Proof checking and logic programming

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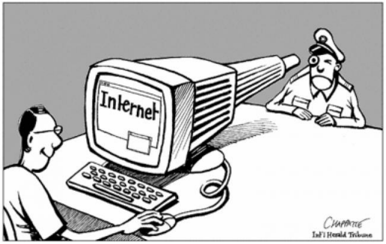
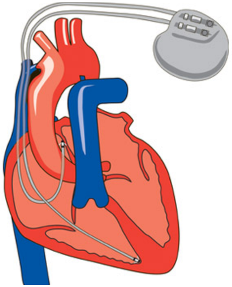
Joint work with Roberto Blanco, Kaustuv Chaudhuri, Zakaria Chihani, Quentin Heath, Stefan Hetzl, Danko Ilik, Chuck Liang, Tomer Libal, Marco Volpe, Fabien Renaud, Giselle Reis, Alwen Tiu.

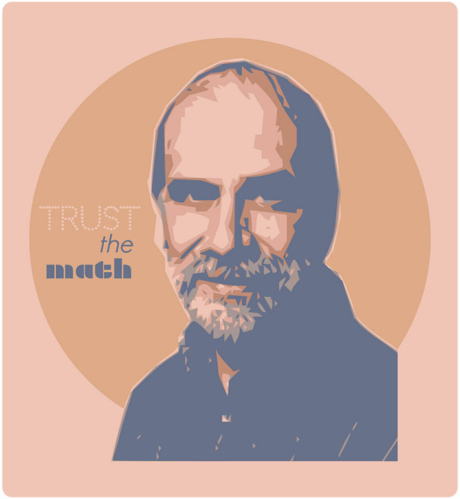
See papers in: CADE 2013, CPP 2011/13/15, PxTP 2013/15, Tableaux 2015.

What can we trust?





In cryptology: Trust the math



Bruce Schneier

In software correctness: Trust the proof!

With software systems, there are so many things to trust.

- printers and parsers
- type checkers, type inference, abstract interpretation
- compilers
- verification condition generators
- theorem provers

All this seems overwhelming. Our challenge here:

provide the framework so that we can at least trust proofs.

We restriction our of attention to **formal proofs**, generated and checked by computer tools.

The current situation with formal proofs

Most proof production and checking is technology based.

Most proofs are locked into the technology.

If you change the version number of a prover, it may not recognized its earlier proofs.

Some bridges are now being built between different provers, but these are affected by two version numbers.

In them we can trust

de Bruijn, Huet, Paulson, Boyer, Moore

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Obvious, this model of trust does not scale!

The tradition of proof checking

Most of the major proof checkers / theorem provers, e.g.,

Automath, Boyer and Moore, LCF Tactics/Tacticals,
HOL, Isabelle, Coq, Agda, PVS, etc,

come from a *functional programming* background.

It seems odd that logic programming has played a minor role in this area, even though

- many primitives in this subject are relations, e.g., "Ξ is a proof of B" and
- non-determinism and unification are part of the fabric of proof checking and theorem proving.

The framework I present here can make extensive use of *logic programming* systems and principles.

The vision: The network *is* the prover

Goal: Permit the formal methods community to become a network of communicating provers.

Proof certificates: documents that circulate and denote proofs.

Approach: Provide formal definitions of “proof evidence” so that proof certificates can be checked by *trusted checkers*.

But: There is a wide range of “proof evidence.”

- resolution refutations, natural deduction, tableaux, etc
- proof scripts for steering a theorem prover to a proof
- winning strategies, simulations

The need for frameworks

Three central questions:

- How can we manage so many "proof languages"?
- Will we need just as many proof checkers?
- How does this improve trust?

Computer scientists have seen this kind of problem before.

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Computer scientists have seen this kind of problem before.

We develop *frameworks* to address such questions.

- lexical analysis: finite state machines / transducers
- language syntax: grammars, parsers, attribute grammars, parser generators
- programming languages: denotational and operational semantics

A framework for proof evidence: First pick the logic

Church's Simple Theory of Types (STT) is a good choice for the syntax of formulas.

It is understood well in both the classical and intuitionistic settings.

