On focusing and polarities in linear logic and intuitionistic logic

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Abstract

There are a number of cut-free sequent calculus proof systems known that are complete for first-order intuitionistic logic. Proofs in these different systems can vary a great deal from one another. We are interested in providing a flexible and unifying framework that can collect together important aspects of many of these proof systems. First, we suggest that one way to unify these proof systems is to first translate intuitionistic logic formulas into linear logic formulas, then assign a bias (positive or negative) to atomic formulas, and then examine the nature of focused proofs in the resulting linear logic setting. Second, we provide a single focusing proof system for intuitionistic logic and show that these other intuitionistic proof systems can be accounted for by assigning bias to atomic formulas and by inserting certain markers that halt focusing on formulas. Using either approach, we are able to account for proof search mechanisms that allow for forward-chaining (program-directed search), backward-chaining (goal-directed search), and combinations of these two. The keys to developing this kind of proof system for intuitionistic logic involves using Andreoli's completeness result for focusing proofs and Girard's notion of polarity used in his LC and LU proof systems.

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1 Introduction

We wish to refine our understanding of the structure of proofs in intuitionistic logic. In particular, we will be concerned with how to structure the non-determinism that arises in the search for cut-free proofs. In doing so, we hope to provide a theoretical foundation for understanding proof search mechanisms based on forward-chaining (program-directed search), backward-chaining (goal-directed search), and their combinations. Such a foundation should find application in the building of logic programming and automated deduction systems, and in the study of various term-reduction systems.

In logic programming, computation is modeled by the process of searching for a *cut-free* proof from the "bottom-up": as attempts to prove sequents give rise to attempts to prove additional sequents, the structure of sequents changes. The logic programmer is then able to use this dynamics of changes to encode computation. The role of cut and the cut-elimination theorem is relegated to providing means to reason about computation. Thus, our interest here in developing normal forms for cut-free proofs.

A principle tool for analyzing proof search is Andreoli's completeness theorem for *focused* proofs [And92]. Since that result holds for linear logic, we shall develop a number of translations of intuitionistic logic into linear logic. Girard's original translation [Gir87] of intuitionistic logic into linear logic was based on mapping all occurrences of the intuitionistic implication $B \supset C$ to the formula $!B \multimap C$. Using such a mapping, it is possible to find tight correspondence between intuitionistic proofs and linear logic proofs preserves the dynamics of cut-elimination. If one is interested in capturing only cut-free proofs for, say, first-order logics, then it is possible to predict whether subformulas of the endsequent appear on the left or the right-hand side of sequents in the proof just by observing their occurrences in endsequent. (Such an invariant is not generally true when higher-order quantification is present [MNPS91, Section 5].) In such settings, intuitionistic formulas can be mapped in two different ways depending on whether or not they occur on the left or right of sequents. For example, Hodas and Miller [HM94] used two translations such that

$$[B \supset C]^+ = ![B]^- \multimap [C]^+ \quad \text{and} \quad [B \supset C]^- = [B]^+ \multimap [C]^-.$$

Notice that such a mapping translates the initial sequent $B \supset C \longrightarrow B \supset C$ to $[B]^+ \multimap [C]^- \longrightarrow ![B]^- \multimap [C]^+$, which is provable (but not initial) if we assume, inductively, that $[C]^- \longrightarrow [C]^+$ and $[B]^- \longrightarrow [B]^+$ are provable. Such translations based on a pair of mutually recursive translations is called here *asymmetrical translations*: most translations that map intuitionistic logic into linear logic considered in this paper are asymmetric.

The translations that we describe are all based on assigning answers to the following questions.

- Where to insert exponentials (! or ?)? For example, the only difference between the two translations displayed above for implication is that one contains a ! that the other does not.
- Does a connective get mapped to a multiplicative or additive connective? For example, the translations used in [HM94] resolves that choice for conjunction as

  $$[B \wedge C]^+ = [B]^- \otimes [C]^+ \quad \text{and} \quad [B \wedge C]^- = [B]^+ \,\&\, [C]^-.$$

  Given the prescence of ! in the translations of intuitionistic logic and the fact that that exponential can change an additive to a multiplicative, this choice is occasionally not simply binary.
- What is the proper treatment of atoms? Often atom A is mapped A or to $!A$. A *bias* for atoms must often be assigned so that focusing proofs can be properly defined.

Eventually, we present a new focused proof system for intuitionistic logic. This system will support the mixing of polarities during proof search. That is, it will support both forward and backward chaining. There might be various ways to motivate and validate our design: for example, coherent space semantics is used by some to motivate related proof systems such as LC [Gir91]. Ultimately, the success of our search for an intuitionistic focusing system will result from a better understanding of the notion of polarity: specifically how polarity as defined in LC and LU [Gir93] relate to the operational behavior of focused proofs. The design of our systems will be motivated entirely proof theoretically. That is, the reasons for our choices in design will be operational: the actual process of searching for proofs will often be considered. Our approach has the advantage of being essentially elementary and self contained.

There are close connections between our work and those of Danos, Joinet and Schellinx [DJS95, DJS97]. Many concepts which they initially explored, such as *inductive decorations,* are used throughout our analysis. Where our work diverges from theirs, however, is in the adoptation of Andreoli's focusing calculus as our main instrument of construction (deconstruction?). The work of [DJS97] was based on an in-depth analysis of cut elimination in classical logic. Their LK_p^η system essentially reformulates focusing not as a sequent calculus but as a protocol for constructing proofs. Its connections to polarization and focusing were further explored and extended by Laurent, Quatrini and de Falco [LQdF05] using *polarized proof nets.* It is likely that these alternative formalisms can be adjusted to intuitionistic logic and replace Andreoli's sequent calculus as the means to accomplish our goals. We find Andreoli's approach suitable because our targets are also sequent calculi, which can be easily implemented. Moreover, Andreoli's system defines focusing for *full* linear logic. We demonstrate that this means that it is in fact endowed with the power of LU, and is therefore a suitable choice as a host-logic framework. Our approach to this subject is elementary. We examine the structure of cut-free proofs, basing our constructions on well-established cut elimination theorems and the correctness of the linear focusing calculus. The end result is a compact focusing calculus for full intuitionistic logic, with which we can also derive a similar system for classical logic. Our work can be extended to second order logic, although this paper is mainly concerned with first-order intuitionistic logic.

This paper has two major goals. First, we show that a number of known intuitionistic proof systems (LJT, LJQ, and λRCC) can be derived and proved correct by first translating intuitionistic logic formulas into linear logic and showing that focused proofs of the translated formulas are in one-to-one correspondence to proofs in the original sequent calculus. Thus, the essential character of the original proof system is captured by the way it is translated into linear logic. This approach to proof systems in intuitionistic logic has a parallel with the way that continuation-passing style translations are used to make explicit either call-by-name or call-by-value evaluation [Plo76]: the strategy is encoded in the translation. In the proof search setting, we have a choice of doing goal-directed or program-directed proof search or a combination of both. Which way this is done is captured here via translations into linear logic.

The second major goal is the introduction of a new proof system, LJF, that allows us to capture focused proof search in intuitionistic logic. We illustrate that other proof systems for intuitionisti logic can be accounted for by inserting logical expressions into intuitionistic formulas that prematurely halt the deterministic aspects of proof search in LJF. Finally, we use the new system LJF to derive a focused sequent calculus for classical logic that supports its constructive characteristics (in the sense of LC).

The structure of completeness proofs A standard format will be used to prove the soundness and completeness of a number of proof systems. It is illustrated with the help of Figure 1. The foundation of the format is a translation from intuitionistic logic to linear logic called the "0/1" translation. A bijective mapping (indicated by a two-way arrow) is established between arbitrary LJ proofs (indicated by $\vdash_I$) and focused proofs in linear logic of the translated image (indicated with $\vdash\Uparrow$). Suppose we then wish to proof the soundness and completeness of some other intuitionistic proof system, labeled $\vdash_O$. We device another translation, labeled l/r, that also maps $\vdash_O$ proofs to focused proofs of its image in linear logic ($\vdash\Uparrow$ with respect to the l/r translation). It will then be shown that proofs under the $0/1$ translation can be transformed into proofs under the new translation. As is usually the case, the soundness of $\vdash_O$ with respect to LJ will be trivial (indicated by the arrow from $\vdash_O$ to $\vdash_I$). This completes a "grand tour" that establishes the equivalence among four distinct notions of provability. Consequently, $\vdash_O$ proofs are shown to be complete with respect to intuitionistic logic.

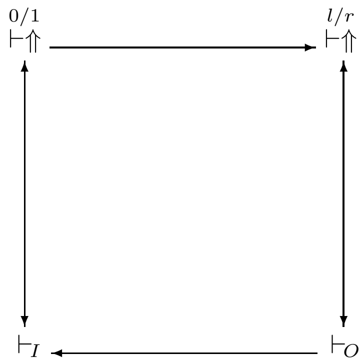

Figure 1: Completeness proof using a “grand tour” through linear logic

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2 Focused proofs in Linear logic

We summarize some key concepts and results from Andreoli paper on focus proofs for linear logic [And92].

A *literal* is either an atomic formula or the linear negation of an atomic formula. A linear logic formula is in *negation normal form* if it does not contain occurrences of $\multimap$ and if all negations have atomic scope. If K is literal, then $K^\perp$ denotes its complement: in particular, if K is $A^\perp$ then $K^\perp$ is A.

2.1 Synchronous and asynchronous proof search

Connectives in linear logic can be classified as either *asynchronous* or *synchronous*. The asynchronous connectives are $\bot$, $\invamp$, $?$, $\top$, $\&$, and $\forall$ while the synchronous connectives are their de Morgan dual, namely, $\mathbf{1}$, $\otimes$, $!$, $\mathbf{0}$, $\oplus$, and $\exists$. Asynchronous connectives are those where the right-introduction rule is always invertible while the synchronous connectives have right-introduction rules that are not alway invertible. Formally, a formula in negation normal form is of three kinds: literal, asynchronous (meaning that its top-level connective is an asynchronous connective), and synchronous (meaning that its top-level connective is a synchronous connective).

