inductive hypothesis $S_m \models A$ and $S_m \not\models B^\perp$ and therefore $S_i \models A \propto B$.
- if $A \supset B \in \Gamma_i$, then it is in all Γ_j for $S_j \succeq S_i$. For each S_j, there are two cases. If Δ_j is non-empty, then either $A \in \Delta_j$ or $B \in \Gamma_j$ (since at least one premise of $\supset L$ is safe), so $S_j \not\models A$ or $S_j \models B$. If Δ_j is empty (i.e., S_j is a k-candidate), then it must be either $B \in \Gamma_j$, or, there exists an $S_m \succeq S_j$ such that $A \in \Delta_m$. By inductive hypothesis, $S_j \models B$ or $S_m \not\models A$. But if $S_m \not\models A$ then by monotonicity $S_j \not\models A$. Thus again we have that $S_j \not\models A$ or $S_j \models B$. Thus $S_i \models A \supset B$.
- if $A \vee^e B \in \Gamma_i$, then $A \vee^e B \in \Gamma^j$ for all classical $K^j \succeq S_i$. Let $K^j = \Gamma^j \vdash$. Then either $A \in \Gamma^j$ or $B \in \Gamma^j$ and so by inductive hypothesis $K^j \models A$ or $K^j \models B$ and so $S_i \models A \vee^e B$. The case of $A \wedge^e B \in \Gamma$ is similar.
- if $A \supset B \in \Delta_i$, then there exists an $S_j \succeq S_i$ such that $A \in \Gamma_j$ and $B \in \Delta_j$, so by inductive hypothesis $S_j \models A$ and $S_j \not\models B$ thus by definition $S_i \not\models A \supset B$.
- if $A \propto B \in \Delta_i$, then since this is a green formula, there exists a $K^j = \Gamma^j \vdash$ such that $K^j \succeq S_i$ and $A \supset B^\perp \in \Gamma^j$. By the same argument already given above, $K^j \models A \supset B^\perp$ so $K^j \not\models A \propto B$.

□

With this lemma, it clearly follows that LPM is semantically complete: if $\nvdash A$ is consistent then $S_1 \not\models A$.

The soundness of LPM is proved by induction on proofs. We show some of the more interesting cases.

- In the case of the *Lift* rule, the inductive hypothesis allows us to assume that for all u, if $u \models L^\perp$ and $u \models \bigwedge \Gamma$ then u is imaginary ($u \models \perp$). Now assume that $v \models \bigwedge \Gamma$. We will show that $v \models L$, which proves the case. Assume that $v \not\models L$. Then there is a classical $k \in \triangle_v$ such that $k \not\models L$, so $k \models L^\perp$. But we also have that $k \models \bigwedge \Gamma$ by monotonicity, so we must have that $k \models \perp$, which contradicts the assumption that k is classical. Note that this argument can also be generalized to the first-order case by the third property of Lemma 6.
- In the case of $\propto L$, assume that for any world u, $u \models A \wedge \bigwedge \Gamma$ implies $u \models B^\perp$. We need to show that if $v \models A \propto B \wedge \bigwedge \Gamma$ then $v \models \perp$ (v is imaginary). But if $v \models A \propto B$ and $\triangle_v$ is non-empty, then for any $k \in \triangle_v$ there is some $w \succeq k$ such that $w \models A$ and $w \not\models B^\perp$. Furthermore, $w \models \bigwedge \Gamma$ by monotonicity since $v \models \bigwedge \Gamma$. But by the assumption, it must be the case that $w \models B^\perp$, a contradiction. Thus $\triangle_v$ is empty and $v \models \perp$ vacuously.
- In the case of $\supset L$, let $\bar{\Delta}$ represent the $\vee$-disjunction of formulas in Δ if Δ is non-empty; otherwise, let $\bar{\Delta}$ represent $\perp$. Assume that $v \models A \supset B$ and $v \models \bigwedge \Gamma$. Then $v \not\models A$ or $v \models B$. But if $v \not\models A$, then by the inductive hypothesis on the left premise ($\Gamma \vdash A$), $v \not\models \bigwedge \Gamma$, a contraction. Thus it must be that $v \models B$, and now by the inductive hypothesis on the right premise ($B, \Gamma \vdash \Delta$), $v \models \bar{\Delta}$.

