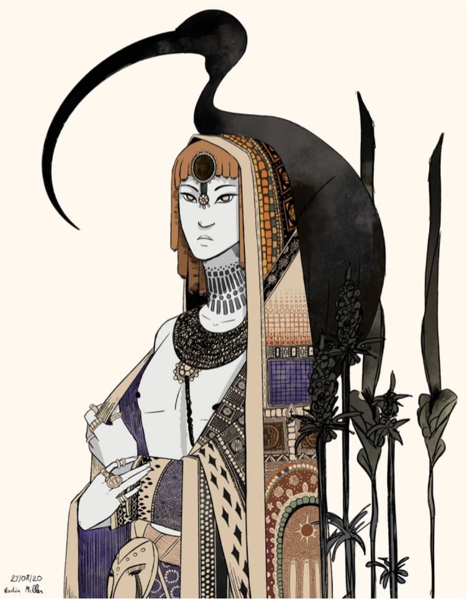
A system of inference based on proof search

Dale Miller

Inria Saclay &
LIX, Institut Polytechnique de Paris

LICS 2023, Boston
29 June 2023



Building complete proofs vs searching among (partial) proofs

Gentzen introduced two proof systems—natural deduction and sequent calculus—in his 1935 paper "Investigations into Logical Deduction".

These systems describe the static structure of *complete* proofs as trees.

- Inference rules take complete proofs to other complete proofs.

We examine here an alternative framework for proof structures based on *search* in a space of *partial proofs*.

- Partial proofs are expanded from their roots.
- $\mathbf{PSF}$ (proof search framework) is given as the inference systems $\mathbf{B}$ and $\mathbf{F}$.

Building complete proofs vs searching among (partial) proofs

Gentzen introduced two proof systems—natural deduction and sequent calculus—in his 1935 paper "Investigations into Logical Deduction".

These systems describe the static structure of *complete* proofs as trees.

- Inference rules take complete proofs to other complete proofs.

We examine here an alternative framework for proof structures based on *search* in a space of *partial proofs*.

- Partial proofs are expanded from their roots.
- $\mathbf{PSF}$ (proof search framework) is given as the inference systems $\mathbf{B}$ and $\mathbf{F}$.

Our analysis:

- is at a conceptual level (similar to Gentzen's 1935 paper),
- provides motivations for many features of linear logic, and
- does not address implementation topics involved in modern automated theorem proving: e.g., unification, resolution refutations, tableaux, saturation, etc.

Proof search using sheets of paper

Prove that $x(x+1)$ is even for natural number x.

Assume:

$\forall n.\ \text{even}\ n \vee \text{odd}\ n$

$\forall n.\ \text{odd}\ n \supset \text{even}\ (s\ n)$

$\forall n, m, p.\ (\text{even}\ n \vee \text{even}\ m) \supset$
$\quad \text{times}\ n\ m\ p \supset \text{even}\ p$

$\vdots$

Hence:

$\forall x, y.\ \text{times}\ x\ (s\ x)\ y \supset \text{even}\ y$

Proof search using sheets of paper

Prove that $x(x+1)$ is even for natural number x.

Assume:

$\forall n.\ \text{even}\ n \vee \text{odd}\ n$

$\forall n.\ \text{odd}\ n \supset \text{even}\ (s\ n)$

$\forall n, m, p.\ (\text{even}\ n \vee \text{even}\ m) \supset$
$\qquad \text{times}\ n\ m\ p \supset \text{even}\ p$

$\text{times}\ x\ (s\ x)\ y$

$\vdots$

Hence:

$\text{even}\ y$

Proof search using sheets of paper

Prove that $x(x+1)$ is even for natural number x.

Assume:

$\forall n.\ \text{even}\ n \vee \text{odd}\ n$

$\forall n.\ \text{odd}\ n \supset \text{even}\ (s\ n)$

$\forall n, m, p.\ (\text{even}\ n \vee \text{even}\ m) \supset$

$\qquad \text{times}\ n\ m\ p \supset \text{even}\ p$

$\text{times}\ x\ (s\ x)\ y$

$\text{even}\ x \vee \text{odd}\ x$

$\vdots$

Hence:

$\text{even}\ y$

Proof search using sheets of paper

Prove that $x(x+1)$ is even for natural number x.

Assume:
$\forall n.\ \mathrm{even}\ n \vee \mathrm{odd}\ n$
$\forall n.\ \mathrm{odd}\ n \supset \mathrm{even}\ (s\ n)$
$\forall n, m, p.\ (\mathrm{even}\ n \vee \mathrm{even}\ m) \supset \mathrm{times}\ n\ m\ p \supset \mathrm{even}\ p$
$\mathrm{times}\ x\ (s\ x)\ y$
$\mathrm{even}\ x$

$\vdots$

Hence:
$\mathrm{even}\ y$

Assume:
$\forall n.\ \mathrm{even}\ n \vee \mathrm{odd}\ n$
$\forall n.\ \mathrm{odd}\ n \supset \mathrm{even}\ (s\ n)$
$\forall n, m, p.\ (\mathrm{even}\ n \vee \mathrm{even}\ m) \supset \mathrm{times}\ n\ m\ p \supset \mathrm{even}\ p$
$\mathrm{times}\ x\ (s\ x)\ y$
$\mathrm{odd}\ x$

$\vdots$

Hence:
$\mathrm{even}\ y$

Proof search using sheets of paper

Prove that $x(x+1)$ is even for natural number x.

Assume:
$\forall n.\ \text{even}\ n \vee \text{odd}\ n$
$\forall n.\ \text{odd}\ n \supset \text{even}\ (s\ n)$
$\forall n, m, p.\ (\text{even}\ n \vee \text{even}\ m) \supset \text{times}\ n\ m\ p \supset \text{even}\ p$
$\text{times}\ x\ (s\ x)\ y$
$\text{even}\ x$

$\vdots$

Hence:
$\text{even}\ y$

Assume:
$\forall n.\ \text{even}\ n \vee \text{odd}\ n$
$\forall n.\ \text{odd}\ n \supset \text{even}\ (s\ n)$
$\forall n, m, p.\ (\text{even}\ n \vee \text{even}\ m) \supset \text{times}\ n\ m\ p \supset \text{even}\ p$
$\text{times}\ x\ (s\ x)\ y$
$\text{odd}\ x$

$\vdots$

Hence:
$\text{even}\ y$

Observations:

1. Occurrences of formulas have two *sense*: as assumption and goal.
2. Some formula occurrences are *permanent*; others may get deleted and/or replaced.
3. One sheet can become 2, also 0 (if an assumption is the goal).

Abstraction of what happens on sheets of paper

$\mathbf{A_1}$
$\mathbf{A_2}$
$\mathbf{A_3}$

$\vdots$

A_0

Conventions:

1. The two senses: hypothesis are blue; goals are red. The vertical dots are no longer needed.

Abstraction of what happens on sheets of paper

$\mathbf{A}_1$
$\mathbf{A}_2$
$\mathbf{A}_3$
$\mathbf{A}_5$

A_4

Conventions:

1. The two senses: hypothesis are blue; goals are red. The vertical dots are no longer needed.
2. *Permanent* items are displayed in **bold**.

Abstraction of what happens on sheets of paper

$\mathbf{A_1}$
$\mathbf{A_2}$
$\mathbf{A_3}$
$\mathbf{A_5}$
A_6

A_4

Conventions:

1. The two senses: hypothesis are blue; goals are red. The vertical dots are no longer needed.
2. *Permanent* items are displayed in **bold**.