In the *focusing proof system* in Figure 2, there are two kind of sequents. In the sequent $\Psi\colon\Delta \Uparrow L$, the "zones" Ψ and Δ are multisets and L is a list. This sequent encodes the usual one-sided sequent $\vdash ?\,\Psi, \Delta, L$ (here, we assume the natural coercion of lists into multisets). This sequent will also always satisfy the invariant that requires Δ to contain only literals and synchronous formulas. In the sequent $\Psi\colon\Delta \Downarrow F$, Ψ is a multiset of formulas and Δ is a multiset of literals and synchronous formulas, while F is a single formula.

Asynchronous phase

$$\frac{\Psi\colon\Delta \Uparrow L}{\Psi\colon\Delta \Uparrow \bot, L}\,[\bot] \qquad \frac{\Psi\colon\Delta \Uparrow F, G, L}{\Psi\colon\Delta \Uparrow F \invamp G, L}\,[\invamp] \qquad \frac{\Psi, F\colon\Delta \Uparrow L}{\Psi\colon\Delta \Uparrow ?\,F, L}\,[?]$$

$$\frac{}{\Psi\colon\Delta \Uparrow \top, L}\,[\top] \qquad \frac{\Psi\colon\Delta \Uparrow F, L \quad \Psi\colon\Delta \Uparrow G, L}{\Psi\colon\Delta \Uparrow F \,\&\, G, L}\,[\&] \qquad \frac{\Psi\colon\Delta \Uparrow B[y/x], L}{\Psi\colon\Delta \Uparrow \forall x.B, L}\,[\forall]$$

$$\frac{\Psi\colon\Delta, F \Uparrow L}{\Psi\colon\Delta \Uparrow F, L}\,[R\Uparrow] \quad \text{provided that F is not asynchronous}$$

Synchronous phase

$$\frac{}{\Psi\colon\cdot \Downarrow \mathbf{1}}\,[\mathbf{1}] \qquad \frac{\Psi\colon\Delta_1 \Downarrow F \quad \Psi\colon\Delta_2 \Downarrow G}{\Psi\colon\Delta_1, \Delta_2 \Downarrow F \otimes G}\,[\otimes] \qquad \frac{\Psi\colon\cdot \Uparrow F}{\Psi\colon\cdot \Downarrow\, !\,F}\,[!]$$

$$\frac{\Psi\colon\Delta \Downarrow F_1}{\Psi\colon\Delta \Downarrow F_1 \oplus F_2}\,[\oplus_l] \qquad \frac{\Psi\colon\Delta \Downarrow F_2}{\Psi\colon\Delta \Downarrow F_1 \oplus F_2}\,[\oplus_r] \qquad \frac{\Psi\colon\Delta \Downarrow B[t/x]}{\Psi\colon\Delta \Downarrow \exists x.B}\,[\exists]$$

$$\frac{\Psi\colon\Delta \Uparrow F}{\Psi\colon\Delta \Downarrow F}\,[R\Downarrow] \quad \text{provided that } F \text{ is either asynchronous or a negative literal}$$

Identity and Decide rules

$$\text{If } K \text{ a positive literal:} \quad \frac{}{\Psi\colon K^\perp \Downarrow K}\,[I_1] \qquad \frac{}{\Psi, K^\perp\colon\cdot \Downarrow K}\,[I_2]$$

$$\text{If } F \text{ is not a negative literal:} \quad \frac{\Psi\colon\Delta \Downarrow F}{\Psi\colon\Delta, F \Uparrow \cdot}\,[D_1] \qquad \frac{\Psi\colon\Delta \Downarrow F}{\Psi, F\colon\Delta \Uparrow \cdot}\,[D_2]$$

Figure 2: The focused proof system LLF for linear logic

For convenience, we shall refer to the focusing proof system for linear logic as *LLF*.

In the original presentation of this focusing system [And92], atoms are considered "positive" and their negations “negative.” This was also the clssification used by Girard for LC [Gir91]. It is, however, a more general treatment to assign an arbitrary *polarity* or *bias* directly to atoms (see further discussion of these notions at the end of this section). For intuitionistic logic and the intuitionistic restriction of linear logic upon which many current systems are based, assigning bias directly to atoms is the only possible choice. For linear and classical logics, this formulation is consistent with the original style of classification.

Thus it is to be interpreted that LLF requires that the atoms are divided into two sets: one for *positive* atoms and the other for *negative* atoms. This polarity of atoms is extended in the natural way to literals: negating a negative atom yields a positive literal and negating a positive atom yields a negative literal.

It is important to observe that in the initial rule, the literal on the right must be positive. Thus, if the literal K is positive, then $\Psi\colon K^\perp \Downarrow K$ has an immediate proof using $[I_1]$, while the sequent $\Psi\colon K \Downarrow K^\perp$ can only be proved with a more involved proof, namely

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Psi\colon K^\perp \Downarrow K}\,I_1}{\Psi\colon K^\perp, K \Uparrow}\,D_1}{\Psi\colon K \Uparrow K^\perp}\,R\Uparrow}{\Psi\colon K \Downarrow K^\perp}\,R\Downarrow$$

$$\dfrac{\dfrac{\dfrac{}{\cdot\colon a^\perp \Downarrow a}\,I_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdot\colon b^\perp \Downarrow b}\,I_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdot\colon c^\perp \Downarrow c}\,I_1}{\cdot\colon c^\perp, c \Uparrow}\,D_1}{\cdot\colon c \Uparrow c^\perp}\,R\Uparrow}{\cdot\colon c \Downarrow c^\perp}\,R\Downarrow}{\cdot\colon c, b^\perp \Downarrow b \otimes c^\perp}\,\otimes}{\cdot\colon b \otimes c^\perp, c, b^\perp \Uparrow \cdot}\,D_1}{\cdot\colon b \otimes c^\perp, c \Downarrow b^\perp}\,R\Downarrow}{\cdot\colon a^\perp, b \otimes c^\perp, c \Downarrow a \otimes b^\perp}\,\otimes}{\cdot\colon a^\perp, a \otimes b^\perp, b \otimes c^\perp, c \Uparrow \cdot}\,D_1 \qquad\qquad \dfrac{\dfrac{}{a \vdash a} \qquad \dfrac{\dfrac{}{b \vdash b} \qquad \dfrac{}{c \vdash c}}{b, b \multimap c \vdash c}}{a, a \multimap b, b \multimap c \vdash c}$$

Figure 3: A focused, one-sided sequent and its corresponding two-sided form.

The choice of polarity for atomic formulas has an effect on the shape of a focused proof. Consider proving the linear logic formula $a \multimap (a \multimap b) \multimap (b \multimap c) \multimap c$. This is provable if and only if the two-sided sequent $a, a \multimap b, b \multimap c \vdash c$ is provable. There are two different proofs of this sequent depending on the order in which the linear implications are introduced. If a, b, and c are positive then the introduction of $b \multimap c$ must appear above the introduction of $a \multimap b$. In particular, Figure 3 contains the focused proof of the one-sided sequent and the corresponding proof of the associated two-sided sequent. If a, b, and c are negative then there is again one focused proof and it corresponds to switching the order on introducing these implications.

The proofs displayed in Figure 3 are sometimes referred to as “bottom-up” or “program-directed”: here, one reasons from the “(logic) program”, which is the left-hand side of the sequent, and attempts to make new logic programs (reading the proof from the bottom). On the other hand, if we switch the order of the $\multimap$ introduction, the resulting proofs are sometimes referred to as “top-down” or “goal-directed”: here, one tries to prove the goal immediately and to produce new sub-goals that are attempted next. It was observed in [Mil96] that all of linear logic could be seen as goal-directed search if all atomic formulas were assigned a negative polarity.

Changes to the polarities assigned to atoms does not affect provability of a linear logic formula: instead it affects the structure of focused proofs. The structuring of cut-free proofs is an underlying concern of this paper.

2.2 Assignment of bias

The term "polarities" is used in two different but related ways. For the sake of discussion, we shall use for the moment two terms to distinguish these two sense. In the work on LU, Girard used a notion of polarity with the three values negative, neutral, and positive: these were used to indicate whether or not a formula can move in and out of different zones within sequents. We shall sometimes refer to this as the "permeability" of a formula and identify negative with *right permeability*, positive with *left permeability*, and neutral with *non-permeability*. In Section 6, we take our first explicit look at permeability. Our immediate interest here and in the next several sections concerns the classification of atoms (and, hence, literals) as positive and negative. For the sake of clarity, we shall refer to this division as "bias" here.

The assignment of bias to atomic formulas within focused proof search affects the shape of proofs but not the existence of proofs. It seems that bias for atoms exists because of the need of focusing to classify all subformula occurrences as being either synchronous or asynchronous. A positive bias literal is treated synchronously: in the $\Downarrow$-phase, encountering a positive literal requires the proof to be completed immediately: no unfocusing of the proof search is allowed. Conversely, a negative bias literal is treated asynchronously: in the $\Downarrow$-phase, encountering a negative literal requires the proof to lose focus, even if the complement of the literal is available in the context.

There appears to be at least three ways to assign bias to atoms.

1. The *atom-based* approach assigns a polarity to all atoms in a fixed and global fashion. This assignment does not need to be stable under substitution. This style assignment is used in [And92].

2. The *predicate-based* approach assigns a polarity first to predicate symbols: atoms then inherent their polarity from the predicate that is their head. Such assignment is stable under first-order substitution. As we shall see in Section 5, the λRCC proof system of [JNS05] can be seen as using this style of assignment.

3. The *proof-occurrence-based* approach assigns polarities to atoms based on their location within proofs. Two different occurrences of the same atom in different (conjunctive) branches of a proof tree can, apparently, be given different polarities. For example, attempting a proof of $\longrightarrow A \wedge (A \supset G)$ will lead to attempting proofs of two different sequent, one for $\longrightarrow A$ and one for $A \longrightarrow G$. The resulting two occurrences of A do not need to have the same bias since they cannot interact (via the initial rule, for example). In fact, one might well wish to treat the bias of these two occurrences differently: if the attempt to prove G results in attempts to prove A, it might be valuable to expect that A is present in the context instead of allowing proof search to reprove A (see the discussion on memo-ization in logic programming in [Mil86]).