The other soundness cases are similar or trivial. We therefore have

Theorem 12 *(Soundness and Completeness of LPM) A formula is provable in LPM if and only if it is valid in all propositional hybrid models.*

The admissibility of several forms of cut follows directly from soundness and completeness: these include

$$\frac{\Gamma \vdash A \quad A, \Gamma' \vdash \Delta'}{\Gamma\Gamma' \vdash \Delta'}\ cut \qquad \frac{\Gamma \vdash A, \Delta \quad A, \Gamma' \vdash B, \Delta'}{\Gamma\Gamma' \vdash B, \Delta\Delta'}\ cut$$

The first cut requires a single conclusion in the left premise and the second cut requires a non-empty right-hand side in the right premise. These restrictions are required because of the multiple roles of the right-hand sides of LPM sequents. A cut between $\Gamma \vdash A, \Delta$ and $A, \Gamma' \vdash \Delta'$ is not generally admissible. In particular, a cut between $\Gamma \vdash A, B$ and $A, \Gamma' \vdash$ is not admissible if B is a red formula, since $\mathbf{u} \models \bot$ does not imply $\mathbf{u} \models B$. These syntactic restrictions are not serious impediments to the application of cut, since one can replace proofs of $\Gamma \vdash$ with proofs of $\Gamma \vdash \bot$.

Besides the analysis of cut-rules for LPM, we can now also derive the decidability of the propositional fragment. Instead of a sequence $S_0, S_1, \ldots$ representing a saturated branch of consistent sequents, we can generate a tree consisting of all such branches without assuming consistency. That is, when there is a choice of which safe premise of an inference rule can be used to form the T_i^* closure, we form two branches for each possibility. In considering the sequent $A \supset B, \Gamma \vdash$, we also regard $A \supset B$ as a generator even if $B \in \Gamma$. A branch can be terminated when an initial sequent (i.e., obviously inconsistent sequent) is encountered. Since each branch is finite and the branching factor is at most two, by König's Lemma, the tree is also finite. Each saturated branch represents a maximal branch in a proof tree. If each branch contains an initial sequent, then the tree represents a proof. Each open branch represents a countermodel. We thus have:

Corollary 13 *The propositional fragment of PIL is decidable.*

8 Related Topics

A logic should be understandable from a variety of perspectives. In this paper, we have presented PIL in traditional terms such as two-sided sequent calculus, semantic tableau, and Hintikka sets. Thus, we have followed an approach similar to that of, for example, Kripke [Kri65] and Fitting [Fit69]. Below we present several related perspectives on the general goal of unifying provability in intuitionistic and classical logics.

8.1 Double Negation Translations

It is known that the double-negation translations of Kolmogorov, Gödel-Gentzen, Kuroda, and certain others are equivalent in the sense that their representations of classical formulas are provably equivalent in intuitionistic logic (see [FO10]). The differences between these translations can be seen in the structure of proofs of the translated formulas. The earliest translation of Kolmogorov places $\sim\sim$ in front of every subformula; e.g, implication becomes $\sim\sim(\sim\sim A \supset \sim\sim B)$. There is little room for intuitionistic structure to survive underneath the excessive double negations. Using fewer double negations, the Gödel-Gentzen translation preserves the "negative" connectives. Kuroda's translation places $\sim\sim$ only at the very outset of a formula (plus the double-negation shift on $\forall$), thereby preserving much of the structure of an intuitionistic proof. By identifying polarities in classical logic, Girard's version of double negation [Gir91] explains LC. A new kind of *classical proof* is formulated that shares the *stoup* of intuitionistic logic. Girard's translation is still logically equivalent to the traditional translations: this is in fact how LC was argued to be correct[6]. These translations allow classical and intuitionistic logics to share varying degrees of structure in their *proofs,* but they do not show how to integrate classical and intuitionistic logics at the level of *formulas and proofs.* They offer no explanation as to the limit of this kind of complete integration, in particular the extent to which cuts are admissible.