Abstraction of what happens on sheets of paper

$\mathbf{A_1}$
$\mathbf{A_2}$
$\mathbf{A_3}$
$\mathbf{A_5}$
$\mathbf{A_7}$

A_4

$\mathbf{A_1}$
$\mathbf{A_2}$
$\mathbf{A_3}$
$\mathbf{A_5}$
$\mathbf{A_8}$

A_4

Conventions:

1. The two senses: hypothesis are blue; goals are red. The vertical dots are no longer needed.
2. *Permanent* items are displayed in **bold**.

Abstraction of what happens on sheets of paper

$\mathbf{A_1}$
$\mathbf{A_2}$
$\mathbf{A_3}$
$\mathbf{A_5}$
$\mathbf{A_7}$

A_4

$\mathbf{A_1}$
$\mathbf{A_2}$
$\mathbf{A_3}$
$\mathbf{A_5}$
$\mathbf{A_8}$

A_4

Conventions:

1. The two senses: hypothesis are blue; goals are red. The vertical dots are no longer needed.
2. *Permanent* items are displayed in **bold**.
3. Sheets are encoded as multisets and not lists.

We shall capture inference systems by first describing an *enriched version of multiset rewriting*.

The distinction between hypothesis and goal is not part of the multiset rewriting system itself: it is added later.

The pre-logical framework

- Formulas will be tagged as "hypothesis" or "goal".
- We abstract away the internal structure of tagged formulas and replace them with *atomic expressions*.
- The current state is simply a set of sheets of paper: i.e., a set of multisets of atomic expressions.
- Our first goals are to describe
  - how state can be encoded as *expressions* and
  - how state *evolves* by applying *rewriting rules*.

Multiplicative features: multiset rewriting

Multisets: $E ::= A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Multiplicative features: multiset rewriting

Multisets: $E ::= A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Entailment between expressions and multisets provides an equality.

$$\frac{}{E \vdash E} \qquad \frac{}{\mathbf{1} \vdash} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

Multiplicative features: multiset rewriting

Multisets: $E ::= \ A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Entailment between expressions and multisets provides an equality.

$$\frac{}{E \vdash E} \qquad \frac{}{\mathbf{1} \vdash} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

Rewriting multiset Δ to multiset Δ' using rule $E_1 \mapsto E_2$ is done in 3 steps.

1. Split Δ into two parts Δ_1 and Δ_2.
2. Determine that E_1 is the same multiset as Δ_1.
3. Identify Δ' with the multiset union of E_2 and Δ_2.

Multiplicative features: multiset rewriting

Multisets: $E ::= \ A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Entailment between expressions and multisets provides an equality.

$$\frac{}{E \vdash E} \qquad \frac{}{\mathbf{1} \vdash} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

Rewriting multiset Δ to multiset Δ' using rule $E_1 \mapsto E_2$ is done in 3 steps.

1. Split Δ into two parts Δ_1 and Δ_2.
2. Determine that E_1 is the same multiset as Δ_1.
3. Identify Δ' with the multiset union of E_2 and Δ_2.

$$\frac{E_1 \vdash \Delta_1 \quad \vdash E_2, \Delta_2}{E_1 \mapsto E_2 \vdash \Delta_1, \Delta_2} \qquad \frac{E_1 \mapsto E_2 \vdash \Delta}{\vdash \Delta} \ \textit{decide} \text{ on } (E_1 \mapsto E_2) \in \mathcal{R}$$

Multiplicative features: multiset rewriting

Multisets: $E ::= \ A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Entailment between expressions and multisets provides an equality.

$$\frac{}{E \vdash E} \qquad \frac{}{\mathbf{1} \vdash} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

Rewriting multiset Δ to multiset Δ' using rule $E_1 \mapsto E_2$ is done in 3 steps.

1. Split Δ into two parts Δ_1 and Δ_2.
2. Determine that E_1 is the same multiset as Δ_1.
3. Identify Δ' with the multiset union of E_2 and Δ_2.

$$\frac{E_1 \vdash \Delta_1 \quad \vdash E_2, \Delta_2}{E_1 \mapsto E_2 \vdash \Delta_1, \Delta_2} \qquad \frac{E_1 \mapsto E_2 \vdash \Delta}{\vdash \Delta} \ \textit{decide} \text{ on } (E_1 \mapsto E_2) \in \mathcal{R}$$

$$\frac{}{\vdash a, a, b}$$

Multiplicative features: multiset rewriting

Multisets: $E ::= \ A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Entailment between expressions and multisets provides an equality.

$$\frac{}{E \vdash E} \qquad \frac{}{\mathbf{1} \vdash} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

Rewriting multiset Δ to multiset Δ' using rule $E_1 \mapsto E_2$ is done in 3 steps.

1. Split Δ into two parts Δ_1 and Δ_2.
2. Determine that E_1 is the same multiset as Δ_1.
3. Identify Δ' with the multiset union of E_2 and Δ_2.

$$\frac{E_1 \vdash \Delta_1 \quad \vdash E_2, \Delta_2}{E_1 \mapsto E_2 \vdash \Delta_1, \Delta_2} \qquad \frac{E_1 \mapsto E_2 \vdash \Delta}{\vdash \Delta}\ \textit{decide}\text{ on } (E_1 \mapsto E_2) \in \mathcal{R}$$

$$\frac{\overline{a \times b \mapsto c \vdash {\color{red}a}, {\color{blue}a, b}}}{\vdash a, a, b}\ \textit{decide}$$

Multiplicative features: multiset rewriting

Multisets: $E ::= \; A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Entailment between expressions and multisets provides an equality.

$$\frac{}{E \vdash E} \qquad \frac{}{\mathbf{1} \vdash} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

Rewriting multiset Δ to multiset Δ' using rule $E_1 \mapsto E_2$ is done in 3 steps.

1. Split Δ into two parts Δ_1 and Δ_2.
2. Determine that E_1 is the same multiset as Δ_1.
3. Identify Δ' with the multiset union of E_2 and Δ_2.

$$\frac{E_1 \vdash \Delta_1 \quad \vdash E_2, \Delta_2}{E_1 \mapsto E_2 \vdash \Delta_1, \Delta_2} \qquad \frac{E_1 \mapsto E_2 \vdash \Delta}{\vdash \Delta} \; \textit{decide} \text{ on } (E_1 \mapsto E_2) \in \mathcal{R}$$

$$\dfrac{\dfrac{\dfrac{}{a \times b \vdash {\color{blue}a, b}} \quad \vdash {\color{red}a}, c}{a \times b \mapsto c \vdash {\color{red}a}, {\color{blue}a, b}}}{\vdash a, a, b} \; \textit{decide}$$

Multiplicative features: multiset rewriting

Multisets: $E ::= \ A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Entailment between expressions and multisets provides an equality.

$$\frac{}{E \vdash E} \qquad \frac{}{\mathbf{1} \vdash} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

Rewriting multiset Δ to multiset Δ' using rule $E_1 \mapsto E_2$ is done in 3 steps.