In this note, we follow the *atom-based* approach, which subsumes the predicate-based approach.

2.3 The meaning of bias

One could say that the "meaning" of bias is to simply reduce non-determinism in the search for proof. A better answer might be used to justify the predicate-based assignment scheme: predicates are non-logical constants can be seen as higher-order (eigen-) variables in a higher-order logic. As such, predicates can be substituted by λ-abstracted formulas. Such an analysis would naturally be based on looking at focused proofs for logics involing higher-order quantifiers, which we leave for future work (see Section 10).

If a predicate that is assigned a positive bias is instantiated with a synchronous (abstracted) formula, then it should be the case that a focused proof is mapped to an almost focused proof (and dually for instantiating a negative predicate with an asynchronous formula).

Another way to provide meaning to an "unknown" predicate is via definitions, in the sense of [Gir92, SH93, MM00]. In that approach, predicates are interpreted as fixed point definitions: during proof search, an atomic formula can be unfolded according to its definition. In such a setting, if a given predicate is instantiated with a synchronous formula, that atom would be assigned a positive polarity (dually for asynchronous formulas and negative polarities).

Much of our debate on how to assign bias to atoms and what bias might mean (apart from proof search) can be avoided if we can, in fact, avoid atoms entirely. The notion of definitions mentioned above allows for one approach to avoiding atoms: atoms are simply expressions that do not refer to themselves but to their meaning. If such unfoldings can be guaranteed to be noetherian, then one needs no instances of the initial rule. A more extreme situation might be the one taken in Ludics by Girard: there formulas are replaced by structures with which one interacts to arbitrary depth [Gir01]. The structures corresponding to formulas are not built from atoms at all.

Given this discussion about bias above, we adopt the following uses of the term "polarity" in this report. In linear logic, the notion of *asynchronous* and *synchronous* are used for non-atomic formulas. The notion of bias is used for atomic formulas. We shall describe a formula as being of *positive polarity* if it is either a non-atomic synchronous formula or a positive bias atom. We shall describe a formula as being of *negative polarity* if it is either a non-atomic asynchronous formula or a negative bias atom. In intuitionistic logic, the notion of positive polarity will be equated to left permeability: we shall also assign all positive bias atoms to this class. Negative polarity will be identified with non-permeable formulas, which will always include the negative bias atoms. Similarly, in classical logic, the notion of positive polarity will be equated to left permeability: we shall also assign all positive bias atoms to this class. Negative polarity will be identified with right-permeable formulas and these formulas always include the negative bias atoms. These classifications are further explained in Section 6.2.

B	B^1	B^0
A	A	A
true	1	$\top$
false	0	0
$P \wedge Q$	$!(P^1 \& Q^1)$	$!\,P^0 \&\, !\,Q^0 = (?(P^0)^\perp \oplus ?(Q^0)^\perp)^\perp$
$P \vee Q$	$!\,P^1 \oplus !\,Q^1$	$!\,P^0 \oplus !\,Q^0 = (?(P^0)^\perp \&\, ?(Q^0)^\perp)^\perp$
$P \supset Q$	$!(!\,P^0 \multimap Q^1) = !(?(P^0)^\perp \parr Q^1)$	$!\,P^1 \multimap !\,Q^0 = (!\,P^1 \otimes ?\,Q^{0^\perp})^\perp$
$\neg P$	$!(!\,P^0 \multimap 0) = !(0 \parr ?(P^0)^\perp)$	$!\,P^1 \multimap 0 = (!\,P^1 \otimes \top)^\perp$
$\exists x P$	$\exists x\,!\,P^1$	$\exists x\,!\,P^0 = (\forall x\,?(P^0)^\perp)^\perp$
$\forall x P$	$!\,\forall x P^1$	$\forall x\,!\,P^0 = (\exists x\,?(P^0)^\perp)^\perp$

Table 1: The 0/1 translation used to encode LJ proofs into linear logic.

3 Translating LJ Proofs

A principal aim of this paper is to show that focusing in linear logic can be used to derive systems in logics that linear logic is designed to embed. Our first task is to show that arbitrary LJ proofs can be mirrored as focused proofs with the appropriate translation to linear logic. Focusing yields a fine degree of control over proof search, *including* the freedom to loose deterministic behavior. More significantly, however, our translation will be used as an anchor for future translations. It will serve as a key link to intuitionistic provability, and allow completeness theorems for various intuitionistic systems to be proved via a "grand tour" through linear logic.

The "0/1" translation given in Table 1 is our *source translation* for intuitionistic logic, defined on formulas of the forms

$$true \mid false \mid A \mid P \wedge Q \mid P \vee Q \mid P \supset Q \mid \exists x P \mid \forall x P \mid \neg P$$

where A represents atomic formulas. For the convenience of moving between one- and two-sided sequents, we display the dual of most of the formulas in the B^0 column. The two-sided intuitionistic sequent $\Gamma \vdash_I G$ is translated into the two-sided linear logic sequent $!\Gamma^0 \vdash_{LL} G^1$ and the one-sided linear logic sequent $\vdash_{LL} ?(\Gamma^0)^\perp, G^1$. Here, we follow the usual convention of applying exponentials, negations, and translations to sets and multiset of formulas: for example, $!\,\Gamma^0 = \{!\,D^0 \mid D \in \Gamma\}$.

Readers familiar with Girard's original translation [Gir87] or the Hodas-Miller translation in [HM94] may be concerned with the prolific use of the exponential ! in the above translation. Indeed, if we are interested only in preserving *provability,* then most of these exponential are unnecessary. However, our interest lies also in preserving the structure of *proofs,* not just in the traditional presentation of linear logic but also in the focusing calculus LLF. Using the terminology of [DJS95] we seek *inductive decoration strategies*: *i.e.*, we wish to define one-to-one mappings between focused proofs and unrestricted LJ proofs.

We shall soon consider other translations of intuitionistic logic into linear logic that strengthen the 0/1-translation along the following, inter-related items.

1. Various !'s can be dropped: for example, $P \supset Q$ can be translated using the patterns $!\,P \multimap !\,Q$, $P \multimap !\,Q$, or $P \multimap Q$.

2. In the above translation, the polarity of atoms is irrelevant: in translations with fewer exponentials, the polarity of atoms can play an important role.

3. The intuitionistic conjunction is mapped to the additive conjunction &. Other translations can use the multiplicative conjunction $\otimes$. The & provides only "part of the story" of the intuitionistic conjunction.

The 0/1-translation places ! in front of asynchronous connectives on the right-hand side for two purposes. First, the ! stops all right-side formulas from being treated asynchronously: in particular, no formula is decomposed until selected by a "decide" inference rule. In this way, we can map arbitrary LJ proofs, which have no focusing discipline, into focused proofs. It should be observed, however, that instead of using $!\,B$

for this purpose, we could have used $B \otimes 1$. The $!$ is chosen because it will also allow us to study $\otimes$ as an alternative interpretation of $\wedge$.

Intuitionistic negation In considering intuitionistic logic there is a choice as to whether to accept intuitionistic negation as a connective. Such a choice is possible: for example, Table 1 contains the clauses $(\neg P)^0 = !\,P^1 \multimap 0$ and $(\neg P)^1 = !\,P^0 \multimap 0$. To account for intuitionistic sequents with empty right-hand sides, we must translate the empty right multiset as 0. That is, if $()$ denotes the empty multiset then $()^0 = 1$ and $()^1 = 0$. The intuitionistic inferences

$$\frac{\Gamma \vdash_I P}{\neg P, \Gamma \vdash_I} \neg L \quad \text{and} \quad \frac{P, \Gamma \vdash_I}{\Gamma \vdash_I \neg P} \neg R$$

are encoded via the derivations

$$\cfrac{\cfrac{\cfrac{!\,\Gamma^0 \vdash_{LL} P}{!\,\Gamma^0 \vdash_{LL} !\,P}\,!R \qquad 0 \vdash_{LL} 0}{!\,P^1 \multimap 0, !\,\Gamma^0 \vdash_{LL} 0} \multimap L}{!(!\,P^1 \multimap 0), !\,\Gamma^0 \vdash_{LL} 0}\,!L \qquad \text{and} \qquad \frac{!\,P^0, !\,\Gamma^0 \vdash_{LL} 0}{!\,\Gamma \vdash_{LL} !\,P^0 \multimap 0} \multimap R$$

The specialized weakening rule for the intuitionistic right-side formula is admissible under this translation since

$$\frac{!\,\Gamma \vdash_{LL} 0 \quad 0 \vdash_{LL} P}{!\,\Gamma \vdash_I P}\, Cut$$

After cut elimination, when all cuts have been permuted above the left inference rules, there will remain sequents of the form $0, \Delta \vdash_{LL} Q$ at the leafs of the proof tree. We shall therefore adopt the version of intuitionistic sequent calculus that does not make explicit use of right-weakening or of the $\neg R$ and $\neg L$ rules. Instead, we include the inference rule

$$\frac{}{\mathit{false}, \Gamma \vdash P}\, \mathit{falseL}$$

Consistent with this choice, we will not use $\neg$ as a connective but define $\neg A$ as $A \supset \mathit{false}$.

Although nominally a multiset, the unbounded context of a focusing sequent is in truth treated additively: it never decreases during proof search. For example, for the $\supset L$ rule, the principal formula $A \supset B$ is retained in the context in both premises and we will not be able to directly account for the well known optimization that the implication does not need to be carried to the right premise. However, as our translation will remain faithful (sound and complete) to intuitionistic provability, one can expect these variations to remain admissible in the intuitionistic calculi that we derive. We therefore consider such issues orthogonal to the main aims of this paper.

The precise intuitionistic calculus for which we define a translation is found in Figure 4. The context Γ is a set in this calculus. Contraction and weakening will not need to be made explicit.

This variant of LJ bares some close resemblances to the "*G3i*" calculus found in [TS96]. However, the additive version of $\wedge L$ is used instead of

$$\frac{A, B, \Gamma \vdash_I R}{A \wedge B, \Gamma \vdash_I R} \wedge L$$

Such variations are common in intuitionistic calculi. One of the aims of this paper is to further clarify these variations. We show when one version of $\wedge$ is preferable to the other, and eventually offer a system that includes both.