The question remains, however, as to whether a double-negation type of translation from PIL to intuitionistic logic exists. It may be said that the most important addition to intuitionistic logic

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[6] In an earlier work [LM09], the authors used a modified version of the LC translation to formulate a *focused* proof system for classical logic called LKF. The correctness of this system was likewise proved by a reduction to the Gödel-Gentzen translation.

found in PIL is the distinction between 0 and $\bot$ (see section 3.3). The LP "classical" sequent $\Gamma \vdash_{\bullet} R$ is equivalent to $\Gamma \vdash_{\circ} R \vee^e \bot$, and $R \vee^e \bot$ is equivalent to $(R \supset \bot) \supset \bot$. Of course $(R \supset \bot) \supset \bot$ is a form of double-negation. There is a clear difference between this form of double negation and $(R \supset 0) \supset 0$: intuitionistic implication is collapsed into a classical one when placed inside the scope of $\sim\sim$. Most of the known double-negation translations can also use minimal-logic negation (with an adjustment required in the case of Kuroda's translation - see [FO10]). The PIL negation $A \supset \bot$ is closer to minimal-logic negation than to intuitionistic negation. However, there is still a difference if *false* in the sense of minimal logic is to coexist with that of intuitionistic logic. If the minimal *false* is treated as an arbitrary propositional atom ϕ, then for every choice of this atom there are formulas of PIL that cannot be accurately translated using a minimal-logic double negation. In particular, $A \vee^e (A \supset \bot)$ is valid in PIL but not $A \vee^e (A \supset \phi)$. If we translated the classical disjunction in $A \vee^e (A \supset \phi)$ using a double negation in minimal logic, the resulting formula would be equivalent to $((A \vee (A \supset \phi)) \supset \phi) \supset \phi$, which is provable. This problem aside, however, it should be possible to translate PIL into intuitionistic logic as long as intuitionistic logic is extended to include both $\bot$ and 0 as distinct logical constants. *There must be at least one green formula.*

When devising the translation one must still be careful not to produce forms that would collapse intuitionistic implication. It is not enough to simply "do not translate the intuitionistic connectives" since these connectives can join classical subformulas, and vice versa. Thus we are lead to define the translation in the form $(A \supset B)^t = A^t \supset B^t$, etc. $A \supset B$ must not be translated into a form that is equivalent to $\neg\neg(\neg\neg A \supset \neg\neg B)$, which is suggested by Kolmogorov's translation and is equivalent to a classical implication even if $\neg$ is taken to represent minimal negation. One possible translation, suggested by the algebraic analysis of section 3.3 and labeled $\approx$, is the following (for the propositional fragment). Here, a again represents only atomic formulas.

$$\begin{array}{lll}
\top^{\approx} = 1^{\approx} = 1 & 0^{\approx} = 0 & \bot^{\approx} = \bot \\
a^{\approx} = a & a^{\bot\approx} = a \supset \bot & \text{for atomic } a \\
(A \vee B)^{\approx} = A^{\approx} \vee B^{\approx} & (A \wedge^e B)^{\approx} = (A^{\bot\approx} \vee B^{\bot\approx}) \supset \bot & \\
(A \wedge B)^{\approx} = A^{\approx} \wedge B^{\approx} & (A \vee^e B)^{\approx} = (A^{\bot\approx} \wedge B^{\bot\approx}) \supset \bot & \\
(A \supset B)^{\approx} = A^{\approx} \supset B^{\approx} & (A \propto B)^{\approx} = (A^{\approx} \supset B^{\bot\approx}) \supset \bot &
\end{array}$$

This translation is distinct from known double-negation translations not only because it preserves the intuitionistic (red) connectives and 0, but also because it relies on the De Morgan negation $A^{\bot}$.

The fact that it may be possible to embed PIL inside another logic does not negate its significance. Many examples of embedding one logic into another exist. An embedding does not obviate all the properties of a logic. In particular, the admissibility of cut in the embedded logic does not immediately follow from the corresponding property in the embedding logic: we still have to show that cut-free proofs exist that observe the invariants of the embedded logic. Likewise, the semantics of the embedding logic may not be fine-grained enough to explain the properties of the embedded logic. We have at the very least simplified the integration of classical and intuitionistic logics by providing proof systems that enable it, and refined semantics that explain the limits of this integration. In PIL the classical and intuitionistic components of both formulas and proofs are clearly distinguished. The proof system LP has the structure of LC without losing the most important connectives of intuitionistic logic.