1. Split Δ into two parts Δ_1 and Δ_2.
2. Determine that E_1 is the same multiset as Δ_1.
3. Identify Δ' with the multiset union of E_2 and Δ_2.

$$\frac{E_1 \vdash \Delta_1 \quad \vdash E_2, \Delta_2}{E_1 \mapsto E_2 \vdash \Delta_1, \Delta_2} \qquad \frac{E_1 \mapsto E_2 \vdash \Delta}{\vdash \Delta}\ \textit{decide} \text{ on } (E_1 \mapsto E_2) \in \mathcal{R}$$

$$\frac{\dfrac{\dfrac{\dfrac{}{a \vdash a} \quad \dfrac{}{b \vdash b}}{a \times b \vdash {\color{blue}a, b}} \quad \vdash {\color{red}a}, c}{a \times b \mapsto c \vdash {\color{red}a}, {\color{blue}a, b}}}{\vdash a, a, b}\ \textit{decide}$$

Multiplicative features: multiset rewriting

Multisets: $E ::= \ A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

Entailment between expressions and multisets provides an equality.

$$\frac{}{E \vdash E} \qquad \frac{}{\mathbf{1} \vdash} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

Rewriting multiset Δ to multiset Δ' using rule $E_1 \mapsto E_2$ is done in 3 steps.

1. Split Δ into two parts Δ_1 and Δ_2.
2. Determine that E_1 is the same multiset as Δ_1.
3. Identify Δ' with the multiset union of E_2 and Δ_2.

$$\frac{E_1 \vdash \Delta_1 \quad \vdash E_2, \Delta_2}{E_1 \mapsto E_2 \vdash \Delta_1, \Delta_2} \qquad \frac{E_1 \mapsto E_2 \vdash \Delta}{\vdash \Delta} \; \textit{decide} \text{ on } (E_1 \mapsto E_2) \in \mathcal{R}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{a \vdash a} \quad \cfrac{}{b \vdash b}}{a \times b \vdash {\color{blue}a, b}} \quad \vdash {\color{red}a}, c}{a \times b \mapsto c \vdash {\color{red}a}, {\color{blue}a, b}}}{\vdash a, a, b} \; \textit{decide} \qquad \longrightarrow \qquad \frac{\vdash a, c}{\vdash a, a, b}$$

Additive feature: copying of multisets

We also need to be able to copy the content of a sheet. To this end, we add the following operators on expressions.

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{E_i \vdash \Delta}{E_1 + E_2 \vdash \Delta}$$

Distributivity holds: the inference systems will not be able to distinguish $E_1 \times (E_2 + E_3)$ from $(E_1 \times E_2) + (E_1 \times E_3)$.

Note that

- The $\times$ on the right builds contexts (by becoming a comma).
- The $\times$ on the left splits contexts.
- The $+$ on the right accumulates branches.
- The $+$ on the left selects a branch.

These two senses for $\times$ and $+$ allow us to prove results similar to the elimination of non-atomic initials and cuts.

Additional features: the linear and classical realms

As motivated before, some atomic expressions can remain in all evolutions of a multiset; others can be deleted and replaced.

Atomic expressions will belong to the *linear* or the *classical realms*.

Non-atomic expressions are not classified either way.

Additional features: the linear and classical realms

As motivated before, some atomic expressions can remain in all evolutions of a multiset; others can be deleted and replaced.

Atomic expressions will belong to the *linear* or the *classical realms*.

Non-atomic expressions are not classified either way.

Atomic expressions in the classical realm have a *superpower*: they can appear any number of times. Technically, the *contraction* and *weakening* rules can be applied to them.

Additional feature: debts

From a distributed computing perspective, multiset rewriting should be more flexible.

- When applying the rule $a \times b \mapsto c$, we must locate in Δ both a and b.
- Δ might be very large and distributed across a network.
- a might be found quickly, but finding b could take time.
- During the search for b, concurrent rewritings might happen on other machines. We cannot tell whether b is present or it might become present due to additional multiset rewritings.

Additional feature: debts

From a distributed computing perspective, multiset rewriting should be more flexible.

- When applying the rule $a \times b \mapsto c$, we must locate in Δ both a and b.
- Δ might be very large and distributed across a network.
- a might be found quickly, but finding b could take time.
- During the search for b, concurrent rewritings might happen on other machines. We cannot tell whether b is present or it might become present due to additional multiset rewritings.

Instead of *blocking* all processing until b is found, we might allow a *debt* to be registered in our multiset and then later resolved that debt with the eventual discovery or production of b.

All atomic expressions will have a "credit rating". If b's rating is positive, then a debt can be constructed.

This debt mechanism will help account for *bottom-up* and *top-down reasoning*.

Expressions and Rules

The variable A ranges over some fixed set of atomic expressions.

Expressions and rules are defined inductively.

$$E ::= A \mid \mathbf{0} \mid E_1 + E_2 \mid \mathbf{1} \mid E_1 \times E_2$$
$$R ::= A \mid \mathbf{0} \mid R_1 + R_2 \mid \mathbf{1} \mid R_1 \times R_2 \mid R \mapsto E \mid R \Mapsto E$$

$\mapsto$ and $\Mapsto$ associate to the left; $+$ and $\times$ associate to the right.

A *debt* is an expression of the form $\overline{A}$.

Γ ranges over multisets containing R-expressions.

Δ ranges over multisets that can contain both E-expressions and debts.

$\mathcal{R}$ denotes some countable set of R-expressions.

Bias assignments

A *bias assignment* $\delta(\cdot)$ maps atomic expressions to $\{-2, -1, +1, +2\}$.

If $\delta(A) > 0$, then A can be converted into a debt.

A is in the *linear realm* if $\delta(A)$ is ± 1 and in the *classical realm* if $\delta(A)$ is ± 2.

$\boldsymbol{S}$ ranges over atomic expressions in the classical realm.

Υ ranges over finite multisets of atomic expressions in the classical realm.

The basic inference system: $\mathbf{B}$

RIGHT RULES

$$\frac{}{\Gamma \vdash \mathbf{0}, \Delta} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \mathbf{1}, \Delta} \qquad \frac{\Gamma \vdash E_1, E_2, \Delta}{\Gamma \vdash E_1 \times E_2, \Delta}$$

LEFT RULES

$$\frac{}{\mathbf{1} \vdash} \qquad \frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{R_i \vdash \Delta}{R_1 + R_2 \vdash \Delta}$$

$$\frac{R \vdash \Delta_1 \quad \vdash E, \Delta_2}{R \mapsto E \vdash \Delta_1, \Delta_2} \qquad \frac{R \vdash \Upsilon \quad \vdash E, \Delta}{R \Mapsto E \vdash \Upsilon, \Delta}$$

The basic inference system: $\mathbf{B}$

Right rules

$$\frac{}{\Gamma \vdash \mathbf{0}, \Delta} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \mathbf{1}, \Delta} \qquad \frac{\Gamma \vdash E_1, E_2, \Delta}{\Gamma \vdash E_1 \times E_2, \Delta}$$

Left rules

$$\frac{}{\mathbf{1} \vdash} \qquad \frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{R_i \vdash \Delta}{R_1 + R_2 \vdash \Delta}$$

$$\frac{R \vdash \Delta_1 \quad \vdash E, \Delta_2}{R \mapsto E \vdash \Delta_1, \Delta_2} \qquad \frac{R \vdash \Upsilon \quad \vdash E, \Delta}{R \Mapsto E \vdash \Upsilon, \Delta}$$

$$\frac{R \vdash \Delta}{\vdash R}\ \mathit{decide},\ R \in \mathcal{R}$$

The basic inference system: $\mathbf{B}$

RIGHT RULES

$$\frac{}{\Gamma \vdash \mathbf{0}, \Delta} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \mathbf{1}, \Delta} \qquad \frac{\Gamma \vdash E_1, E_2, \Delta}{\Gamma \vdash E_1 \times E_2, \Delta}$$