Bijective mappings between proof systems We define a bijective translation from focused proofs of translated sequents of intuitionistic proofs. The bijection is modulo some structural rules: specifically weakenings above the leafs and contractions below the root. Implicitly the procedure also provides an

$$\frac{}{A, \Gamma \vdash_I A}\ ID,\ A \text{ atomic} \qquad \frac{}{false, \Gamma \vdash_I R}\ falseL \qquad \frac{}{\Gamma \vdash_I true}\ trueR$$

$$\frac{A_i, \Gamma \vdash_I R}{A_1 \wedge A_2, \Gamma \vdash_I R}\ \wedge L \qquad \frac{\Gamma \vdash_I A \quad \Gamma \vdash_I B}{\Gamma \vdash_I A \wedge B}\ \wedge R$$

$$\frac{A, \Gamma \vdash_I R \quad B, \Gamma \vdash_I R}{A \vee B, \Gamma \vdash_I R}\ \vee L \qquad \frac{\Gamma \vdash_I A_i}{\Gamma \vdash_I A_1 \vee A_2}\ \vee R$$

$$\frac{\Gamma \vdash_I A \quad B, \Gamma \vdash_I R}{A \supset B, \Gamma \vdash_I R}\ \supset L \qquad \frac{\Gamma, A \vdash_I B}{\Gamma \vdash_I A \supset B}\ \supset R$$

$$\frac{A[t/x], \Gamma \vdash_I R}{\forall x A, \Gamma \vdash_I R}\ \forall L \qquad \frac{\Gamma \vdash_I A}{\Gamma \vdash_I \forall y A}\ \forall R \qquad \frac{A, \Gamma \vdash_I R}{\exists y A, \Gamma \vdash_I R}\ \exists L \qquad \frac{\Gamma \vdash_I A[t/x]}{\Gamma \vdash_I \exists x A}\ \exists R$$

y not free in Γ, R.

Figure 4: An Intuitionistic Calculus based on "*G3i*"

inductive proof of the equivalence of provability in the two systems. The implied inductive measure is the height of proofs.

In stating the following theorem, we employ the one-sided sequents used by Andreoli but stay with two-sided sequents with intuitionistic proofs. $\vdash$ will be used to represent sequents LLF. We also note that formulas under the 0/1 translation do not contain forms $A^{\perp}$, so they cannot move between the left and right-hand sides of sequents. That is, the meaning of $(A^0)^{\perp}$ in a one-side focusing sequent is unambiguously interpreted as indicating that A belongs to the left side of the corresponding intuitionistic sequent.

In the following Proposition, we consider cut-free, atomically closed proofs in intuitionistic logic.

Proposition 1 (Full completeness for the 0/1 translation) *Let $(\Gamma^0)^{\perp}$ denote the multiset $\{?(D^0)^{\perp} \mid D \in \Gamma\}$. Assume that all atomic formulas have been assigned a positive or negative polarity. Then $\vdash (\Gamma^0)^{\perp} :\Uparrow R^1$ is provable if and only if $\Gamma \vdash_I R$ is provable. Furthermore, there is a one-to-one mapping between focused linear logic proofs and cut-free, intuitionistic proofs as defined in Figure 4.*

Proof We define the following invertible procedure that recursively maps focused proofs to $\vdash_I$ proofs. The procedure is organized into four sub-procedures: τA, τB, τBr and τBl, with τA the top-level procedure. Each sub-procedure is responsible for translating a class of focused proofs to a proof of $\Gamma \vdash_I R$.

τA: translates proofs of $\vdash (\Gamma^0)^{\perp} :\Uparrow R^1$

τB: translates proofs of $\vdash (\Gamma^0)^{\perp} : R^1 \Uparrow$

τBr: translates proofs of $\vdash (\Gamma^0)^{\perp} :\Downarrow R^1$

τBl: translates proofs of $\vdash (\Gamma^0)^{\perp} : R^1 \Downarrow (D^0)^{\perp}$, where $(D^0)^{\perp} \in (\Gamma^0)^{\perp}$.

Procedure τA: All R^1 formulas are either atoms or synchronous. Thus the last rule of the focused proof must be $R \Uparrow$:

$$\frac{\begin{matrix}\Pi \\ \vdash (\Gamma^0)^{\perp} : R^1 \Uparrow\end{matrix}}{\vdash (\Gamma^0)^{\perp} :\Uparrow R^1}\ R \Uparrow$$

Procedure τA calls τB on the subproof Π and return the result. The symbol $\longmapsto$ will be used to indicate the mapping between proofs.

Procedure τB**:** (the *dispatch* procedure) Depending on whether or not the last inference rule of the focused proof is $[D_1]$ or $[D_2]$, we compute the intuitionistic proof along one of these lines:

$$\dfrac{\overset{\Pi}{\vdash (\Gamma^0)^\perp :\Downarrow R^1}}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow}\ [D_1] \quad \longmapsto \quad \begin{array}{c}\tau Br(\Pi)\\ \Gamma \vdash_I R\end{array}$$

$$\dfrac{\overset{\Pi}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow (D^0)^\perp}}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow}\ [D_2] \quad \longmapsto \quad \begin{array}{c}\tau Bl(\Pi)\\ \Gamma \vdash_I R\end{array} \quad \text{where } (D^0)^\perp \in (\Gamma^0)^\perp.$$

The two sub-procedures τBl and τBr are defined below.
Procedure τBr**:** Consider the different ways that a focused proof of $\vdash (\Gamma^0)^\perp :\Downarrow R^1$ can be terminated. We do this by induction on the structure of the formula R.

1. Positive atom. If R is a positive literal (atom) B, then the sequent $\vdash (\Gamma^0)^\perp :\Downarrow B$ is mapped as follows[1]:

   $$\dfrac{}{\vdash (\Gamma^0)^\perp :\Downarrow B}\ I_2,\ B^\perp \in (\Gamma^0)^\perp \quad \longmapsto \quad \dfrac{}{\Gamma \vdash_I B}\ ID$$

   Note that this is not necessarily where focusing is *forced,* because this rule may be situated directly above a $[D_1]$ or $[D_2]$ rule. Also note that if B had negative polarity, then $B^\perp$ must be selected for focus from the context, resulting in an $[I_1]$ rule (see procedure τBl).

2. If R is a negative literal, then it cannot be selected for focus. There is nothing more to consider here.

3. The unique proof of the sequent $\vdash (\Gamma^0)^\perp :\Downarrow 1$ maps to ($\longmapsto$) the $\Gamma \vdash_I$ *true*.

4. $R^1 = (G_1 \wedge G2)^1 = !(G_1^1 \& G_2^1)$.

   $$\dfrac{\dfrac{\overset{\Pi_1}{\vdash (\Gamma^0)^\perp ; \Uparrow G_1^1} \quad \overset{\Pi_2}{\vdash (\Gamma^0)^\perp ; \Uparrow G_2^1}}{\vdash (\Gamma^0)^\perp :\Uparrow G_1^1 \& G_2^1}\ \&}{\vdash (\Gamma^0)^\perp :\Downarrow !(G_1^1 \& G_2^1)}\ ! \quad \longmapsto \quad \dfrac{\overset{\tau A(\Pi_1)}{\Gamma \vdash_I G_1} \quad \overset{\tau A(\Pi_2)}{\Gamma \vdash_I G_2}}{\Gamma \vdash_I G_1 \wedge G_2}\ \wedge R$$

5. $R^1 = (\forall x G)^1 = !\forall x G^1$. We assume the usual conditions on the eigenvariable introduction.

   $$\dfrac{\dfrac{\overset{\Pi}{\vdash (\Gamma^0)^\perp :\Uparrow G^1}}{\vdash (\Gamma^0)^\perp :\Uparrow \forall x G^1}\ \forall}{\vdash (\Gamma^0)^\perp :\Downarrow !\forall x G^1}\ ! \quad \longmapsto \quad \dfrac{\overset{\tau A(\Pi)}{\Gamma \vdash_I G}}{\Gamma \vdash_I \forall x G}\ \forall R$$

6. For $R^1 = (D \supset G)^1 = !(!D^0 \multimap G^1) = !(G^1 \parr ?(D^0)^\perp)$:

   $$\dfrac{\dfrac{\dfrac{\overset{\Pi}{\vdash (D^0)^\perp, (\Gamma^0)^\perp :\Uparrow G^1}}{\vdash (\Gamma^0)^\perp :\Uparrow G^1, ?(D^0)^\perp}\ ?}{\vdash (\Gamma^0)^\perp :\Uparrow G^1 \parr ?(D^0)^\perp}\ \parr}{\vdash (\Gamma^0)^\perp :\Downarrow !(G^1 \parr ?(D^0)^\perp)}\ ! \quad \longmapsto \quad \dfrac{\overset{\tau A(\Pi)}{D, \Gamma \vdash_I G}}{\Gamma \vdash_I D \supset G}\ \supset R$$

[1] In embeddings of intuitionistic logic, linear negation is only used to distinguish between left and right occurrences. Intuitionistic negation is also a defined connective. The term "atom" is thus often used in place of "literal" when referring to intuitionistic formulas.

7. $R^1 = (G_1 \vee G_2)^1 = !G_1^1 \oplus !G_2^1$:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_i\\ \vdash (\Gamma^0)^\perp :\Uparrow G_i^1\end{array}}{\vdash (\Gamma^0)^\perp :\Downarrow !G_i^1}\;!}{\vdash (\Gamma^0)^\perp :\Downarrow !G_1^1 \oplus !G_2^1}\;\oplus \quad\longmapsto\quad \dfrac{\begin{array}{c}\tau A(\Pi_i)\\ \Gamma \vdash_I G_i\end{array}}{\Gamma \vdash_I G_1 \vee G_2}\;\vee R$$

8. $R^1 = (\exists x G)^1 = \exists x !G^1$:

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdash (\Gamma^0)^\perp :\Uparrow G^1[t/x]\end{array}}{\vdash (\Gamma^0)^\perp :\Downarrow !G^1[t/x]}\;!}{\vdash (\Gamma^0)^\perp :\Downarrow \exists x !G^1}\;\exists \quad\longmapsto\quad \dfrac{\begin{array}{c}\tau A(\Pi)\\ \Gamma \vdash_I G[t/x]\end{array}}{\Gamma \vdash_I \exists x G}\;\exists R$$

Procedure τBl: This procedure translates proofs of $\vdash (\Gamma^0)^\perp : R^1 \Downarrow (D^0)^\perp$ under the assumption that $(D^0)^\perp \in (\Gamma^0)^\perp$.