8.2 Linear Logic

We have designed a combination of intuitionistic and classical logics and we have tried to validate that design using principles taken from the literatures of these two logics. It is possible, however, to use a third logic, linear logic, to give a proof-theoretic "semantics" for PIL, especially when PIL is in the form of LP. Green formulas have the proof-theoretic characteristics in LP of ?-formulas and red formulas correspond to !-formulas. The dual connectives $\supset$ and $\propto$ reflect Girard's decomposition of intuitionistic implication into $\parr$ and its dual $\otimes$. The formula $A \propto B$ can be interpreted as roughly $?(!A \otimes ?B)$ (of course the exact translation would have to be recursive). The translations of the classically oriented connectives are based on the linear logic interpretation of LC. For example, $A \vee^e B$ is roughly equivalent to $?A \parr ?B$ and $A \vee B$ is $!A \oplus !B$. The translation of atomic formulas will have to assume the existence of atoms with the inherent property $a \equiv !a$, but this extension of linear logic is known [Gir93, Lau02][7].

Despite this translation, to describe PIL as simply a fragment of linear logic would leave too many questions unanswered. For example, we still would have to show that a proof system can be formulated for the logic that preserves cut-elimination. We also would have to justify separately the decidability of the propositional fragment, which does not hold in linear logic. Semantically, if we simply used an interpretation of linear logic, such as phase spaces, to interpret PIL, then something is sure to be lost in the translation. A Heyting algebra with an embedded boolean algebra is a more specific structure than a phase space or even a topolinear space. Had we used linear logic as the starting point, most of the work presented in this paper, in some equivalent form, would still have to be completed.

One of the ways to view linear logic is that it generalizes the principles of Gentzen in terms of the conditions under which cut-elimination is possible. Given that LP extends LJ and LC and preserves cut-elimination, it should not be surprising that a translation of LP into linear logic exists. Moreover, the translation will *largely* preserve the structure of proofs (with some difficulties concerning 0). However, such a translation will not explain LPM. The semantics of linear logic emphasize the interpretation of *proofs* as opposed to just formulas. However, to say that a Kripke-style semantics is merely a "truth-value" semantics, with little impact on structural proof theory, is not correct. The Beth-Fitting tableau, and the sequent calculus of Dragalin, are proof systems that correspond directly to Kripke semantics. They cannot be easily understood as valid intuitionistic systems outside of Kripke's view of intuitionistic logic (e.g., the disjunction and existence properties are no longer obvious). In LPM, contraction and weakening are available on both the left- and right-hand sides and $\vee$ appears as a multiplicative. The explanation that intuitionistic logic is "*classical on the left, linear on the right*" no longer applies. There is a possibility that if linear logic is extended with *subexponentials* [NM09], i.e., extra pairs of exponentials that are weaker than ! and ?, that this kind of intuitionistic proof system can then be captured. But our goal here is to combine intuitionistic logic with classical logic. Thus, instead of seeking a refinement of linear logic, we sought a refinement of Kripke semantics.

8.3 The Unified Logic LU

Using themes from LC and linear logic, Girard developed the LU proof system [Gir93] that contains classical, intuitionistic, and linear logics as fragments. Where well-formed formulas in different logics intersect, LU proofs in different proof fragments can mix via cut-elimination. PIL has many of the same motivations that Girard had for LU: in fact, the fragment of the LU sequent calculus that contains intuitionistic and classical logics is similar to LP. There are, however, a range of differences between LU and PIL. For example, LU contains a great many inference rules but our use of the *Signal* rule allowed PIL to have a much smaller proof system. LU does not contain a dual to intuitionistic implication. In fact, LU can be translated into linear logic and it is clear from the translation tables that the De Morgan dualities do not always hold for arbitrary mixtures of polarities, even for conjunction and disjunction. LU's ability to mix connectives from different logics in the same formula is limited as is its (largely unexplored) ability to perform cut-elimination across different logics. Thus, LU does not encompass PIL.