LEFT RULES

$$\frac{}{\mathbf{1} \vdash} \qquad \frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{R_i \vdash \Delta}{R_1 + R_2 \vdash \Delta}$$

$$\frac{R \vdash \Delta_1 \quad \vdash E, \Delta_2}{R \mapsto E \vdash \Delta_1, \Delta_2} \qquad \frac{R \vdash \Upsilon \quad \vdash E, \Delta}{R \Mapsto E \vdash \Upsilon, \Delta}$$

$$\frac{R \vdash \Delta}{\vdash R} \; decide, \; R \in \mathcal{R}$$

$$\frac{\vdash \bar{A}, \Delta}{A \vdash \Delta} \; debit_1, \; \text{if } \delta(A) = +1 \qquad \frac{\vdash \bar{S}, \Upsilon}{S \vdash \Upsilon} \; debit_2, \; \text{if } \delta(S) = +2$$

The basic inference system: $\mathbf{B}$

RIGHT RULES

$$\frac{}{\Gamma \vdash \mathbf{0}, \Delta} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \mathbf{1}, \Delta} \qquad \frac{\Gamma \vdash E_1, E_2, \Delta}{\Gamma \vdash E_1 \times E_2, \Delta}$$

LEFT RULES

$$\frac{}{\mathbf{1} \vdash} \qquad \frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{R_i \vdash \Delta}{R_1 + R_2 \vdash \Delta}$$

$$\frac{R \vdash \Delta_1 \quad \vdash E, \Delta_2}{R \mapsto E \vdash \Delta_1, \Delta_2} \qquad \frac{R \vdash \Upsilon \quad \vdash E, \Delta}{R \Mapsto E \vdash \Upsilon, \Delta}$$

$$\frac{R \vdash \Delta}{\vdash R} \; \mathit{decide}, \;\; R \in \mathcal{R}$$

$$\frac{\vdash \bar{A}, \Delta}{A \vdash \Delta} \; \mathit{debit}_1, \;\text{ if } \delta(A) = +1 \qquad \frac{\vdash \bar{\boldsymbol{S}}, \Upsilon}{\boldsymbol{S} \vdash \Upsilon} \; \mathit{debit}_2, \;\text{ if } \delta(\boldsymbol{S}) = +2$$

IDENTITY RULES

$$\frac{}{E \vdash E} \; \mathit{init} \qquad \frac{}{\vdash \bar{A}, A} \; \mathit{iou}$$

STRUCTURAL RULES

$$\frac{\Gamma \vdash \Delta, \boldsymbol{S}, \boldsymbol{S}}{\Gamma \vdash \Delta, \boldsymbol{S}} \; \text{contract} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \boldsymbol{S}} \; \text{weaken}$$

Some meta-theory of $\mathbf{B}$

Proposition

The $\mathbf{B}$ *proof system is still complete if the init rule is restricted to atomic expressions, i.e.,* $A \vdash A$ *instead of* $E \vdash E$*.*

Proposition

The following inference rule is admissible in $\mathbf{B}$*.*

$$\frac{\Gamma \vdash \Delta_1, E \qquad E \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}\ \mathit{clip}$$

Proposition

If $\delta(A) = +1$*, the following* dclip_1 *rule is admissible.*

$$\frac{\vdash \Delta_1, A \qquad \vdash \Delta_2, \overline{A}}{\vdash \Delta_1, \Delta_2}\ \mathit{dclip}_1$$

If $\delta(\boldsymbol{S}) = +2$*, the following* dclip_2 *rule is admissible.*

$$\frac{\vdash \Delta, \boldsymbol{S} \qquad \vdash \Upsilon, \overline{\boldsymbol{S}}}{\vdash \Delta, \Upsilon}\ \mathit{dclip}_2$$

Some meta theory of $\mathbf{B}$ (cont)

Proposition

If $\vdash \Delta$ *has a* $\mathbf{B}$*-proof, it has a* $\mathbf{B}$*-proof without the debit*$_1$ *and debit*$_2$ *rules.*

- If all debts are eventually paid, we can reorganize the proof so that the payments precede the formation of a debt.
- Of course, these proofs might vary a great deal in structure.

Proposition

The right rules are invertible. In particular, if E *is not atomic and the sequent* $\vdash E, \Delta$ *is provable, then there is a proof of this sequent in which the last inference rule is an introduction rule for* E*.*

Removing more non-determinism from $\mathbf{B}$

1. The right rules are invertible: done in any order and to exhaustion.
   - A $\mathbf{B}$-proof Ξ is *reduced* if every occurrence of the *decide* rule has a right-hand side containing only atomic expressions or debts.
   - Proposition: If the sequent $\vdash \Delta$ has a $\mathbf{B}$-proof, it has a reduced proof.
2. There are two ways to prove $A \vdash A$ when $\delta(A) = +1$: *init* or a combination of debit_1 and *iou*. This has a simple resolution.
3. Major issue:
   The structural rules seem all wrong from the proof search perspective.

Structural rules: a major revision is needed

$$\frac{\Gamma \vdash \Delta, \boldsymbol{S}, \boldsymbol{S}}{\Gamma \vdash \Delta, \boldsymbol{S}} \text{ contract} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \boldsymbol{S}} \text{ weaken}$$

These can be applied almost anytime! We need a better treatment.

Structural rules: a major revision is needed

$$\frac{\Gamma \vdash \Delta, \boldsymbol{S}, \boldsymbol{S}}{\Gamma \vdash \Delta, \boldsymbol{S}}\ \text{contract} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \boldsymbol{S}}\ \text{weaken}$$

These can be applied almost anytime! We need a better treatment.

Consider again a multiplicative and an additive rule.

$$\frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta}$$

In the multiplicative rule, every side-expression occurrence in the conclusion (a member of $\Delta_1 \cup \Delta_2$) also occurs in a *unique* premise.

In an additive rule, every side-expression occurrence in the conclusion (a member of Δ) occurs in *every* premise.

Structural rules: a major revision is needed

$$\frac{\Gamma \vdash \Delta, S, S}{\Gamma \vdash \Delta, S}\ \text{contract} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, S}\ \text{weaken}$$

These can be applied almost anytime! We need a better treatment.

Consider again a multiplicative and an additive rule.

$$\frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta}$$

In the multiplicative rule, every side-expression occurrence in the conclusion (a member of $\Delta_1 \cup \Delta_2$) also occurs in a *unique* premise.

In an additive rule, every side-expression occurrence in the conclusion (a member of Δ) occurs in *every* premise.

New treatment: Classical realm atomic expressions are *treated additively*, even in multiplicative rules.

This new treatment of structural rules produces rules of the following form.

$$\frac{}{\mathbf{1} \vdash \Upsilon} \qquad \frac{R_1 \vdash \mathcal{A}_1, \Upsilon \quad R_2 \vdash \mathcal{A}_2, \Upsilon}{R_1 \times R_2 \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon}$$

Here, $\mathcal{A}_1$ and $\mathcal{A}_2$ have only linear realm atomic expressions or debts.

