Advances in Linear Logic

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The foundational role played by logic in various areas of computer science, such as programming languages, databases, natural language understanding, and automated reasoning, is now a common place observation. When a major new advance is made in our understanding of logic, we can expect to see that advance ripple into these various areas of computer science. Such rippling is attested to by the large number of papers on linear logic that have appeared in both the computer science and matematical logic literatures during the ten years since the introduction of linear logic by J.-Y. Girard in 1987. The book under review is the refereed proceedings of the first international meeting on linear logic held at Cornell University, June 1993. (A second international meeting was held at Keio University in Tokyo, March 1996.) This collection of papers provides an excellent place to find a general introduction to linear logic, to gain an overview of the literature, and to learn about the recent advances and research directions.

This volume starts with the general introduction article by Girard titled *Linear logic: its syntax and semantics*. This article presents the proof theory of linear logic using both sequent calculus and proof nets and then develops some of its semantic models. This article is a gentle and readable introduction to linear logic.

The remaining 15 papers are combinations of overviews and research papers. They are grouped into the following five parts.

PART I. CATEGORIES AND SEMANTICS. Category theory and denotational semantics have often guided the development and rationale of new logics. This has been true and continues to be true for linear logic. In 1958, J. Lambek presented one of the precursors of linear logic by considering a Gentzen sequent calculus without any structural rules. He provides an overview of his early work on such a *substructural logic* in *Bilinear logic in algebra and linguistics* and discusses his motivations for it, which come from category theory and natural language. In *Questions and answers — A category arising in linear logic, complexity theory, and set theory*, A. Blass points out that a category used by V. de Paiva to provide a model of linear logic and a category used by P. Vojtáš to analyze cardinal characteristics of the continuum are the same, and he points out how that category can be related to game theory semantics and how it can be used to provide natural extensions to linear logic. The paper *Hypercoherences: a strongly stable model of linear logic* by T. Erhard is a slight revision of a paper with the same title appearing in **Mathematical Structures in Computer Science** 1993. In it, Erhard presents a semantics for linear logic using the concept of *strong stability*, a generalization of an earlier semantic treatment of sequentiality.

Part II. Complexity and Expressivity. Because linear logic sequent proofs treat formulas as resources, a wide variety of abstract machines can be modeled by encoding a computation's state with sequents and a computation's actions on state with inference rules. A lot of received recent attention has been given to the problem of determining the complexity of computations that can be supported by various subsets of linear logic. In *Deciding provability of linear logic formulas*, P. D. Lincoln surveys the known results about the decidability of various subsets of linear logic. One surprise: the fragment containing just the logical constants and connectives (no quantifiers and no non-logical symbols) is undecidable. In *The direct simulation of Minsky machines in linear logic*, M. I. Kanovich provides a direct and natural encoding of many-counter Minsky machines and uses this encoding to provide information about various decidability results. Given that certain subsets of linear logic correspond to natural complexity classes and that game theory can be used both to provide a semantics for linear logic and for characterizing some complexity classes (in the setting of stochastic interactions), P. D. Lincoln, J. Mitchell, and A. Scedrov in *Stochastic interaction and linear logic* provide an intuitive and direct semantics for the non-modal linear logic connectives in terms of interaction. Using a generalization of linear logic proposed by Girard called *Unified Logic*, C. Fouqueré and J. Vauzeilles present in *Inheritance with exceptions: an attempt at formalization with linear connective in Unified Logic* an encoding of taxonomic networks that allow exceptions.

Part III. Proof Theory. Classical and intuitionistic logics allow formulas to be freely reused and discarded in the process of building a proof. Linear logic allows such free use of formulas only if they are explicitly marked by either the "reuse" modal ! or its de Morgan dual ? (collectively called *exponentials*). Understanding the strength and subtle of these modal operators is a current research area. In *On the fine structure of the exponential rule*, S. Martini and A. Masini explore various subsets of linear logic by providing different restrictions on the use of the ! modal. In *LKQ and LKT: Sequent calculi for second order logic based upon dual linear decompositions of classical implications*, V. Danos, J. B. Joinet, and H. Schellinx develop various translations of classical and intuitionistic second-order logics into second-order linear logic using combinations of modal operators.

Part IV. Proof Nets. Linear logic has a simple sequent calculus presentation that makes some of its relationships to other logic clear. The development of a corresponding notion of natural deduction for linear logic, called *proof nets*, is a challenge and is one of the novelties of linear logic. The multiplicative fragment of linear logic has been given a clear and elegant proof net presentation by

Girard. Inspired by that representation of proof and of the notion of parallel computation that results from considering cut-elimination on them, Y. Lafont developed a simple model of computing that he calls *interaction nets* and which he overviews in the paper *From proof nets to interaction nets.* In *Subnets of Proof-nets in MLL^-* by G. Bellin and J. van de Wiele present some observations about the structure of subnets and use them to provide some new proofs of known results regarding proof nets. The one structural rule kept by linear logic is that of exchange, a rule that says that the order of formulas in a sequent is not important. This rule results in the fact that multiplicative conjunction and disjunction are commutative. In *Noncommutative proof nets*, V. M. Abrusci presents a graph theoretic approach to defining proof nets for a noncommutative variant of proof nets for multiplicative linear logic. The graph theoretic analysis of proof nets can be replaced by one using homology algebra, as is shown by F. Métayer in *Volume of multiplicative formulas and provability.*

PART V. GEOMETRY OF INTERACTION. The *geometry of interaction* has the goal of providing a new style of semantics for logic, going beyond the model-theory-versus-proofs used in classical logic and the constructive interpretation of proofs used in intuitionistic logic. Instead, a mathematics of interaction and of information interchange is sought. Proof nets have provided a starting point for this investigation. In *Proof-nets and the Hilbert space*, V. Danos and L. Regnier have used operators on Hilbert spaces to provide interpretations of λ-terms and proof nets. In *Geometry of interaction III: accommodating the additives*, J. Y. Girard extends his previous work on the geometry of interaction by showing how the additive connectives of linear logic, generally the difficult connectives to address with this style of semantics, can be handled. Although he outlines how Hilbert spaces and C^* algebras can be used to analyze interaction, he gives a more concrete interpretation model using the behavior of resolution on simple, binary Horn clauses.