Propositional, first-order, and higher-order logics are easily identifiable sublogics of STT (many others too).

Earliest notion of formal proof

Frege, Hilbert, Church, Gödel, etc, made extensive use of the following notion of proof:

> *A proof is a list of formulas, each one of which is either an* axiom *or the conclusion of an* inference rule *whose premises come earlier in the list.*

While granting us trust, there is little useful structure here.

The first programmable proof checker

Lecture Notes in Computer Science

Edited by G. Goos and J. Hartmanis

78

Michael J. Gordon
Arthur J. Milner
Christopher P. Wadsworth

Edinburgh LCF

Springer-Verlag
Berlin Heidelberg New York

LCF/ML (1979) viewed proofs as slight generalizations of such lists.

ML provided types, abstract datatypes, and higher-order programming in order to increase confidence in proof checking.

Many provers today (HOL, Coq, Isabelle) follow LCF principles.

More recent advances: Atoms and molecules of inference

Atoms of inference

- Gentzen's **sequent calculus** first provided these: introduction, identity, and structural rules.
- Girard's **linear logic** refined our understanding of these atoms.
- To account for first-order structure, we also need **fixed points** and **equality**.

Rules of Chemistry

- **Focused proof systems** show us that some atoms stick together while other atoms form boundaries.

Molecules of inference

- Collections of atomic inference rules that stick together form synthetic inference rules.

Features enabled for proof certificates

- Simple checkers can be implemented.
  Only the atoms of inference and the rules of chemistry (both small and closed sets) need to be implemented in a checker of certificates.

- Certificates support a wide range of proof systems.
  The molecules of inference can be engineered into a wide range of inference rules.

- Certificates are based (ultimately) on proof theory.
  Immediate by design.

- Proof details can be elided.
  Search using atoms will match search in the space of molecules: that is, the checker will not invent new molecules.

An analogy between two frameworks: SOS and FPC

Structural Operational Semantics (SOS)

1. There are many programming languages.
2. SOS can define the semantics of many of them.
3. Logic programming can provide prototype interpreters.
4. Compliant compilers can be built based on the semantics.

An analogy between two frameworks: SOS and FPC

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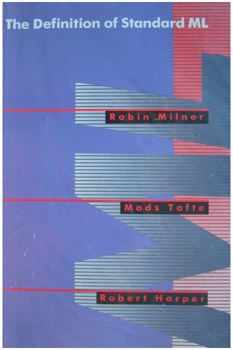

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Foundational Proof Certificates (FPC)

1. There are many forms of proof evidence.
2. FPC can define the semantics of many of them.
3. Logic programming can provide prototype checkers.
4. Compliant checkers can be built based on the semantics.

Clerks and experts: the office workflow analogy

Imagine an accounting office that needs to check if a certain mound of financial documents (provided by a **client**) represents a legal tax transaction (as judged by the **kernel**).

Experts look into the mound and extract information and

- *decide* which transactions to dig into and
- *release* their findings for storage and later reconsideration.

Clerks take information released by the experts and perform some computations on them, including their *indexing* and *storing*.

Focused proofs alternate between two phases: *positive* (experts are active) and *negative* (clerks are active).

The terms *decide*, *store*, and *release* come from proof theory.

A proof certificate format defines workflow and the duties of the clerks and experts.

Proof checking and proof reconstruction

Clearly, (determinate) computation is built into this paradigm: the clerks can perform such computation.

Proof *reconstruction* might be needed when invoking not-so-expert experts (or ambiguous tax forms).

Non-deterministic computation is part of the mix: non-determinism is an important resource that is useful for proof-compression.

The *LKneg* proof system

Use invertible rules where possible. In propositional classical logic, both conjunction and disjunction can be given invertible rules.

$$\frac{\vdash \cdot; B}{\vdash B}\ \mathit{start} \qquad \frac{\vdash \Delta, L; \Gamma}{\vdash \Delta; L, \Gamma}\ \mathit{store} \qquad \frac{}{\vdash \Delta, A, \neg A; \cdot}\ \mathit{init}$$

$$\frac{\vdash \Delta; \Gamma}{\vdash \Delta; \mathit{false}, \Gamma} \quad \frac{\vdash \Delta; B, C, \Gamma}{\vdash \Delta; B \vee C, \Gamma} \quad \frac{}{\vdash \Delta; \mathit{true}, \Gamma} \quad \frac{\vdash \Delta; B, \Gamma \quad \vdash \Delta; C, \Gamma}{\vdash \Delta; B \wedge C, \Gamma}$$

Here, A is an atom, L a literal, Δ a multiset of literals, and Γ a list of formulas. Sequents have two *zones*.