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

$$\frac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \textit{decide}, \ \ R \in \mathcal{R}$$

Three kinds of sequents: $\vdash \Delta \qquad \Downarrow R \vdash \mathcal{A}, \Upsilon \qquad \vdash E \Downarrow \mathcal{A}, \Upsilon$

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

$$\frac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \mathit{decide},\ \ R \in \mathcal{R}$$

$$\frac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \qquad \frac{\Downarrow R_1 \vdash \mathcal{A}_1, \Upsilon \quad \Downarrow R_2 \vdash \mathcal{A}_2, \Upsilon}{\Downarrow R_1 \times R_2 \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R_i \vdash \mathcal{A}, \Upsilon}{\Downarrow R_1 + R_2 \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\Downarrow R \vdash \mathcal{A}_1, \Upsilon \quad \vdash E \Downarrow \mathcal{A}_2, \Upsilon}{\Downarrow R \mapsto E \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R \vdash \Upsilon \quad \vdash E \Downarrow \mathcal{A}, \Upsilon}{\Downarrow R \Mapsto E \vdash \mathcal{A}, \Upsilon}$$

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

$$\frac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \textit{decide},\ \ R \in \mathcal{R}$$

$$\frac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \qquad \frac{\Downarrow R_1 \vdash \mathcal{A}_1, \Upsilon \quad \Downarrow R_2 \vdash \mathcal{A}_2, \Upsilon}{\Downarrow R_1 \times R_2 \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R_i \vdash \mathcal{A}, \Upsilon}{\Downarrow R_1 + R_2 \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\Downarrow R \vdash \mathcal{A}_1, \Upsilon \quad \vdash E \Downarrow \mathcal{A}_2, \Upsilon}{\Downarrow R \mapsto E \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R \vdash \Upsilon \quad \vdash E \Downarrow \mathcal{A}, \Upsilon}{\Downarrow R \Mapsto E \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\vdash \overline{A}, \mathcal{A}, \Upsilon}{\Downarrow A \vdash \mathcal{A}, \Upsilon}\ \textit{debit}_1,\ \text{if } \delta(A) = +1 \qquad \frac{\vdash \bar{A}, \Upsilon}{\Downarrow A \vdash \Upsilon}\ \textit{debit}_2,\ \text{if } \delta(A) = +2$$

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

$$\frac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \mathit{decide},\ \ R \in \mathcal{R}$$

$$\frac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \qquad \frac{\Downarrow R_1 \vdash \mathcal{A}_1, \Upsilon \quad \Downarrow R_2 \vdash \mathcal{A}_2, \Upsilon}{\Downarrow R_1 \times R_2 \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R_i \vdash \mathcal{A}, \Upsilon}{\Downarrow R_1 + R_2 \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\Downarrow R \vdash \mathcal{A}_1, \Upsilon \quad \vdash E \Downarrow \mathcal{A}_2, \Upsilon}{\Downarrow R \mapsto E \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R \vdash \Upsilon \quad \vdash E \Downarrow \mathcal{A}, \Upsilon}{\Downarrow R \Mapsto E \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\vdash \overline{A}, \mathcal{A}, \Upsilon}{\Downarrow A \vdash \mathcal{A}, \Upsilon}\ \mathit{debit}_1,\ \text{if } \delta(A) = +1 \qquad \frac{\vdash \bar{A}, \Upsilon}{\Downarrow A \vdash \Upsilon}\ \mathit{debit}_2,\ \text{if } \delta(A) = +2$$

$$\frac{\delta(A) < 0}{\Downarrow A \vdash A, \Upsilon}\ \mathit{initL} \qquad \frac{\delta(A) > 0}{\vdash A \Downarrow \overline{A}, \Upsilon}\ \mathit{initR} \qquad \frac{\vdash E, \mathcal{A}, \Upsilon}{\vdash E \Downarrow \mathcal{A}, \Upsilon}\ \mathit{release}\dagger \qquad \frac{\delta(A) > 0}{\vdash \overline{A}, A, \Upsilon}\ \mathit{iou}$$

The proviso † for *release*: E is either not atomic or it is atomic and $\delta(E) < 0$.

Synthetic inference rules in $\mathbf{F}$

$\vdash \mathcal{A}, \Upsilon$ is a *border sequent*: only of atomic expressions and debts.

A *synthetic rule* is built from right phases above a left phase: their premises and conclusions are border sequents.

$$
\begin{array}{cl}
\underline{\vdash \mathcal{A}', \Upsilon'} & \\
\vdots & \qquad \Leftarrow \text{right phase} \\
\overline{\underline{\vdash \Delta, \Upsilon}}\ \dagger & \\
\cdots \vdots \cdots \qquad \vdots \qquad \cdots \vdots \cdots & \qquad \Leftarrow \text{left phase} \\
\hline
\dfrac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \textit{decide} &
\end{array}
$$

$\dagger$ is either *release*, $\textit{debit}_1$, or $\textit{debit}_2$.

The *right phase* is invertible and additive.

The *left phase* is not invertible and multiplicative.

Different levels of adequacy when encoding proof systems

$\mathbf{F}$ presents an *assembly language* for inference. We want to *compile* inference rules into $\mathbf{F}$ and preserve the proof search semantics.

Three levels of adequacy of encodings are natural to identify.

1. *Relative completeness*: a formula has a proof in one system if it has a proof in the other system.
2. *Full completeness of proofs*: the complete proofs in one system naturally correspond to proofs in the other system.
3. *Full completeness of inference rules*: every inference rule is in one-to-one correspondence with those in the other system.

All encodings in this talk are at this highest level of adequacy:

> *A set of rules $\mathcal{R}$ encodes a proof system* **P** *means that a synthetic inference rules in $\mathbf{F}$ for $R \in \mathcal{R}$ corresponds to an inference rule in the* **P**, *and vice versa.*

Encoding sequents of formulas

Two-sided sequents are of the form

$$B_1, \ldots, B_n \vdash C_1, \ldots, C_m$$

which we encode as the expression

$$\lfloor B_1 \rfloor \times \cdots \times \lfloor B_n \rfloor \times \lceil C_1 \rceil \times \cdots \times \lceil C_m \rceil$$

or, equivalently, by the multiset

$$\lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil.$$

In *classical logic*, formulas on the left and right are subject to weakening and contraction: thus, $\delta(\lfloor \cdot \rfloor) = \pm 2$ and $\delta(\lceil \cdot \rceil) = \pm 2$.

In *intuitionistic logic*, only the formulas on the left are subject to weakening and contraction: thus, $\delta(\lfloor \cdot \rfloor) = \pm 2$ and $\delta(\lceil \cdot \rceil) = \pm 1$.

Rules for classical and intuitionistic logic $\mathcal{R}_1$

$$
\begin{array}{lrcl}
(\supset L) & \lfloor A \supset B \rfloor & \mapsto & \lceil A \rceil \Rrightarrow \lfloor B \rfloor \\
(\supset R) & \lceil A \supset B \rceil & \mapsto & \lfloor A \rfloor \times \lceil B \rceil \\
(\wedge L) & \lfloor A \wedge B \rfloor & \mapsto & \lfloor A \rfloor \\
(\wedge L) & \lfloor A \wedge B \rfloor & \mapsto & \lfloor B \rfloor \\
(\wedge R) & \lceil A \wedge B \rceil & \mapsto & \lceil A \rceil + \lceil B \rceil \\
(\vee L) & \lfloor A \vee B \rfloor & \mapsto & \lfloor A \rfloor + \lfloor B \rfloor \\
(\vee R) & \lceil A \vee B \rceil & \mapsto & \lceil A \rceil \\
(\vee R) & \lceil A \vee B \rceil & \mapsto & \lceil B \rceil \\
(\perp L) & \lfloor \perp \rfloor & \mapsto & \mathbf{0} \\
(\top R) & \lceil \top \rceil & \mapsto & \mathbf{0} \\
(Id_1) & \lfloor C \rfloor \times \lceil C \rceil & & \\
(Id_2) & \mathbf{1} & \mapsto & \lceil C \rceil \Rrightarrow \lfloor C \rfloor
\end{array}
$$

Note that the two $(\wedge L)$ rules can be written as one R-formula.

$$(\lfloor A \wedge B \rfloor \mapsto \lfloor A \rfloor) + (\lfloor A \wedge B \rfloor \mapsto \lfloor B \rfloor)$$

Choosing the correct bias assignment for sequent calculi

Using the bias assignment that returns only negative numbers, then we get *sequent calculi* similar to Gentzen's $\mathbf{LK}$ and $\mathbf{LJ}$.

