A positive perspective on term representation: work in progress

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Outline

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- What to do with terms? Equality, substitution, evaluation, etc.

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  - Focused proof system *LJF*:
    - ▷ Focusing: large-scale rules.
    - ▷ Polarization: flexibility on forms of proofs (terms).

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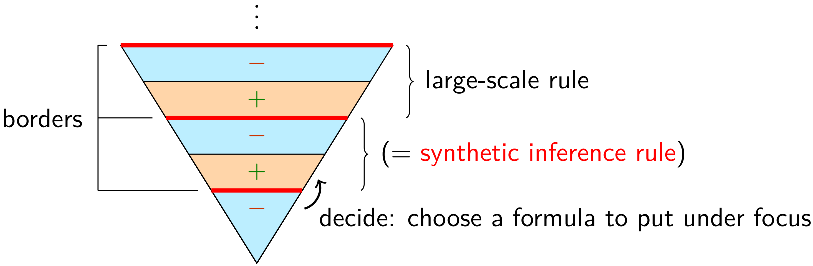
- Large-scale rules (not phases!): *synthetic inference rules*.

Two-phase structure, borders, and large-scale rules



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\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,ax \qquad \dfrac{}{\vdash B, B^\perp}\,ax}{\vdash A^\perp, B^\perp, A \otimes B}\,\otimes}{\vdash A^\perp, B^\perp \oplus (C^\perp \otimes D^\perp), A \otimes B}\,\oplus_1
\qquad
\dfrac{\dfrac{}{\vdash A^\perp, A}\,ax \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash C, C^\perp}\,ax \qquad \dfrac{}{\vdash D^\perp, D}\,ax}{\vdash C, D, C^\perp \otimes D^\perp}\,\otimes}{\vdash C, D, B^\perp \oplus (C^\perp \otimes D^\perp)}\,\oplus_2}{\vdash C \parr D, B^\perp \oplus (C^\perp \otimes D^\perp)}\,\parr}{\vdash A^\perp, B^\perp \oplus (C^\perp \otimes D^\perp), A \otimes (C \parr D)}\,\otimes
}{
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- A polarized formula (resp. theory) is a formula (resp. theory) together with an atomic bias assignment $\delta : \mathcal{A} \to \{+, -\}$.
- Different polarizations do not affect provability, but give different forms of proofs.
  - ▷ If a sequent is provable in *LJF* for some polarization, then it is provable for all such polarizations.

The *LJF* system with only implication

Decide, Release, and Store Rules

$$\frac{N, \Gamma \Downarrow N \vdash A}{N, \Gamma \vdash A}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow}{\Gamma \vdash P}\ D_r \qquad \frac{\Gamma \Uparrow P \vdash A}{\Gamma \Downarrow P \vdash A}\ R_l \qquad \frac{\Gamma \vdash N \Uparrow}{\Gamma \vdash N \Downarrow}\ R_r$$

$$\frac{\Gamma, C \Uparrow \Theta \vdash \Delta' \Uparrow \Delta}{\Gamma \Uparrow \Theta, C \vdash \Delta' \Uparrow \Delta}\ S_l \qquad \frac{\Gamma \Uparrow \Theta \vdash A}{\Gamma \Uparrow \Theta \vdash A \Uparrow}\ S_r$$

Initial Rules

$$\frac{A \text{ positive}}{A, \Gamma \vdash A \Downarrow}\ I_r \qquad \frac{A \text{ negative}}{\Gamma \Downarrow A \vdash A}\ I_l$$

Introduction Rules for Implication

$$\frac{\Gamma \vdash B \Downarrow \quad \Gamma \Downarrow B' \vdash A}{\Gamma \Downarrow B \supset B' \vdash A} \supset L \qquad \frac{\Gamma \Uparrow \Theta, B \vdash B' \Uparrow}{\Gamma \Uparrow \Theta \vdash B \supset B' \Uparrow} \supset R$$

Synthetic inference rules

Synthetic inference rule = large-scale rule = $\Downarrow$-phase + $\Uparrow$-phase

Definition

A *left synthetic inference rule* for B is an inference rule of the form

$$\frac{\Gamma_1 \vdash A_1 \quad \dots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A}\ B$$

justified by a derivation (in LJF) of the form

$$\begin{array}{c} \Gamma_1 \vdash A_1 \qquad \dots \qquad \Gamma_n \vdash A_n \\ \vdots\ \Uparrow \text{phase} \\ \\ \vdots\ \Downarrow \text{phase} \\ \dfrac{\Gamma \Downarrow B \vdash A}{\Gamma \vdash A}\ D_l \end{array}$$

Axioms as rules

Definition

Let $\mathcal{T}$ be a finite polarized theory of order 2 or less, We define $LJ\langle\mathcal{T}\rangle$ to be the extension of LJ with the left synthetic inference rules for $\mathcal{T}$. More precisely, for every left synthetic inference rule

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with $B \in \mathcal{T}$, the inference rule

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Theorem

$\mathcal{T}, \Gamma \vdash B$ provable in $LJ \Leftrightarrow \Gamma \vdash B$ provable in $LJ\langle\mathcal{T}\rangle$.

An example

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- ▷ a shortest proof of linear size

Annotating rules and proofs

Consider the inference rules in the previous example and annotate them.

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Consider the proofs of $a_0 \vdash a_4$.