This proof system provides a decision procedure (resembling conjunctive normal forms).

A small (constant sized) certificate is possible.

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Consider proving $(p \vee C) \vee \neg p$ for large C.

The *LKpos* proof system

Non-invertible rules are used here.

$$\frac{\vdash B; \cdot; B}{\vdash B}\ \textit{start} \qquad \frac{\vdash B; \mathcal{N}, \neg A; B}{\vdash B; \mathcal{N}; \neg A}\ \textit{restart} \qquad \frac{}{\vdash B; \mathcal{N}, \neg A; A}\ \textit{init}$$

$$\frac{\vdash B; \mathcal{N}; B_i}{\vdash B; \mathcal{N}; B_1 \vee B_2} \qquad \frac{}{\vdash B; \mathcal{N}; \textit{true}} \qquad \frac{\vdash B; \mathcal{N}; B_1 \quad \vdash B; \mathcal{N}; B_2}{\vdash B; \mathcal{N}; B_1 \wedge B_2}$$

Here, A is an atom and $\mathcal{N}$ is a multiset of negated atoms.
Sequents have three *zones*.

The $\vee$ rule *consumes* some external information or some non-determinism.

An *oracle string*, a series of bits used to indicate whether to go left or right, can be a proof certificate.

A proof in *LKpos*

Let C have several alternations of conjunction and disjunction.

Let $B = (p \vee C) \vee \neg p$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B; \neg p; p}\ \mathit{init}}{\vdash B; \neg p; p \vee C}\ *}{\vdash B; \neg p; (p \vee C) \vee \neg p}\ *}{\vdash B; \cdot\ ; \neg p}\ \mathit{restart}}{\vdash B; \cdot\ ; (p \vee C) \vee \neg p}\ *}{\vdash B}\ \mathit{start}
$$

The subformula C is avoided. Clever choices $*$ are injected at these points: right, left, left. We have a small certificate and small checking time. In general, these certificates may grow large.

Combining the *LKneg* and *LKpos* proof systems

Introduce two versions of conjunction, disjunction, and their units.

$$t^-, t^+, f^-, f^+, \vee^-, \vee^+, \wedge^-, \wedge^+$$

The inference rules for negative connectives are invertible.

These polarized connectives also exist in linear logic.

Introduce the two kinds of sequent, namely,
$\vdash \Theta \Uparrow \Gamma$: for invertible (negative) rules (Γ a list of formulas)
$\vdash \Theta \Downarrow B$: for non-invertible (positive) rules (B a formula)

LKF : a focused proof systems for classical logic

$$\frac{}{\vdash \Theta \Uparrow \Gamma, t^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, B \qquad \vdash \Theta \Uparrow \Gamma, B'}{\vdash \Theta \Uparrow \Gamma, B \wedge^- B'} \qquad \frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, f^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, B, B'}{\vdash \Theta \Uparrow \Gamma, B \vee^- B'}$$

$$\frac{}{\vdash \Theta \Downarrow t^+} \qquad \frac{\vdash \Theta \Downarrow B \qquad \vdash \Theta \Downarrow B'}{\vdash \Theta \Downarrow B \wedge^+ B'} \qquad \frac{\vdash \Theta \Downarrow B_i}{\vdash \Theta \Downarrow B_1 \vee^+ B_2}$$

$$\overset{\text{Init}}{\frac{}{\vdash \neg A, \Theta \Downarrow A}} \qquad \overset{\text{Store}}{\frac{\vdash \Theta, C \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, C}} \qquad \overset{\text{Release}}{\frac{\vdash \Theta \Uparrow N}{\vdash \Theta \Downarrow N}} \qquad \overset{\text{Decide}}{\frac{\vdash P, \Theta \Downarrow P}{\vdash P, \Theta \Uparrow \cdot}}$$

P is a positive formula; N is a negative formula;
A is an atom; C positive formula or negative literal

Results about LKF

Let B be a propositional logic formula and let $\hat{B}$ result from B by placing $+$ or $-$ on t, f, $\wedge$, and $\vee$ (there are exponentially many such placements).