- If $\delta(\lfloor \cdot \rfloor) = -2$ and $\delta(\lceil \cdot \rceil) = -1$, then deciding on $(\supset L)$ yields

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow \lfloor A \supset B \rfloor \vdash \lfloor A \supset B \rfloor, \Upsilon} \quad \vdash \lceil A \rceil, \lfloor A \supset B \rfloor, \Upsilon}{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \vdash \lceil A \rceil, \lfloor A \supset B \rfloor, \Upsilon} \qquad \vdash \lfloor B \rfloor, \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \Mapsto \lfloor B \rfloor \vdash \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}}{\vdash \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}$$

which encodes (assuming that $\mathcal{A}$ is $\{\lceil C \rceil\}$).

$$\dfrac{A \supset B, \Gamma \vdash A \qquad A \supset B, B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}$$

Choosing the correct bias assignment for sequent calculi

Using the bias assignment that returns only negative numbers, then we get *sequent calculi* similar to Gentzen's $\mathbf{LK}$ and $\mathbf{LJ}$.

- If $\delta(\lfloor\cdot\rfloor) = -2$ and $\delta(\lceil\cdot\rceil) = -1$, then deciding on $(\supset L)$ yields

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow\lfloor A\supset B\rfloor \vdash \lfloor A\supset B\rfloor, \Upsilon} \quad \vdash \lceil A\rceil, \lfloor A\supset B\rfloor, \Upsilon}{\Downarrow\lfloor A\supset B\rfloor \mapsto \lceil A\rceil \vdash \lceil A\rceil, \lfloor A\supset B\rfloor, \Upsilon} \qquad \vdash \lfloor B\rfloor, \lfloor A\supset B\rfloor, \mathcal{A}, \Upsilon}{\Downarrow\lfloor A\supset B\rfloor \mapsto \lceil A\rceil \Mapsto \lfloor B\rfloor \vdash \lfloor A\supset B\rfloor, \mathcal{A}, \Upsilon}}{\vdash \lfloor A\supset B\rfloor, \mathcal{A}, \Upsilon}$$

  which encodes (assuming that $\mathcal{A}$ is $\{\lceil C\rceil\}$).

$$\dfrac{A\supset B, \Gamma\vdash A \qquad A\supset B, B, \Gamma\vdash C}{A\supset B, \Gamma\vdash C}$$

- If we set $\delta(\lfloor\cdot\rfloor) = -2$ and $\delta(\lceil\cdot\rceil) = -2$, then we have

$$\dfrac{A\supset B, \Gamma\vdash A, \Psi \qquad A\supset B, B, \Gamma\vdash \Psi}{A\supset B, \Gamma\vdash \Psi}\ .$$

The two identity rules: initial and cut

The (Id_1) and (Id_2) rules have special roles.

$$\dfrac{\dfrac{}{\Downarrow \lfloor C \rfloor \vdash \lfloor C \rfloor, \Upsilon}\ \mathit{initL} \quad \dfrac{}{\Downarrow \lceil C \rceil \vdash \lceil C \rceil, \Upsilon}\ \mathit{initL}}{\dfrac{\Downarrow \lfloor C \rfloor \times \lceil C \rceil \vdash \lfloor C \rfloor, \lceil C \rceil, \Upsilon}{\vdash \lfloor C \rfloor, \lceil C \rceil, \Upsilon}\ \mathit{decide}\ Id_1}$$

$$\dfrac{\dfrac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \quad \vdash \lceil C \rceil, \Upsilon \quad \vdash \lfloor C \rfloor, \mathcal{A}, \Upsilon}{\dfrac{\Downarrow \mathbf{1} \Rrightarrow \lceil C \rceil \mapsto \lfloor C \rfloor \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \mathit{decide}\ Id_2}$$

These justify the synthetic rules

$$\dfrac{}{\vdash \lfloor C \rfloor, \lceil C \rceil, \Upsilon} \qquad \dfrac{\vdash \lceil C \rceil, \Upsilon \quad \vdash \lfloor C \rfloor, \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}$$

In the intuitionistic setting, the variable Υ contains only $\lfloor \cdot \rfloor$ atomic expressions while $\mathcal{A}$ contains only a single expression, which is of the form $\lceil \cdot \rceil$.

(Id_1) and (Id_2) encode the *init* and *cut* rules of sequent calculus.

Negative bias encodes sequent calculi

Proposition

Let $\delta(\lfloor \cdot \rfloor) = -2$.

1. *If* $\delta(\lceil \cdot \rceil) = -1$ *then* $\mathcal{R}_1$ *encodes (essentially) Gentzen's* $\mathbf{LJ}$ *proof system.*
2. *If* $\delta(\lceil \cdot \rceil) = -2$, *then* $\mathcal{R}_1$ *encodes (essentially) Gentzen's* $\mathbf{LK}$ *proof system.*

Natural deduction for intuitionistic logic

$$\frac{\Gamma \vdash A \supset B \downarrow \quad \Gamma \vdash A \uparrow}{\Gamma \vdash B \downarrow}\ [\supset E] \qquad \frac{\Gamma, A \vdash B \uparrow}{\Gamma \vdash A \supset B \uparrow}\ [\supset I]$$

$$\frac{\Gamma \vdash A \wedge B \downarrow}{\Gamma \vdash A \downarrow}\ [\wedge E] \qquad \frac{\Gamma \vdash A \wedge B \downarrow}{\Gamma \vdash B \downarrow}\ [\wedge E] \qquad \frac{\Gamma \vdash A \uparrow \quad \Gamma \vdash B \uparrow}{\Gamma \vdash A \wedge B \uparrow}\ [\wedge I]$$

$$\frac{}{\Gamma \vdash \top \uparrow}\ [\top I] \qquad \frac{\Gamma \vdash \bot \downarrow}{\Gamma \vdash C \uparrow}\ [\bot E]$$

$$\frac{}{\Gamma, A \vdash A \downarrow}\ [\mathrm{I}] \qquad \frac{\Gamma \vdash A \downarrow}{\Gamma \vdash A \uparrow}\ [\mathrm{M}] \qquad \frac{\Gamma \vdash A \uparrow}{\Gamma \vdash A \downarrow}\ [\mathrm{S}]$$

Natural deduction in the style of Sieg and Byrnes (*Studia Logica*, 1998).

A proof is *normal* if it does not contain the switch rule [*S*].

$\mathcal{R}_1$ can also capture natural deduction

Let $\delta(\lfloor \cdot \rfloor) = +2$ and $\delta(\lceil \cdot \rceil) = -1$.

The $\uparrow$ and $\downarrow$ judgments are encoded as follows.

- $\Gamma \vdash C \uparrow$ is encoded using $\vdash \lfloor \Gamma \rfloor, \lceil C \rceil$.
- $\Gamma \vdash C \downarrow$ is encode using $\vdash \lfloor \Gamma \rfloor, \overline{\lfloor C \rfloor}$.

$\mathcal{R}_1$ can also capture natural deduction

Let $\delta(\lfloor \cdot \rfloor) = +2$ and $\delta(\lceil \cdot \rceil) = -1$.

The $\uparrow$ and $\downarrow$ judgments are encoded as follows.

- $\Gamma \vdash C \uparrow$ is encoded using $\vdash \lfloor \Gamma \rfloor, \lceil C \rceil$.
- $\Gamma \vdash C \downarrow$ is encode using $\vdash \lfloor \Gamma \rfloor, \overline{\lfloor C \rfloor}$.