The intuitionistic fragment of LU contains a "stoup" on the left-hand side in addition to the right-side stoup. This refinement is significant, but it is a characteristic of *focusing* and not a characteristic of the logic itself. Once focusing is formulated for a logic the left-side stoup is not only recovered but will take on a stronger form. Indeed the stoup, as a form of focusing, is available *as an option* in both classical and linear logics. In intuitionistic logic, it is also optional on the left-hand side. The right-side stoup of intuitionistic logic is the only one that is necessitated by the logic itself.

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[7] The constants $\top$ and 1 translate to themselves. However, a translation of formulas should be accompanied by a translation of "*equivalence.*" In linear logic, $\top \multimap 1$ does not hold but $!\top \multimap 1$ does.

Another difference between PIL and LU, and linear logic upon which it is largely based, is the approach to the definition of "polarity." In LU and linear logic, one can define this concept in terms of the availability of structural rules: "positive"formulas are those such that $A \equiv\ !A$ and "negatives" are such that $B \equiv\ ?B$. This definition is acceptable in linear logic because the question of where structural rules can be applied has an unambiguous answer. The same is not true in intuitionistic logic. The multiple-conclusion sequent calculus admits contraction everywhere. Gentzen's sequent calculus can be formulated as either having exactly one formula on the right-hand side, or at most one. There are also *contraction free* versions of intuitionistic proof systems [Dyc92]. Trying to define an intuitionistic notion of "polarity" in terms of proof-theoretic features is highly problematic. The meaning of polarity in PIL, however, is defined model theoretically.

8.4 Polarized Linear Logic

Despite the similarity in name, PIL is not subsumed by Polarized Linear Logic (LLP) [Lau02]. Like LLP, PIL may also be seen as a generalization of LC, at least in the form LP, but it is a different kind of generalization. A fundamental difference between our approach and that of LC and LLP is our separation of *positive/negative* polarization from *red/green* polarization. Only in the LC fragment do these polarities coincide. Only the red/green polarization is directly meaningful in our semantics. In PIL, the connectives $\supset$ and Π, and their duals $\propto$ and Σ, exist on a different *axis* of polarization than LC (see the diagram in Figure 1 of Section 2). The form of polarization that defines LLP is a generalization of that of LC and still does not account for the polarization of the purely intuitionistic connectives. Also, LLP formulas are restricted in that negative connectives may only join negative subformulas, and dually for positives. For other formulas, a polarity switch in the form $?A$ or $!B$ is required before they can be joined. PIL places no restriction on how formulas of different polarities can be composed. For example, the law of excluded middle in the form $A \vee^e \sim A$, if translated into linear logic, would be something equivalent to $?A \parr ?!(?A^{\perp} \parr 0)$. Such a form is not allowed in LLP.

8.5 Focusing

Another concept related to polarization and likewise originated from the study of linear logic is *focusing* [And92]. The LP proof system is not focused, at least not in the sense of the *synchronous* (positive) versus *asynchronous* (negative) duality. On the other hand, the LP invariant in the $\vdash_{\circ}$ mode means that *red-polarized introduction rules must be applied exhaustively until a green-polarized subformula is encountered* (reading proofs up from the endsequent). Thus, there is some "focusing" behavior along the red-green polarities instead of the positive-negative ones.

An early version of PIL was in fact developed that took the form of a sequent calculus that focused in the positive-negative sense. However, this system was not entirely satisfactory because it relied too much on a translation to linear logic, which imposed unnatural restrictions on syntax. In linear logic, focusing must stop before a ! or a ?. This implies that a translation of PIL into linear logic needs to remove as many of these operators as possible in order to preserve focusing. For example, $?(?A \parr ?B)$ need to be recognized as equivalent to $?A \parr ?B$ and $?(!A \otimes ?B)$ as equivalent to $?(!A \otimes B)$. In terms of the semantics of linear logic, these transformations also allow for the preservation of associativity at the denotational level. However, there is also a price to be paid when these operators are removed: *is "$A \parr B$" supposed to represent an intuitionistic implication, or a classical disjunction?* Without the exponential operators, the information is no longer available to restrict linear logic proofs to mimic proofs in the logic that it embeds. The cost of focusing (and denotational associativity) was a loss of "full adequacy" of the translation. In order to accurately represent a PIL focused proof in linear logic, we had to restrict intuitionistic implication $A \supset B$ to require that A must be red-polarized. This meant that $\sim(A \vee^e B)$ had to be written in the form $\sim((A \vee^e B) \wedge 1)$ instead. Other restrictions were also imposed, removing forms that would require a "double-exponential" such as $?!(?A \parr B)$. These restrictions to the syntax of PIL seemed reasonable when viewed in the context of the proof theory and semantics of linear logic. However, they make little sense in terms of intuitionistic and classical logics. Focusing in