1. Identity: Here B is a negative literal (atom).

$$\dfrac{}{\vdash (\Gamma^0)^\perp : B \Downarrow B^\perp}\; I_1,\ B^\perp \in (\Gamma^0)^\perp \quad\longmapsto\quad \dfrac{}{\Gamma \vdash_I B}\; ID$$

2. If B is a positive literal, then $B^\perp$ could not have been selected for focus.

3. No case also for $(D'^0)^\perp = (true^0)^\perp = \top^\perp = 0$.

4. $(D^0)^\perp = (false^0)^\perp = 0^\perp = \top$:

$$\dfrac{\dfrac{}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow \top}\;\top}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow \top}\; R\Downarrow \quad\longmapsto\quad \dfrac{}{\Gamma, \mathit{false} \vdash_I R}\; \mathit{falseL}$$

5. $(D'^0)^\perp = ((D_1 \wedge D_2)^0)^\perp = (!D_1^0 \& !D_2^0)^\perp = ?(D_1^0)^\perp \oplus ?(D_2^0)^\perp$:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_i\\ \vdash (\Gamma^0)^\perp, (D_i^0)^\perp : R^1 \Uparrow\end{array}}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(D_i^0)^\perp}\;?}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D_i^0)^\perp}\; R\Downarrow}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D_1^0)^\perp \oplus ?(D_2^0)^\perp}\;\oplus \quad\longmapsto\quad \dfrac{\begin{array}{c}\tau B(\Pi_i)\\ \Gamma, D_i \vdash_I R\end{array}}{\Gamma, D_1 \wedge D_2 \vdash_I R}\;\wedge L$$

The choice in $i = 1, 2$ is reflected in the $\vdash_I$ proof.

6. $(D^0)^\perp = ((P \vee Q)^0)^\perp = (!P^0 \oplus !Q^0)\perp = ?(P^0)^\perp \& ?(Q^0)^\perp$:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdash (\Gamma^0)^\perp, (P^0)^\perp : R^1 \Uparrow\end{array}}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(P^0)^\perp}\;? \quad \dfrac{\begin{array}{c}\Pi_2\\ \vdash (\Gamma^0)^\perp, (Q^0)^\perp : R^1 \Uparrow\end{array}}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(Q^0)^\perp}\;?}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(P^0)^\perp \& ?(Q^0)^\perp}\;\&}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(P^0)^\perp \& ?(Q^0)^\perp}\; R\Downarrow \quad\longmapsto\quad \dfrac{\begin{array}{c}\tau B(\Pi_1)\\ \Gamma, P \vdash_I R\end{array} \quad \begin{array}{c}\tau B(\Pi_2)\\ \Gamma, Q \vdash_I R\end{array}}{\Gamma, P \vee Q \vdash_I R}\;\vee L$$

7. $(D'^0)^\perp = ((G \supset D)^0)^\perp = (!G^1 \multimap !D^0)^\perp = !G^1 \otimes ?(D^0)^\perp$:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash (\Gamma^0)^\perp :\Uparrow G^1\end{matrix}}{\vdash (\Gamma^0)^\perp :\Downarrow !G^1}\;! \qquad \dfrac{\dfrac{\begin{matrix}\Pi_2\\ \vdash (\Gamma^0)^\perp, (D^0)^\perp : R^1 \Uparrow\end{matrix}}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(D^0)^\perp}\;?}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D^0)^\perp}\;R\Downarrow}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow !G^1 \otimes ?(D^0)^\perp}\;\otimes \qquad \longmapsto \qquad \dfrac{\begin{matrix}\tau A(\Pi_1)\\ \Gamma \vdash_I G\end{matrix} \quad \begin{matrix}\tau B(\Pi_2)\\ \Gamma, D \vdash_I R\end{matrix}}{\Gamma, D \supset G \vdash_I R}\;\supset L$$

Note that the splitting of the context is determined by the presence of ! in front of G^1 (focusing forces immediate promotion). The cases for $\neg A$ are absorbed into the general case for implication.

8. $(D'^0)^\perp = ((\exists x D)^0)^\perp = (\exists x !D^0)^\perp = \forall x\, ?(D^0)^\perp$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdash (\Gamma^0)^\perp, (D^0)^\perp : R^1 \Uparrow\end{matrix}}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(D^0)^\perp}\;?}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow \forall x\, ?(D^0)^\perp}\;\forall}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow \forall x\, ?(D^0)^\perp}\;R\Downarrow \qquad \longmapsto \qquad \dfrac{\begin{matrix}\tau B(\Pi)\\ \Gamma, D \vdash_I R\end{matrix}}{\Gamma, \exists x D \vdash_I R}\;\exists L$$

Implicit here is the usual restriction on variable x.

9. $(D'^0)^\perp = ((\forall x D)^0)^\perp = (\forall x !D^0)^\perp = \exists x\, ?(D^0)^\perp$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi\\ \vdash (\Gamma^0)^\perp, (D^0)^\perp[t/x] : R^1 \Uparrow\end{matrix}}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(D^0)^\perp[t/x]}\;?}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D^0)^\perp[t/x]}\;R\Downarrow}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow \exists x\, ?(D^0)^\perp}\;\exists \qquad \longmapsto \qquad \dfrac{\begin{matrix}\tau B(\Pi)\\ \Gamma, D[t/x] \vdash_I R\end{matrix}}{\Gamma, \forall x D \vdash_I R}\;\forall L$$

This completes the construction of the procedure. That it is one-to-one follows from the deterministic nature of each case of the procedure. The choices made in deciding between selection left or right rules and in deciding instances of the $\exists R$ and $\wedge L$ rules are reflected in the corresponding focused proofs. We sketch the structure of the inverse translation below:

1. τA^{-1}:

$$\begin{matrix}\Psi\\ \Gamma \vdash_I G\end{matrix} \qquad \longmapsto \qquad \dfrac{\begin{matrix}\tau B^{-1}(\Psi)\\ \vdash (\Gamma^0)^\perp : G^1 \Uparrow\end{matrix}}{\vdash (\Gamma^0)^\perp :\Uparrow G^1}\;R\Uparrow$$

2. τB^{-1}:

$$\begin{matrix}\Psi\\ \Gamma \vdash_I G\end{matrix} \quad \longmapsto \quad \dfrac{\begin{matrix}\tau Br^{-1}(\Psi)\\ \vdash (\Gamma^0)^\perp :\Downarrow G^1\end{matrix}}{\vdash (\Gamma^0)^\perp : G^1 \Uparrow}\;[D_1] \quad \text{if the last rule in } \Psi \text{ is a right rule}$$

$$\begin{matrix}\Psi\\ \Gamma \vdash_I G\end{matrix} \quad \longmapsto \quad \dfrac{\begin{matrix}\tau Bl^{-1}(\Psi)\\ \vdash (\Gamma^0)^\perp : G^1 \Downarrow (D^0)^\perp\end{matrix}}{\vdash (\Gamma^0)^\perp : G^1 \Uparrow}\;[D_2] \quad \text{if the last rule in } \Psi \text{ is a left rule}$$

3. τBr^{-1} and τBl^{-1} : Invert the translations given above, dispatching on the forms of the principal formula of the last inference rule in the $\vdash_I$ proof. Recursively call τA^{-1} or τB^{-1} on the premises of the $\vdash_I$ proof as the constructions indicate.

□

An important characteristic of this translation is that the polarity of atoms does not affect the structure of proofs, as they will in later translations.

We note that this translation procedure remains bijective even in the presence of 0. Even a proof such as

$$\frac{\mathit{false} \vdash_I A \quad \mathit{false}, B \vdash_I C}{\mathit{false}, A \supset B \vdash_I C} \supset L$$

has *exactly one* corresponding focused proof. The ! exponential forces contexts to be split in the "right way," preventing occurrences of multiple-conclusion sequents even when 0 is present in the context. Accounting for 0 will cause some disruption when we consider alternative translations.

4 Translating LJQ and LJT

In this section we consider two sequent calculi for intuitionistic logic that have been based on different focusing strategies. We show that we can prove the soundness and completeness of these systems within the framework of defining translations of intuitionistic logic into linear logic.

4.1 LJQ'

We first consider the proof system LJQ' for intuitionistic logic that is given in [DL06], which generalizes the LJQ calculus of Herbelin [Her95], itself a derivative of LKQ [DJS95]. This calculus was designed to include strong characteristics of focusing. There are two style of sequents, $\Gamma \rightarrow A$ and $\Gamma \Rightarrow A$, which represent focused and unfocused sequents. Rules for this proof system are given in Figure 5.

$$\frac{}{\Gamma, \mathit{false} \Rightarrow R}\ \mathit{Lfalse} \qquad \frac{\Gamma \rightarrow R}{\Gamma \Rightarrow R}\ \mathit{Der} \qquad \frac{}{\Gamma, C \rightarrow C}\ \mathit{Ax},\ \text{atomic } C$$

$$\frac{\Gamma \rightarrow A \quad \Gamma, B \Rightarrow R}{\Gamma, A \supset B \Rightarrow R}\ L\supset' \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \rightarrow A \supset B}\ R\supset'$$

$$\frac{\Gamma, A \Rightarrow R \quad \Gamma, B \Rightarrow R}{\Gamma, A \vee B \Rightarrow R}\ L\vee' \qquad \frac{\Gamma \rightarrow A_i}{\Gamma \rightarrow A_1 \vee A_2}\ R\vee'$$

$$\frac{\Gamma, A, B \Rightarrow R}{\Gamma, A \wedge B \Rightarrow R}\ L\wedge' \qquad \frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B}\ R\wedge'$$

Figure 5: Proof rules for LJQ'.