Using *decide* on the R-formula $(\supset L)$ yields

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \overline{\lfloor A \supset B \rfloor}, \Upsilon}{\Downarrow \lfloor A \supset B \rfloor \vdash \Upsilon}\ \mathit{debit}_2 \quad \dfrac{\vdash \lceil A \rceil, \Upsilon}{\vdash \lceil A \rceil \Downarrow \Upsilon}\ \mathit{release}}{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \vdash \Upsilon} \qquad \dfrac{}{\vdash \lfloor B \rfloor \Downarrow \overline{\lfloor B \rfloor}, \Upsilon}\ \mathit{initR}}{\Downarrow (\lfloor A \supset B \rfloor \mapsto \lceil A \rceil) \Rrightarrow \lfloor B \rfloor \vdash \overline{\lfloor B \rfloor}, \Upsilon}\ \mathit{decide}}{\vdash \overline{\lfloor B \rfloor}, \Upsilon}$$

$\mathcal{R}_1$ can also capture natural deduction

Let $\delta(\lfloor \cdot \rfloor) = +2$ and $\delta(\lceil \cdot \rceil) = -1$.

The $\uparrow$ and $\downarrow$ judgments are encoded as follows.

- $\Gamma \vdash C \uparrow$ is encoded using $\vdash \lfloor \Gamma \rfloor, \lceil C \rceil$.
- $\Gamma \vdash C \downarrow$ is encode using $\vdash \lfloor \Gamma \rfloor, \overline{\lfloor C \rfloor}$.

Using *decide* on the R-formula $(\supset L)$ yields

$$
\dfrac{\dfrac{\dfrac{\vdash \overline{\lfloor A \supset B \rfloor}, \Upsilon}{\Downarrow \lfloor A \supset B \rfloor \vdash \Upsilon}\ \mathit{debit}_2 \quad \dfrac{\vdash \lceil A \rceil, \Upsilon}{\vdash \lceil A \rceil \Downarrow \Upsilon}\ \mathit{release}}{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \vdash \Upsilon} \quad \dfrac{}{\vdash \lfloor B \rfloor \Downarrow \overline{\lfloor B \rfloor}, \Upsilon}\ \mathit{initR}}{\dfrac{\Downarrow(\lfloor A \supset B \rfloor \mapsto \lceil A \rceil) \Mapsto \lfloor B \rfloor \vdash \overline{\lfloor B \rfloor}, \Upsilon}{\vdash \overline{\lfloor B \rfloor}, \Upsilon}\ \mathit{decide}}
$$

This yields the synthetic rule, which encodes the $[\supset E]$ inference rule.

$$
\dfrac{\vdash \overline{\lfloor A \supset B \rfloor}, \Upsilon \quad \vdash \lceil A \rceil, \Upsilon}{\vdash \overline{\lfloor B \rfloor}, \Upsilon}
$$

The [M] and [S] rules

Deciding on (Id_1) and (Id_2), respectively, yields

$$\dfrac{\dfrac{\dfrac{\vdash \overline{\lfloor B \rfloor}, \Upsilon}{\Downarrow \lfloor B \rfloor \vdash \Upsilon}\ \textit{debit}_2 \qquad \dfrac{}{\Downarrow \lceil B \rceil \vdash \lceil B \rceil, \Upsilon}\ \textit{initL}}{\Downarrow \lfloor B \rfloor \times \lceil B \rceil \vdash \lceil B \rceil, \Upsilon}}{\vdash \lceil B \rceil, \Upsilon}\ \textit{decide}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \quad \dfrac{\vdash \lceil B \rceil, \Upsilon}{\vdash \lceil B \rceil \Downarrow \Upsilon}\ \textit{release}}{\Downarrow \mathbf{1} \mapsto \lceil B \rceil \vdash \Upsilon} \qquad \dfrac{}{\vdash \lfloor B \rfloor \Downarrow \overline{\lfloor B \rfloor}, \Upsilon}\ \textit{initL}}{\Downarrow \mathbf{1} \mapsto \lceil B \rceil \Mapsto \lfloor B \rfloor \vdash \overline{\lfloor B \rfloor}, \Upsilon}}{\vdash \overline{\lfloor B \rfloor}, \Upsilon}\ \textit{decide}$$

The $[M]$ and $[S]$ rules

Deciding on (Id_1) and (Id_2), respectively, yields

$$\dfrac{\dfrac{\dfrac{\vdash \overline{\lfloor B \rfloor}, \boldsymbol{\Upsilon}}{\Downarrow \lfloor B \rfloor \vdash \boldsymbol{\Upsilon}}\ \mathit{debit}_2 \qquad \dfrac{}{\Downarrow \lceil B \rceil \vdash \lceil B \rceil, \boldsymbol{\Upsilon}}\ \mathit{initL}}{\Downarrow \lfloor B \rfloor \times \lceil B \rceil \vdash \lceil B \rceil, \boldsymbol{\Upsilon}}}{\vdash \lceil B \rceil, \boldsymbol{\Upsilon}}\ \mathit{decide}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow \mathbf{1} \vdash \boldsymbol{\Upsilon}} \qquad \dfrac{\vdash \lceil B \rceil, \boldsymbol{\Upsilon}}{\vdash \lceil B \rceil \Downarrow \boldsymbol{\Upsilon}}\ \mathit{release}}{\Downarrow \mathbf{1} \mapsto \lceil B \rceil \vdash \boldsymbol{\Upsilon}} \qquad \dfrac{}{\vdash \lfloor B \rfloor \Downarrow \overline{\lfloor B \rfloor}, \boldsymbol{\Upsilon}}\ \mathit{initL}}{\Downarrow \mathbf{1} \mapsto \lceil B \rceil \Mapsto \lfloor B \rfloor \vdash \overline{\lfloor B \rfloor}, \boldsymbol{\Upsilon}}}{\vdash \overline{\lfloor B \rfloor}, \boldsymbol{\Upsilon}}\ \mathit{decide}$$

and these yield the two synthetic rules (encoding $[M]$ and $[S]$)

$$\dfrac{\vdash \overline{\lfloor B \rfloor}, \boldsymbol{\Upsilon}}{\vdash \lceil B \rceil, \boldsymbol{\Upsilon}} \quad \text{and} \quad \dfrac{\vdash \lceil B \rceil, \boldsymbol{\Upsilon}}{\vdash \overline{\lfloor B \rfloor}, \boldsymbol{\Upsilon}}\ .$$

The R-expression (Id_2) corresponds to cut in sequent calculus and to the switch $[S]$ in natural deduction.

Encoding natural deduction

Proposition

Assume that $\delta(\lceil \cdot \rceil) = -1$ *and* $\delta(\lfloor \cdot \rfloor) = +2$. *Then*

- $\Gamma \vdash C \uparrow$ *if and only if* $\vdash \lfloor \Gamma \rfloor, \lceil C \rceil$ *is provable using* $\mathcal{R}_1$, *and*
- $\Gamma \vdash C \downarrow$ *if and only if* $\vdash \lfloor \Gamma \rfloor, \overline{\lfloor C \rfloor}$ *is provable using* $\mathcal{R}_1$.

Normal proofs are captured by removing (Id_2) *from consideration.*

Encoding natural deduction

Proposition

Assume that $\delta(\lceil \cdot \rceil) = -1$ *and* $\delta(\lfloor \cdot \rfloor) = +2$. *Then*

- $\Gamma \vdash C \uparrow$ *if and only if* $\vdash \lfloor \Gamma \rfloor, \lceil C \rceil$ *is provable using* $\mathcal{R}_1$, *and*
- $\Gamma \vdash C \downarrow$ *if and only if* $\vdash \lfloor \Gamma \rfloor, \overline{\lfloor C \rfloor}$ *is provable using* $\mathcal{R}_1$.