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The term annotating the unique proof is

$$\begin{array}{l}
(E_4\ (E_3\ (E_2\ (E_1\ d_0)\ (E_1\ d_0)) \\
\qquad\qquad (E_2\ (E_1\ d_0)\ (E_1\ d_0))) \\
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We fix a theory $\mathcal{T} = \{\Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D\}$ and consider proofs of sequents of the form $\mathcal{T}, x_1 : D, \cdots, x_k : D \vdash t : D$

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When D is given the positive bias, we have the following synthetic inference rules:

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and the initial rule.

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and the initial rule.

Two formats for untyped λ-terms

Two different polarity assignments give **two different term structures**:

- D is negative:

x	`nvar x`	x
$\Phi\ t\ u$	`napp t u`	tu
$\Psi\ (\lambda x.t)$	`nabs (x\ t)`	$\lambda x.t$

$\rightarrow$ Top-down / tree-like structure

- D is positive:

x	`pvar x`	x
$\Phi\ x\ y\ (\lambda z.t)$	`papp x y (z\ t)`	$\mathit{name}\ z = xy\ \mathit{in}\ t$
$\Psi\ (\lambda x.t)\,(\lambda y.s)$	`pabs (x\ t) (y\ s)`	$\mathit{name}\ y = \lambda x.t\ \mathit{in}\ s$

$\rightarrow$ Bottom-up / DAG structure

Some examples for the positive-bias syntax

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name y = app x x in name z = app y y in z
```

- Arguments of `app` are all names

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name z = abs (x\ name y1 = app y y in y1) in z
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▷ Parallel naming

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name y1 = app x x and y2 = app y y in
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Cut-elimination for $LJ\langle\mathcal{T}\rangle$

The following theorem[1] states that cut is admissible for the extensions of LJ with polarized theories based on synthetic inference rules.

Theorem (Cut admissibility for $LJ\langle\mathcal{T}\rangle$)

Let $\mathcal{T}$ be a finite polarized theory of order 2 or less. Then the cut rule is admissible for the proof system $LJ\langle\mathcal{T}\rangle$.

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When we restrict to atomic cut formulas, the cut elimination procedure can be presented in a big-step style.

- Cuts are permuted with synthetic rules instead of *LJF* rules.

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Untyped λ-terms (substitution)

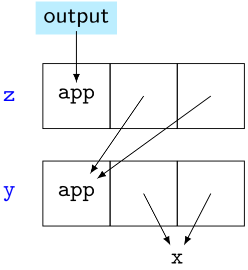
The cut-elimination procedure of *LJF* gives us the following definitions of substitutions.

```
type nsubst, psubst   tm -> (val -> tm) -> tm -> o.
```

```
nsubst T (x\ nvar x) T.
nsubst T (x\ nvar Y) (nvar Y).
nsubst T (x\ napp (R x) (S x)) (napp R' S') :-
   nsubst T R R', nsubst T S S'.
nsubst T (x\ nabs y\ R x y) (nabs y\ R' y) :-
   pi y\ nsubst T (x\ R x y) (R' y).
```

```
psubst (papp U V K) R (papp U V H) :- pi x\ psubst
    (K x) R (H x).
psubst (pabs S K)   R (pabs S H)   :- pi x\ psubst
    (K x) R (H x).
psubst (pvar U)     R (R U).
```

An example

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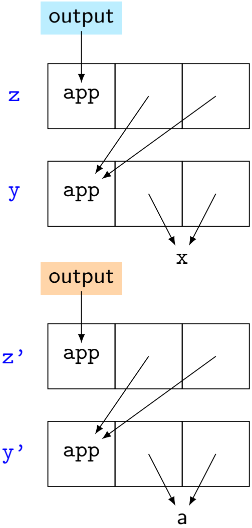


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```
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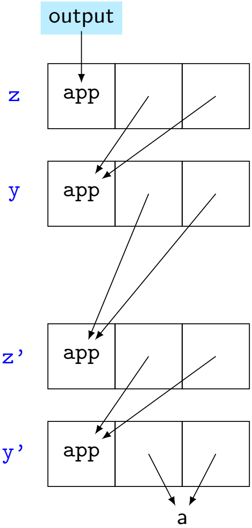


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Equality on terms

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When should two such expressions be considered the same?

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"White box" approach:

- ▷ Look at the actual syntax of proof expressions.
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"White box" approach:

- ▷ Look at the actual syntax of proof expressions.
  $\Rightarrow$ not working since we have two different sets of synthetic inference rules.

"Black box" approach:

- ▷ Describe *traces* by probing a term: exponential cost.
  $\hookrightarrow$ Bisimulation on graphical representations.

Graphical representations

The positive-bias syntax is closely related to some graphical representations.

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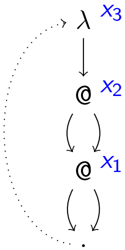


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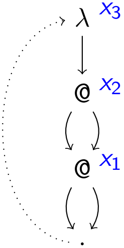


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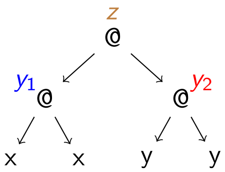
Bisimulation on graphs allows to check sharing equality in linear time[2].

[2] Andrea Condoluci, Beniamino Accattoli, and Claudio Sacerdoti Coen. **Sharing equality is linear.** *PPDP 2019*.

Graphical representations and parallel naming

Parallel naming can be captured by graphical representations:

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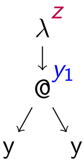


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- Proof-theoretic methods for checking term equality.