Theorem. [Liang & M, TCS 2009]

- If B is a tautology then every $\hat{B}$ has an LKF proof.
- If some $\hat{B}$ has an LKF proof, then B is a tautology.

The different polarizations do not change *provability* but can radically change the *proofs*.

Also:

- Negative (non-atomic) formulas are treated linearly (never weakened nor contracted).
- Only positive formulas are contracted (in the Decide rule).

Example: deciding on a simple clause

Assume that Θ contains the formula $a \wedge^+ b \wedge^+ \neg c$ and that we have a derivation that Decides on this formula.

$$\dfrac{\dfrac{\dfrac{}{\vdash \Theta \Downarrow a}\,\mathit{Init} \quad \dfrac{}{\vdash \Theta \Downarrow b}\,\mathit{Init} \quad \dfrac{\dfrac{\vdash \Theta, \neg c \Uparrow \cdot}{\vdash \Theta \Uparrow \neg c}\,\mathit{Store}}{\vdash \Theta \Downarrow \neg c}\,\mathit{Release}}{\vdash \Theta \Downarrow a \wedge^+ b \wedge^+ \neg c}\,\wedge^+}{\vdash \Theta \Uparrow \cdot}\,\mathit{Decide}$$

This derivation is possible iff Θ is of the form $\neg a, \neg b, \Theta'$. Thus, the "macro-rule" is

$$\dfrac{\vdash \neg a, \neg b, \neg c, \Theta' \Uparrow \cdot}{\vdash \neg a, \neg b, \Theta' \Uparrow \cdot}$$

Example: Resolution as a proof certificate

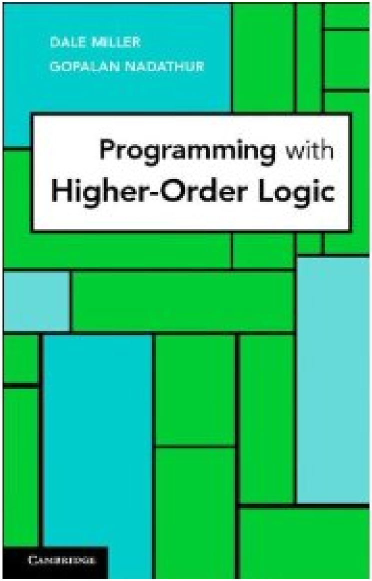
- A *clause:* $\forall x_1 \ldots \forall x_n [L_1 \vee \cdots \vee L_m]$
- C_3 is a *resolution* of C_1 and C_2 if we chose the mgu of two complementary literals, one from each of C_1 and C_2, etc.
- If C_3 is a resolvent of C_1 and C_2 then $\vdash \neg C_1, \neg C_2 \Uparrow C_3$ has a short proof (decide depth 2 or less).

Translate a refutation of $C_1, \ldots, C_n$ into a (focused) sequent proof with small holes:

$$\dfrac{\dfrac{\Xi}{\vdash \neg C_1, \neg C_2 \Uparrow C_{n+1}} \qquad \dfrac{\dfrac{\vdots}{\vdash \neg C_1, \ldots, \neg C_n, \neg C_{n+1} \Uparrow \cdot}}{\vdash \neg C_1, \ldots, \neg C_n \Uparrow \neg C_{n+1}}\ \mathit{Store}}{\vdash \neg C_1, \ldots, \neg C_n \Uparrow \cdot}\ \mathit{Cut}$$

Here, Ξ can be replaced with a "hole" bounded by depth 2.

Reference proof checking in λProlog



Logic programming can check proofs in sequent calculus.

Proof reconstruction requires unification and (bounded) proof search.

The λProlog programming language [M & Nadathur, 1986, 2012] also include types, abstract datatypes, and higher-order programming.