PIL requires a re-examination of this principle independently of focusing in linear logic, and we leave this to another occasion.

8.6 Dual-Intuitionistic Logic

Our use of a dual connective to intuitionistic implication may invoke comparisons to some versions of "Dual-Intuitionistic Logic" [Rau74], some of which also contains a "dual" to intuitionistic implication. However, the main aim of PIL is to mix classical and intuitionistic reasoning, which requires a *De Morgan dual* to all connectives. In the Kripke semantics defined for dual-intuitionistic logic (see [Gor00]), the dual of intuitionistic implication is interpreted as being valid in a possible world if it is valid in all worlds *in the past,* i.e., beneath the current world. While such a definition preserves monotonicity, it is not the De Morgan dual of intuitionistic implication. We have shown how such a dual can be defined in our semantics without destroying monotonicity.

8.7 Other Kripke and Algebraic Semantics

We chose a Kripke-like semantics for PIL because it provided some insights into particular aspects of mixing classical and intuitionistic logic (*e.g.*, double-negation translations). As we mentioned before, the papers [ILH10, Vel76] also generalize Kripke semantics to allow for worlds that are inconsistent. There are also a number of papers that dig into various semantic and algebraic considerations behind intermediate and non-classical logics. It will be particularly interesting to see how well the semantic models of this paper might fit into the broad algebraic setting described in papers such as [CST09, CGT11].

8.8 LKU and ICL

Finally, we briefly mention two pieces of our own related work. In [LM11] we presented LKU, a unified, focused proof system. LKU extends our understanding of what is possible in focused proof systems and it provides generalized criteria for cut-elimination and completeness. We also formulated a mixed intuitionistic-classical logic as a fragment of LKU in which intuitionistic connectives can join classical subformulas, but *not vice versa.* This limitation was due to the fact that polarization in LKU is based on fundamentally classical principles and thus did not provide for the full inclusion of the intuitionistic connectives.

The semantic analysis of PIL has led to the development of a related logic we currently call *Intuitionistic Control Logic* (ICL). The choice of this name is related to its ability to extend the Curry-Howard isomorphism and represent programming language control operators such as *call/cc* while preserving much of intuitionistic logic. Semantically the idea is exceedingly simple: restrict to *rooted* Kripke frames and designate the root to be the only classical world, which means that all other worlds are imaginary. In this (propositional) logic, the excluded middle in the form $A \vee (A \supset \bot)$ holds *without a second version of* $\vee$. Polarization and negation in the form $A^{\perp}$ are no longer used. However, unlike PIL, which combines classical and intuitionistic logics, ICL has more of the characteristics of an intermediate logic. It is not a fragment of PIL. ICL will be presented in another paper.

9 Conclusion

We have introduced a logic that combines intuitionistic logic with classical logic. One might say that, since intuitionistic proofs are classical proofs, classical logic is already such a combination: but this sense of combination misses the point about the kind of invariants one wishes to have in the proofs of a logic. On the other hand, by virtue of traditional double-negation translations, one can argue that intuitionistic logic is already such a combination: but such translations do not clarify how classical and intuitionistic formulas can *mix.* We wish to know whether the critical intuitionistic connectives of implication and universal quantification can retain their strength when combined with classical connectives. One might also say that linear logic is already such a combination: but using linear logic for this purpose also leaves some questions unanswered, such as, why should a combination of classical and intuitionistic logics lose the decidability of its propositional fragment?