Normal proofs are captured by removing (Id_2) *from consideration.*

The following rules can also be captured.

$$\frac{\Gamma \vdash A \vee B \downarrow \quad \Gamma, A \vdash C \uparrow\!(\downarrow) \quad \Gamma, B \vdash C \uparrow\!(\downarrow)}{\Gamma \vdash C \uparrow\!(\downarrow)} \ [\vee E]$$

$$\frac{\Gamma \vdash A_i \uparrow}{\Gamma \vdash A_1 \vee A_2 \uparrow} \ [\vee I]$$

Other proof systems and logics

The paper in the proceedings discusses additional proof systems:

- Generalized elimination rules [Schroeder-Heister, 1984], [von Plato, 2001]
- Free deduction for classical logic [Parigot 1992]
- Sequent calculus for linear logic: uses four tags, not just the two used here.
- Quantificational logic

Future work: by accommodating a feature similar to *sub-exponentials* from linear logic, several more proof systems can be accommodated.

- Multi-conclusion proof systems for intuitionistic logic [Maehara, 1954]
- **G1m**, **LJQ***, etc [Nigam, Pimentel and Reis, 2011].

First-order quantification

Early logical frameworks (λProlog, LF) were notable for their treatment of quantification via *binder mobility*: term-level bindings move to formula-level bindings (quantifier) to proof-level bindings (eigenvariables).

- Add quantified expressions and rules: $\boldsymbol{Q}\, x.(E\, x)$ and $\boldsymbol{Q}\, x.(R\, x)$.
- Sequents are enriched: Σ binds over sequents: $\Sigma : \Gamma \vdash \Delta$ and $\Sigma : \Downarrow \Gamma \vdash \Delta$.
- Add two rules to $\mathbf{B}$ (in the first rule, $x \notin \Sigma$).

$$\frac{\Sigma, x : \Gamma \vdash E\, x, \Delta}{\Sigma : \Gamma \vdash \boldsymbol{Q}\, x.E\, x, \Delta} \qquad \frac{\Sigma : \Gamma, R\, t \vdash \Delta \quad t \text{ is a } \Sigma\text{-term}}{\Sigma : \Gamma, \boldsymbol{Q}\, x.R\, x \vdash \Delta}$$

First-order quantification

Early logical frameworks (λProlog, LF) were notable for their treatment of quantification via *binder mobility*: term-level bindings move to formula-level bindings (quantifier) to proof-level bindings (eigenvariables).

- Add quantified expressions and rules: $\boldsymbol{Q}\, x.(E\, x)$ and $\boldsymbol{Q}\, x.(R\, x)$.
- Sequents are enriched: Σ binds over sequents: $\Sigma : \Gamma \vdash \Delta$ and $\Sigma :\Downarrow \Gamma \vdash \Delta$.
- Add two rules to $\mathbf{B}$ (in the first rule, $x \notin \Sigma$).

$$\frac{\Sigma, x : \Gamma \vdash E\, x, \Delta}{\Sigma : \Gamma \vdash \boldsymbol{Q}\, x.E\, x, \Delta} \qquad \frac{\Sigma : \Gamma, R\, t \vdash \Delta \quad t \text{ is a } \Sigma\text{-term}}{\Sigma : \Gamma, \boldsymbol{Q}\, x.R\, x \vdash \Delta}$$

The rule $(\supset L)$ in $\mathcal{R}_1$ can be written more explicitly as

$$\boldsymbol{Q}\, A.\boldsymbol{Q}\, B.\lfloor A \supset B \rfloor \mapsto \lceil A \rceil \Mapsto \lfloor B \rfloor$$

We can now add the following to $\mathcal{R}_1$.

$$\begin{array}{rlcl} \boldsymbol{Q}\, B.\boldsymbol{Q}\, t. & \lfloor \forall x.B\, x \rfloor & \mapsto & \lfloor B\, t \rfloor \\ \boldsymbol{Q}\, B. & \lceil \forall x.B\, x \rceil & \mapsto & \boldsymbol{Q}\, x.\lceil B\, x \rceil \\ \boldsymbol{Q}\, B. & \lfloor \exists x.B\, x \rfloor & \mapsto & \boldsymbol{Q}\, x.\lfloor B\, x \rfloor \\ \boldsymbol{Q}\, B.\boldsymbol{Q}\, t. & \lceil \exists x.B\, x \rceil & \mapsto & \lceil B\, t \rceil \end{array}$$

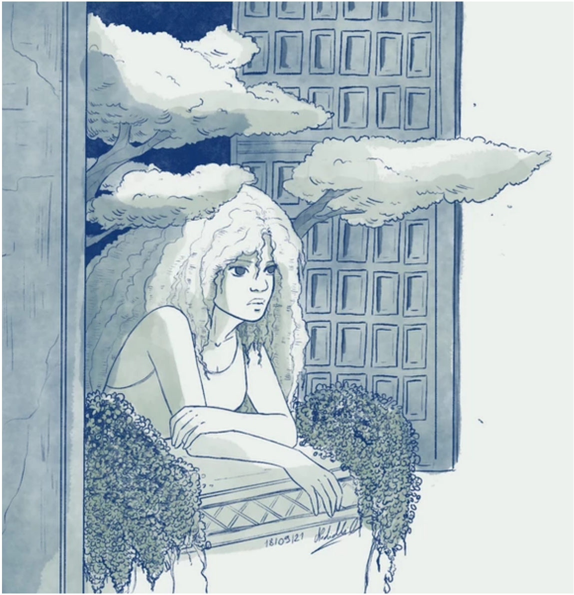
Related work

- Logical frameworks in the 1980s and 1990s: Framework based on intuitionistic logic and typed λ-terms.
  - E.g., λProlog, LF.
- Frameworks based on linear logic (with subexponentials)
  - M, Pimentel, Nigam, and Reis et al. [1996-2014] have considered many proof systems and logic.
  - Sufficient (and decidable) conditions that ensure that a sequent calculus for a first-order logic has the cut-elimination property.
  - Various implementations have been developed.
- This paper grew out of the desire to supplant linear logic with something more basic and pre-logical.
- There are related approaches using algebraic and model-theoretic semantics as frameworks: e.g., A. Avron and I. Lev [IJCAR 2001].

Conclusion

$\mathbf{PSF}$ is a framework for specifying proof systems.

- It separates the semantics of inference rules into two parts:
  - the rule, i.e., $\lfloor A \supset B \rfloor \mapsto \lceil A \rceil \Rrightarrow \lfloor B \rfloor$
  - the bias assignment, i.e., values for $\delta(\lceil \cdot \rceil)$ and $\delta(\lfloor \cdot \rfloor)$.
- Inference rules in, say, $\mathbf{NJ}$ and $\mathbf{LK}$ are identified as synthetic inference rules containing two phases of $\mathbf{PSF}$ rewriting steps.
- Many features shared with linear logic appear naturally.
  - Inference rules are characterized as multiplicative and additive.
  - The tagged formulas are either deletable or permanent.
  - Importance of contraction and weakening.
- Centrality of don't-know-nondeterminism and don't-care-nondeterminism
- In $\mathbf{PSF}$, cut-elimination is used to reason about the framework instead of specifying computation *à la* Curry-Howard correspondence.



Questions?

Art by Nadia Miller