From inference rules to λProlog clauses

We first "instrument" the inference rules with terms denoting proof certificates and add premises that invoke "clerks" and "experts".

$$\frac{\Xi_1 \vdash \Theta \Uparrow \Gamma, A \qquad \Xi_2 \vdash \Theta \Uparrow \Gamma, B \qquad \wedge\mathrm{clerk}(\Xi_0, \Xi_1, \Xi_2)}{\Xi_0 \vdash \Theta \Uparrow \Gamma, A \wedge^- B}$$

$$\frac{\Xi_1 \vdash \Theta \Downarrow B_i \qquad \vee\mathrm{expert}(\Xi_0, \Xi_1, i)}{\Xi_0 \vdash \Theta \Downarrow B_1 \vee^+ B_2}$$

Turning inference rules sideways yields logic programs.

The resulting logic program is the *kernel* of the checker.

Soundness of the kernel is reduced to soundness of the logic programming implementation.

An FPC: Checking by conjunctive normal form

```
type lit        index.
type cnf         cert.

andNeg_kc       cnf cnf cnf.
orNeg_kc            cnf cnf.
false_kc            cnf cnf.
release_ke          cnf cnf.
initial_ke              cnf lit.
decide_ke           cnf cnf lit.
store_kc            cnf cnf lit.
```

The token `cnf` is just passed around during the checking. The only items that are stored are literals and they are all indexed the same.

An FPC: Checking binary resolution

```
type idx                    int -> index.
type lit                           index.
kind resol                          type.
type resol    int -> int -> int -> resol.
type dl               list int    -> cert.
type ddone                          cert.
type rdone                          cert.
type rlist            list resol -> cert.
type rlisti   int ->  list resol -> cert.

orNeg_kc    (dl L) _  (dl L).
false_kc    (dl L) (dl L).
store_kc    (dl L) C  lit (dl L).
decide_ke   (dl [I]) (idx I) (dl []).
decide_ke   (dl [I,J]) (idx I) (dl [J]).
decide_ke   (dl [J,I]) (idx I) (dl [J])
all_kc      (dl L) (x\ dl L).
true_ke     (dl L).
some_ke     (dl L) _  (dl L).
andPos_ke   (dl L) _  (dl L) (dl L).
release_ke  (dl L) (dl L).
initial_ke  (dl L) _.
decide_ke   (dl L) _ ddone.
initial_ke  ddone _.

false_kc  (rlist R) (rlist R).
store_kc  (rlisti K R) _ (idx K) (rlist R).
true_ke   rdone.
decide_ke (rlist [])  (idx I) rdone.
cut_ke    (rlist [(resol I J K) |R]) CutForm (dl [I,J]) (rlisti K R).
```

Grammars, Parsers, and Logic Programming

Grammars are declarative.

LP can turn context free grammars into parsers.

- Definite clause grammars
- Issues of search are central: avoid left-recursion, use tabled deduction or Earley deduction, etc.

If you restrict to subclasses (e.g., LALR(1)) then specialize tools (e.g., YACC) can replace LP, gaining efficiency and acceptance.

Why trust YACC to generate a correct parser?

FPC, Checkers, and Logic Programming

LP can turn an FPC into a checker by adding it to the kernel that implements the focused sequent calculus.

More logic than first order Horn clause logic is needed.

- Support for binders to capture quantification in formulas and eigenvariables in proofs.
- Unification must be sound (turn on occur-check)
- Handling induction/co-induction requires generalized forms of negation-as-failure.

Processing FPCs will require the possibility of performing arbitrary deterministic and non-deterministic computations.

Program analysis and transformation will likely be key to improving the performance of checkers.

The ProofCert project: recent results and next steps

Formal definition of the FPC framework for first-order logic.

Many example proof certificate formats are defined:

- Classical: resolution, expansion trees, matings, CNF, etc.
- Intuitionistic: natural deduction, various typed λ-calculus.
- Also: Frege systems, equality reasoning, etc.

Implemented a reference kernel (using λProlog / Teyjus)

The intuitionistic checker can "host" the classical kernel, so only one kernel is needed.

Next: designing certificates for

- model checking and inductive/co-inductive reasoning and
- modal logic proofs.

Thank you