We give a translation of intuitionistic logic into linear logic in such a way that focused proofs of the translated formula correspond to LJQ' proofs. For LJQ' we devise the "j/q" translation (for lack of better name) for intuitionistic formulas, which is given in Table 2. A crucial aspect of the translation not shown in the table is that *all atoms are assigned positive polarity.* The "$\otimes 1$" device rewrites any formula into synchronous form. This is an acceptable alternative to ! in certain cases. By appealing to cut-elimination and the soundness and completeness of focusing, it is an easy matter to establish the completeness of this translation.

F	F^q (right)	F^j (left)	$(F^j)^\perp$
atom T	T	T	$T^\perp$
false	0	0	$\top$
$A \wedge B$	$A^q \otimes B^q$	$!A^j \otimes !B^j$	$?(A^j)^\perp \mathbin{⅋} ?(B^j)^\perp$
$A \vee B$	$A^q \oplus B^q$	$!A^j \oplus !B^j$	$?(A^j)^\perp \mathbin{\&} ?(B^j)^\perp$
$A \supset B$	$(!A^j \multimap B^q) \otimes 1$	$A^q \multimap !B^j$	$A^q \otimes ?(B^j)^\perp$

Table 2: The j/q translation for LJQ'

Proposition 2 *If* $\vdash (\Gamma^0)^\perp :\Uparrow R^1$ *is provable then* $\vdash (\Gamma^j)^\perp :\Uparrow R^q$ *is provable.*

Proof For each formula F it can be shown that $!F^j \vdash_{LL} !F^0$ and $F^1 \vdash_{LL} F^q$ are both provable. The proof is by mutual induction. We show two cases:

$$\dfrac{\dfrac{\dfrac{A^1 \vdash_{LL} A^q}{A^1 \& B^1 \vdash_{LL} A^q}\ \&L}{!(A^1 \& B^1) \vdash_{LL} A^q}\ !L \quad \dfrac{\dfrac{B^1 \vdash_{LL} B^q}{A^1 \& B^1 \vdash_{LL} B^q}\ \&L}{!(A^1 \& B^1) \vdash_{LL} B^q}\ !L}{\dfrac{!(A^1 \& B^1), !(A^1 \& B^1) \vdash_{LL} A^q \otimes B^q}{!(A^1 \& B^1) \vdash_{LL} A^q \otimes B^q}\ C}\ \otimes$$

$$\dfrac{\dfrac{\dfrac{!A^1 \vdash_{LL} A^q \quad !B^j \vdash_{LL} !B^0}{A^q \multimap !B^j \vdash_{LL} !A^1 \multimap !B^0}\ \multimap R}{!(A^q \multimap !B^j) \vdash_{LL} !A^1 \multimap !B^0}\ !L}{!(A^q \multimap !B^j) \vdash_{LL} !(!A^1 \multimap !B^0)}\ !R$$

The remaining premises in both cases follow by inductive hypothesis. Other cases are similar. Thus by repeated applications of cut and cut-elimination (and the completeness of focusing), a proof of $\vdash (\Gamma^0)^\perp :\Uparrow R^1$ can be transformed into one of $\vdash (\Gamma^j)^\perp :\Uparrow R^q$. □

Since the j/q translation has fewer uses of !, we can no longer rely on this connective to guarantee that the splitting of contexts will preserve sequents in intuitionistic form. The possible presence of 0 further complicates matters, forcing us into some compromises. The following lemma is required.

Lemma 3 *If there is no proof of* $!\Gamma^j \vdash_{LL} 0$ *then there is also no proof of* $!\Gamma^j \vdash_{LL}$ *or of* $!\Gamma^j \vdash_{LL} \Delta^q$ *with more than one formula in* Δ^q.

Proof Proceed by contradiction. Assume there is such a proof and consider such a proof of minimal height. Instances of the initial rule are immediately ruled out by the assumptions of the lemma. By inspecting the premises of each possible inference figure for the final rule of the proof, we see that in each case at least one premise will also have an empty right-hand side (or a right-hand side with more than one formula), which contradicts the shortest-height assumption. □

The following technical lemma is also needed because of 0 and the removal of ! from the left-side translation of $\supset$. It states that focused proofs with non-intuitionistic sequents can be mapped into the "right form". The intricacy of this lemma is due to the direction of the focus arrow: it is $\Downarrow$.

Lemma 4 *If there is a proof of* $\vdash (\Gamma^j)^\perp : R^q \Downarrow A^q$ *then there is also a proof of* $\vdash (\Gamma^j)^\perp :\Downarrow A^q$.

Proof The proof of this lemma will appeal to cut-elimination directly within the LLF focusing calculus. The *focused cut* rule was given by Andreoli in [And92] as

$$\dfrac{\vdash \Gamma : \Delta_1 \Downarrow H \quad \vdash \Gamma : \Delta_2 \Downarrow H^\perp}{\vdash \Gamma : \Delta_1 \Delta_2 \Uparrow}\ \Downarrow Cut$$

Cut-elimination was also proved for this form of cut.

If $\vdash (\Gamma^j)^\perp : R^q \Downarrow A^q$ is provable then, depending on whether A^q is asynchronous or not, either $\vdash (\Gamma^j)^\perp : R^q \Uparrow A^q$ or $\vdash (\Gamma^j)^\perp : R^q, A^q \Uparrow$ is also provable, which by Lemma 3 means that $\vdash (\Gamma^j)^\perp :\Uparrow 0$ is provable.

First, it can be ruled out that $\vdash (\Gamma^j)^\perp : R^q \Downarrow A^q$ is the conclusion of an initial rule because the translation excludes the possibility that $R^q = (A^q)^\perp$.

We show that in the proof of $\vdash (\Gamma^j)^\perp : R^q \Downarrow A^q$, R^q need never be selected for focus by a $[D_1]$ rule. Suppose there is such a selection. Then it must occur in a subproof with final rule of the form

$$\dfrac{\vdash (\Theta^j)^\perp : \Delta \Downarrow R^q}{\vdash (\Theta^j)^\perp : R^q, \Delta \Uparrow}\ [D_1]$$

By a simple observation on LLF inference rules, $(\Theta^j)^\perp$ is a superset of $(\Gamma^j)^\perp$. Thus $!\Theta \vdash_{LL} 0$ is also provable. Then by a cut with $0 \vdash_{LL} (R^q)^\perp$, $\Theta \vdash_{LL} (R^q)^\perp$ and consequently $\vdash (\Theta^j)^\perp :\Uparrow (R^q)^\perp$ are provable. R^q is either a synchronous formula or a positive atom: it cannot be a negative atom since then the above $[D_1]$ rule would not be valid; if it where asynchronous, then it is in the wrong section of the sequent. Therefore $(R^q)^\perp$ is either an asynchronous formula or a negative atom.The following inference is therefore valid:

$$\frac{\vdash (\Theta^j)^\perp :\Uparrow (R^q)^\perp}{\vdash (\Theta^j)^\perp :\Downarrow (R^q)^\perp}\ R\Downarrow$$

Then by a focused cut we have:

$$\frac{\vdash (\Theta^j)^\perp : \Delta \Downarrow R^q \quad \vdash (\Theta^j)^\perp :\Downarrow (R^q)^\perp}{\vdash (\Theta^j)^\perp : \Delta \Uparrow}\ \Downarrow Cut$$

Thus subproofs of $\vdash (\Theta^j)^\perp : R^q, \Delta \Uparrow$ can be replaced by proofs of $\vdash (\Theta^j)^\perp : \Delta \Uparrow$. By repeated uses of focused cut-elimination, we therefore have that $\vdash (\Theta^j)^\perp :\Downarrow A^q$ is also provable. □

The above lemma underscores the problems one can encounter with intuitionistic negation. A perfect bijection between focused proofs and LJQ' proofs cannot be formed, since a focused linear logic proof with an inconsistent context may now involve none-intuitionistic sequents. The following theorem will still allow us to make the same meta-level arguments, however.

Theorem 5 *Let* $(\Gamma^j)^\perp = \{(D^j)^\perp \mid D \in \Gamma\}$ *and let all atoms be assigned positive polarity. The following holds:*

1. $\vdash (\Gamma^j)^\perp :\Uparrow R^q$ *is provable if and only if* $\Gamma \Rightarrow R$ *is provable.*
2. *There is an injective mapping from proofs of* $\Gamma \Rightarrow R$ *to focused proofs of* $\vdash (\Gamma^j)^\perp :\Uparrow R^q$.
3. *There is a bijective mapping between proofs of* $\Gamma \Rightarrow R$ *and proofs of* $\vdash (\Gamma^j)^\perp :\Uparrow R^q$ *that do not contain sequents of the form* $\vdash (\Delta^j)^\perp :\Uparrow$.

Proof We outline a proof of Part 3 that will also provide proofs of Parts 1 and 2. The proof is again by a constructive translation. Since all R^q (positive) formulas are synchronous or atomic, a focused proof must end in

$$\frac{\vdash (\Gamma^j)^\perp : R^q \Uparrow}{\vdash (\Gamma^j)^\perp :\Uparrow R^q}\ R\Uparrow$$

We define two mutually recursive procedures:

1. $\tau^{\Rightarrow}$ translates proofs of $\vdash (\Gamma^j)^\perp : R^q \Uparrow$ to proofs of $\Gamma \Rightarrow R$
2. $\tau^{\rightarrow}$ translates proofs of $\vdash (\Gamma^j)^\perp :\Downarrow R^q$ to proofs of $\Gamma \rightarrow R$

The top-level procedure is $\tau^{\Rightarrow}$. The mutual-inductive measure is the height of either proofs category of proof[2].