PIL can be summarized as adding the distinction between 0 and $\perp$, a concept borrowed from linear logic, to intuitionistic logic. From this perspective it can be said that the traditional form of intuitionistic logic was *missing its other half.* The addition of $\perp$ allows intuitionistic logic to unite with its other half, without becoming classical logic or linear logic. We have shown both semantically and proof-theoretically that PIL stands on its own as a new logic.

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References

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. of Logic and Computation*, 2(3):297–347, 1992.

[Bet59] E. W. Beth. *The Foundations of Mathematics.* North Holland, Amsterdam, 1959. Also published by Harper & Row, New York (1966).

[CGT11] Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. MacNeille completions of FL-algebras. *Algebra Universalis*, 66(4):405–420, 2011.

[CH96] Luis F. Del Cerro and Andreas Herzig. Combining classical and intuitionistic logic, or: intuitionistic implication as a conditional. In *Frontiers of Combining Systems: 1st International Workshop*, pages 93–102. Kluwer Academic Publishers, 1996.

[CST09] Agata Ciabattoni, Lutz Straßburger, and Kazushige Terui. Expanding the realm of systematic proof theory. In Erich Grädel and Reinhard Kahle, editors, *Computer Science Logic, CSL'09*, volume 5771 of *Lecture Notes in Computer Science*, pages 163–178. Springer, 2009.

[Dra88] A. G. Dragalin. *Mathematical Intuitionism: Introduction to Proof Theory*, volume 67 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, Rhode Island, 1988.

[Dyc92] Roy Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. *J. of Symbolic Logic*, 57(3):795–807, September 1992.

[Fit69] Melvin C. Fitting. *Intuitionistic Logic Model Theory and Forcing.* North-Holland, 1969.

[FO10] Gilda Ferreira and Paulo Oliva. On various negative translations. In *Proceedings of the Third International Workshop on Classical Logic and Computation*, volume 47 of *Electronic Proceedings in Theoretical Computer Science*, pages 21–33, 2010.

[Gen69] Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland, Amsterdam, 1969. Translation of articles that appeared in 1934-35.

[Gir91] Jean-Yves Girard. A new constructive logic: classical logic. *Math. Structures in Comp. Science*, 1:255–296, 1991.

[Gir93] Jean-Yves Girard. On the unity of logic. *Annals of Pure and Applied Logic*, 59:201–217, 1993.

[Gli29] V. Glivenko. Sur quelques points de la logique de M. Brouwer. *Academie Royale de Belgique, Bulletins de la classe des sciences, ser. 5*, 15:183–188, 1929.

[Gor00] Rajeev Goré. Dual intuitionistic logic revisited. In *TABLEAUX 2000*, volume 1847 of *LNCS*, pages 252–267. Springer, 2000.

[ILH10] Danko Ilik, Gyesik Lee, and Hugo Herbelin. Kripke models for classical logic. *Annals of Pure and Applied Logic*, 161(11):1367–1378, 2010.

[Kri65] S. A. Kripke. Semantical analysis of intuitionistic logic I. *Formal Systems and Recursive Functions*, pages 92–130, North-Holland, 1965.

[Lam08] François Lamarche. Proof nets for intuitionistic linear logic: Essential nets. Technical Report INRIA-00347336, INRIA-Lorraine, December 2008.

[Lau02] Olivier Laurent. *Etude de la polarisation en logique.* PhD thesis, Université Aix-Marseille II, March 2002.

[LM09] Chuck Liang and Dale Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, 2009.

[LM11] Chuck Liang and Dale Miller. A focused approach to combining logics. *Annals of Pure and Applied Logic*, 162(9):679–697, 2011.

[NM09] Vivek Nigam and Dale Miller. Algorithmic specifications in linear logic with subexponentials. In *ACM SIGPLAN Conference on Principles and Practice of Declarative Programming (PPDP)*, pages 129–140, 2009.

[Rau74] C. Rauszer. A formalization of the propositional calculus of H-B logic. *Studia Logica*, 33:23–34, 1974.

[TS96] Anne S. Troelstra and Helmut Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, 1996.

[Vel76] Wim Veldman. An intuitionistic completeness theorem for intuitionistic predicate logic. *Journal of Symbolic Logic*, 41(1):159–166, 1976.