Procedure $\tau^{\Rightarrow}$:

1.

$$\frac{\begin{array}{c}\Pi\\ \vdash (\Gamma^j)^\perp :\Downarrow R^q\end{array}}{\vdash (\Gamma^j)^\perp : R^q \Uparrow}\ [D_1] \qquad \longrightarrow \qquad \frac{\begin{array}{c}\tau^{\rightarrow}(\Pi)\\ \Gamma \rightarrow R\end{array}}{\Gamma \Rightarrow R}\ Der$$

The $[D_2]$ rules represent left rules that picks a formula $(D^j)^\perp$ from $(\Gamma^j)^\perp$ for focus

[2]A subtle but important difference: the inductive measure for the proof of Proposition 1 (for the 0/1 translation) is *only* the height of proofs of the form $\vdash (\Gamma^0)^\perp : R^1 \Uparrow$.

2. $(D^j)^\perp = (\mathit{false}^j)^\perp = \top$

$$\dfrac{\dfrac{\dfrac{}{\vdash (\Gamma^j)^\perp : R^q \Uparrow \top}\ \top}{\vdash (\Gamma^j)^\perp : R^q \Downarrow \top}\ R\Downarrow}{\vdash (\Gamma^j)^\perp : R^q \Uparrow}\ [D_2] \qquad \longrightarrow \qquad \dfrac{}{\Gamma, \mathit{false} \Rightarrow R}\ L\mathit{false}$$

3. $(D^j)^\perp = ((A \wedge B)^j)^\perp = ?(A^j)^\perp \parr ?(B^j)^\perp$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\Pi}{\vdash (\Gamma^j)^\perp, (B^j)^\perp, (A^j)^\perp : R^q \Uparrow}}{\vdash (\Gamma^j)^\perp, (B^j)^\perp : R^q \Uparrow ?(A^j)^\perp}\ ?}{\vdash (\Gamma^j)^\perp : R^q \Uparrow ?(A^j)^\perp, ?(B^j)^\perp}\ ?}{\vdash (\Gamma^j)^\perp : R^q \Uparrow ?(A^j)^\perp \parr ?(B^j)^\perp}\ \parr}{\vdash (\Gamma^j)^\perp : R^q \Downarrow ?(A^j)^\perp \parr ?(B^j)^\perp}\ R\Downarrow}{\vdash (\Gamma^j)^\perp : R^q \Uparrow}\ [D_2] \qquad \longrightarrow \qquad \dfrac{\overset{\tau^{\Rightarrow}(\Pi)}{\Gamma, A, B \Rightarrow R}}{\Gamma, A \wedge B \Rightarrow R}\ L\wedge'$$

4. $(D^j)^\perp = ((A \vee B)^j)^\perp = ?(A^j)^\perp \& ?(B^j)^\perp$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\Pi_1}{\vdash (\Gamma^j)^\perp, (A^j)^\perp : R^q \Uparrow}}{\vdash (\Gamma^j)^\perp : R^q \Uparrow ?(A^j)^\perp}\ ? \quad \dfrac{\overset{\Pi_2}{\vdash (\Gamma^j)^\perp, (B^j)^\perp : R^q \Uparrow}}{\vdash (\Gamma^j)^\perp : R^q \Uparrow ?(B^j)^\perp}\ ?}{\vdash (\Gamma^j)^\perp : R^q \Uparrow ?(A^j)^\perp \& ?(B^j)^\perp}\ \&}{\vdash (\Gamma^j)^\perp : R^q \Downarrow ?(A^j)^\perp \& ?(B^j)^\perp}\ R\Downarrow}{\vdash (\Gamma^j)^\perp : R^q \Uparrow}\ [D_2] \qquad \longrightarrow \qquad \dfrac{\overset{\tau^{\Rightarrow}(\Pi_1)}{\Gamma, A \Rightarrow R} \quad \overset{\tau^{\Rightarrow}(\Pi_2)}{\Gamma, B \Rightarrow R}}{\Gamma, A \vee B \Rightarrow R}\ L\vee'$$

5. $(D^j)^\perp = (A \supset B^j)^\perp = A^q \otimes ?(B^j)^\perp$. Two focused proofs are possible, depending on how the context is split.

   (a)

$$\dfrac{\dfrac{\overset{\Xi}{\vdash (\Gamma^j)^\perp :\Downarrow A^q} \quad \dfrac{\dfrac{\overset{\Pi}{\vdash (\Gamma^j)^\perp, (B^j)^\perp : R^q \Uparrow}}{\vdash (\Gamma^j)^\perp : R^q \Uparrow ?(B^j)^\perp}\ ?}{\vdash (\Gamma^j)^\perp : R^q \Downarrow ?(B^j)^\perp}\ R\Downarrow}{\vdash (\Gamma^j)^\perp : R^q \Downarrow A^q \otimes ?(B^j)^\perp}\ \otimes}{\vdash (\Gamma^j)^\perp : R^q \Uparrow}\ [D_2] \qquad \longrightarrow \qquad \dfrac{\overset{\tau^{\rightarrow}(\Xi)}{\Gamma \rightarrow A} \quad \overset{\tau^{\Rightarrow}(\Pi)}{\Gamma, B \Rightarrow R}}{\Gamma, A \supset B \Rightarrow R}\ L\supset'$$

   (b)

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash (\Gamma^j)^\perp : R^q \Downarrow A^q} \quad \dfrac{\dfrac{\dfrac{\vdots}{\vdash (\Gamma^j)^\perp, (B^j)^\perp :\Uparrow}}{\vdash (\Gamma^j)^\perp :\Uparrow ?(B^j)^\perp}\ ?}{\vdash (\Gamma^j)^\perp :\Downarrow ?(B^j)^\perp}\ R\Downarrow}{\vdash (\Gamma^j)^\perp : R^q \Downarrow A^q \otimes ?(B^j)^\perp}\ \otimes}{\vdash (\Gamma^j)^\perp : R^q \Uparrow}\ [D_2] \qquad \longrightarrow \qquad \dfrac{\overset{\tau^{\rightarrow}(\Xi)}{\Gamma \rightarrow A} \quad \overset{\tau^{\Rightarrow}(\Pi)}{\Gamma, B \Rightarrow R}}{\Gamma, A \supset B \Rightarrow R}\ L\supset'$$

Both proofs translate to the *same* LJQ' proof, thus the mapping is not one to one. In the reverse direction, we map the LJQ' proof only to the first focused proof (5a), thus forming an injection. This mapping is well defined, and preserves soundness, because the existence of proof 5b entails the existence of proof 5a. That is, by Lemma 3, proof 5b is possible only if $!\Gamma^j \vdash_{LL} 0$ is provable. By Lemma 4, it follows that the premises of proof 5b implies the premises of proof 5a (the right premise follows from a cut with $0 \vdash_{LL} R^q$). Thus even focused proofs that appear malformed correspond to valid derivations in LJQ'. Equivalence of provability is preserved despite the absence of a complete bijection.

Procedure $\tau^{\rightarrow}$:

1. $R^q =$ atom C, which has been assigned positive polarity:

$$\frac{}{\vdash (\Gamma^j)^\perp :\Downarrow C,\ C^\perp \in (\Gamma^j)^\perp}\ [I_2] \qquad \longrightarrow \qquad \frac{}{\Gamma, C \to C}\ Ax$$

2. $R^q = (A \wedge B)^q = A^q \otimes B^q$:

$$\frac{\begin{matrix}\Xi_1 \\ \vdash (\Gamma^j)^\perp :\Downarrow A^q\end{matrix} \quad \begin{matrix}\Xi_2 \\ \vdash (\Gamma^j)^\perp :\Downarrow B^q\end{matrix}}{\vdash (\Gamma^j)^\perp :\Downarrow A^q \otimes B^q}\ \otimes \qquad \longrightarrow \qquad \frac{\begin{matrix}\tau^{\rightarrow}(\Xi_1) \\ \Gamma \to A\end{matrix} \quad \begin{matrix}\tau^{\rightarrow}(\Xi_2) \\ \Gamma \to B\end{matrix}}{\Gamma \to A \wedge B}\ R\wedge'$$

3. $R^q = (A_1 \vee A_2)^q = A_1^q \oplus A_2^q$:

$$\frac{\begin{matrix}\Xi_i \\ \vdash (\Gamma^j)^\perp :\Downarrow A_i\end{matrix}}{\vdash (\Gamma^j)^\perp :\Downarrow A_1^q \oplus A_2^q}\ \oplus \qquad \longrightarrow \qquad \frac{\begin{matrix}\tau^{\rightarrow}(\Xi_i) \\ \Gamma \to A_i\end{matrix}}{\Gamma \to A_1 \vee A_2}\ R\vee'$$

4. $R^q = (A \supset B)^q = (!A^j \multimap B^q) \otimes 1 = (B^q \parr ?(A^j)^\perp) \otimes 1$:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\Xi \\ \vdash (\Gamma^j)^\perp, (A^j)^\perp : B^q \Uparrow\end{matrix}}{\vdash (\Gamma^j)^\perp, (A^j)^\perp :\Uparrow B^q}\ R\Uparrow}{\vdash (\Gamma^j)^\perp :\Uparrow B^q, ?(A^j)^\perp}\ ?}{\vdash (\Gamma^j)^\perp :\Uparrow B^q \parr ?(A^j)^\perp}\ \parr}{\vdash (\Gamma^j)^\perp :\Downarrow B^q \parr ?(A^j)^\perp}\ R\Downarrow \quad \dfrac{}{\vdash (\Gamma^j)^\perp :\Downarrow 1}\ 1}{\vdash (\Gamma^j)^\perp :\Downarrow (B^q \parr ?(A^j)^\perp) \otimes 1}\ \otimes \qquad \longrightarrow \qquad \frac{\begin{matrix}\tau^{\Rightarrow}(\Xi) \\ A, \Gamma \Rightarrow B\end{matrix}}{\Gamma \to A \supset B}\ R\supset'$$

Again, the $R \Uparrow$ rule is justified because B^q can only be a synchronous formula or an atom.

□

Unlike in the translation of the $\vdash_I$ proofs, here the two procedures $\tau^{\Rightarrow}$ and $\tau^{\rightarrow}$ are mutually recursive. Focus is *continued* whenever $\tau^{\rightarrow}$ is called recursively. The direction of the focus arrow may remain $\Downarrow$. Since all focusing occurs on the right-hand side, the *Der* rule starts a *critical section* that ends either when a proof is complete or when a $A \supset B$ formula is encountered. The focusing characteristics of LJQ' are clearly reflected by focused proofs in linear logic, given the appropriate translation.

Despite some difficulties caused by 0, we were able to establish meaningful connections between focused proofs and LJQ' proofs. Combined with the almost immediate fact that LJQ' is sound with respect to intuitionistic logic, we have "completed the circle," establishing the equivalence of provability among intuitionistic proofs, LJQ' proofs, and focused proofs under the j/q translation.

Corollary 6 *LJQ' is sound and complete with respect to intuitionistic provability.*

One may be tempted to ask at this point: what is the point of translating one focusing calculus into another? The translations place LJQ' and similar systems in a broader context. Not only important properties such as completeness can be proved, possible variations and enhancements also become apparent. For example, the "$\otimes 1$" device in the positive translation of $(A \supset B)$, which forces all right-formulas to be synchronous, can be eliminated while trivially preserving all soundness and completeness results. The modified translation leads to a new rule at the intuitionistic level:

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R{\supset}''$$

Moreover, this rule can be applied exhaustively before switching to other rules: in a focused proof $(A \supset B)^q$ is asynchronous and in a linear context. Most significantly, all atoms in LJQ have uniform, positive polarity. Backchaining (focusing on the left) is not supported.

It is interesting to observe that LJQ' was formulated with some of the same goals as that of our present effort: to provide a proof-theoretic framework for reasoning about other systems. LJQ' was shown to be capable of encompassing several other systems. Later we present the LJF proof system (Section 6.4) which appears to be more general in this respect: in particular, we show how LJQ' can be captured inside LJF (Section 8).

4.2 LJT'

The complement to LJQ is LJT, also called MJ [Her95, DP98] (a version with full connectives appears in [How98]). This system supports left-side focus and is noted for its ability to distinguish a *stoup* inducing a head-variable in lambda-term reduction. The translation for this calculus is interesting in that it is virtually the same as Girard's original (non-polarized) translation in [Gir87]. This correspondence was also implicit in the analysis of [DJS95]. The only possible need for asymmetry is in forcing asynchronous decompositions on the right hand side to be delayed, since that is not a feature of the stated calculus. That is, we may need to use the $\otimes 1$ device, already seen for LJQ, in translating right-side occurrences of $A \supset B$ and $A \wedge B$, since $\wedge$ now has the & interpretation.

Without this device, the translation is identical to Girard's original. LJT/MJ was developed to give a to one-to-one correspondence between sequent calculus proofs and normalized natural deductions. Girard's original translation was also motivated by normalization, in particular for system F.

Instead of repeating the same process of translating focused proofs to LJT/MJ proofs, this time we extract a "*LJT'*" calculus directly from Girard's original translation. It differs from LJT/MJ only in a few additional focusing features. Girard's translation is repeated here for convenience[3]:

$A^\circ = A$ for atoms A,
$\mathit{false}^\circ = 0$, $\mathit{true}^\circ = 1$,
$(A \wedge B)^\circ = A^\circ \& B^\circ$,
$(A \vee B)^\circ = !\,A^\circ \oplus !\,B^\circ$,
$(A \supset B)^\circ = !\,A^\circ \multimap B^\circ$,
$(\forall x A)^\circ = \forall x A^\circ$, and
$(\exists x A)^\circ = \exists x\, !\,A^\circ$.

The result of Proposition 2 can be similarly stated for this translation. That is, proofs of $!\Gamma^0 \vdash_{LL} R^1$ can be translated into proofs of $!\Gamma^\circ \vdash_{LL} R^\circ$. Due to the position of ! in the translation of implications, Lemmas 3 and 4 are *not* required for this translation. Of course, we also know that the translation is correct from [Gir87]. In LJT/MJ one can arbitrarily switch to the left-focusing phase before asynchronous formulas (on the right) are completely decomposed. To clearly separate these stages in LJT', we use the sequent notation $\Gamma \Rightarrow [R]$ to indicate that R is not asynchronous. Such a sequent corresponds to the focused sequent $\vdash \Gamma^{\circ\perp} : R^\circ \Uparrow$. We also introduce the notation $\Gamma \xrightarrow{A} [R]$ for left-focusing on the "stoup" formula A. Before

[3]The only possible exception to the pure symmetry in Girard's translation would be in treating *true* on the left-hand side as $\top$ instead of 1, since the latter would lead to pointless loops.

$$\frac{}{\Gamma \xrightarrow{A} [A]}\ Ax,\ \text{atom } A \qquad \frac{}{\Gamma \xrightarrow{\mathit{false}} [R]}\ \mathit{falseL} \qquad \frac{}{\Gamma \Rightarrow [\mathit{true}]}\ \mathit{trueR}$$

$$\frac{\Gamma \Rightarrow [R]}{\Gamma \Rightarrow R}\ []_r,\ R \text{ not asynchronous} \qquad \frac{\Gamma, B \xrightarrow{B} [R]}{\Gamma, B \Rightarrow [R]}\ \mathit{Choose}$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma \xrightarrow{A_i} [R]}{\Gamma \xrightarrow{A_1 \wedge A_2} [R]}\ \wedge L$$

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow [A_1 \vee A_2]}\ \vee R \qquad \frac{\Gamma, A \Rightarrow [R] \quad \Gamma, B \Rightarrow [R]}{\Gamma \xrightarrow{A \vee B} [R]}\ \vee L$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ \supset R \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \xrightarrow{B} [R]}{\Gamma \xrightarrow{A \supset B} [R]}\ \supset L$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \forall y A}\ \forall R \qquad \frac{\Gamma \xrightarrow{A[t/x]} [R]}{\Gamma \xrightarrow{\forall x A} [R]}\ \forall L \qquad \frac{\Gamma \Rightarrow A[t/x]}{\Gamma \Rightarrow [\exists x A]}\ \exists R \qquad \frac{\Gamma, A \Rightarrow [R]}{\Gamma \xrightarrow{\exists y A} [R]}\ \exists L$$

In $\forall R$ and $\exists L$, y is not free in Γ, R.

Figure 6: LJT' Induced from Girard's Original Translation

we proceed however, it is necessary to point out that *all atoms of LJT' are assigned negative polarity.* The rules are found in Figure 6.

In this intuitionistic setting, we use the term "asynchronous" for formulas of the forms $\forall x A$, $A \supset B$ and $A \wedge B$. The rules of LJT' are directly derived from the focused proofs. We list a few representative examples:

1. $[]_r$ corresponds to

$$\frac{\vdash \Gamma^{\circ\perp} : R^\circ \Uparrow}{\vdash \Gamma^{\circ\perp} :\Uparrow R^\circ}\ R\Uparrow$$

2. *Choose* (a.k.a. contraction or absorption) corresponds to

$$\frac{\vdash \Gamma^{\circ\perp}, P^{\circ\perp} : R^\circ \Downarrow P^{\circ\perp}}{\vdash \Gamma^{\circ\perp}, P^{\circ\perp} : R^\circ \Uparrow}\ [D_2]$$

3. $\wedge L$ corresponds to

$$\frac{\vdash \Gamma^{\circ\perp} : R^\circ \Downarrow A_i^{\circ\perp}}{\vdash \Gamma^{\circ\perp} : R^\circ \Downarrow A_1^{\circ\perp} \oplus A_2^{\circ\perp}}\ \oplus$$

4. $\supset R$ corresponds to

$$\cfrac{\cfrac{\vdash \Gamma^{\circ\perp}, A^{\circ\perp} :\Uparrow B^\circ}{\vdash \Gamma^{\circ\perp} :\Uparrow B^\circ, ?A^{\circ\perp}}\ ?}{\vdash \Gamma^{\circ\perp} :\Uparrow B^\circ \parr ?A^{\circ\perp}}\ \parr$$

5. $\supset L$ corresponds to

$$\cfrac{\cfrac{\vdash \Gamma^{\circ\perp} :\Uparrow A^\circ}{\vdash \Gamma^{\circ\perp} :\Downarrow !A^\circ}\ ! \qquad \vdash \Gamma^{\circ\perp} : R^\circ \Downarrow B^{\circ\perp}}{\vdash \Gamma^{\circ\perp} : R^\circ \Downarrow !A^\circ \otimes B^{\circ\perp}}\ \otimes$$

6. $\vee L$ corresponds to

$$\dfrac{\dfrac{\vdash \Gamma^{\circ\perp}, A^{\circ\perp} : R^\circ \Uparrow}{\vdash \Gamma^{\circ\perp} : R^\circ \Uparrow ?A^{\circ\perp}}\,? \quad \dfrac{\vdash \Gamma^{\circ\perp}, B^{\circ\perp} : R^\circ \Uparrow}{\vdash \Gamma^{\circ\perp} : R^\circ \Uparrow ?B^{\circ\perp}}\,?}{\dfrac{\vdash \Gamma^{\circ\perp} : R^\circ \Uparrow ?A^{\circ\perp} \& ?B^{\circ\perp}}{\vdash \Gamma^{\circ\perp} : R^\circ \Downarrow ?A^{\circ\perp} \& ?B^{\circ\perp}}\, R\Downarrow}\,\&$$

7. $\exists R$ corresponds to

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma^{\circ\perp} :\Uparrow A^\circ[t/x]}{\vdash \Gamma^{\circ\perp} :\Downarrow !A^\circ[t/x]}\,!}{\vdash \Gamma^{\circ\perp} :\Downarrow \exists x!A^\circ}\,\exists}{\vdash \Gamma^{\circ\perp} : \exists x!A^\circ \Uparrow}\,[D_1]$$

The correspondence is a bijection, as in the case of the 0/1 translation. As is usually the case, the soundness of LJT' is almost immediate. The "grand tour" through linear logic thus yields another completeness result.

Corollary 7 *LJT' is sound and complete with respect to intuitionistic provability.*

LJT' and LJQ' complement each other: one contains only left focus, using only negative atoms, while the other has only right focus with positive atoms. One favors forward-chaining, the other back-chaining. A natural question arises: is it possible to combine the features of the two system into a unified setting. The answer, as will be seen in Section 6, is yes. However, first we will examine another intuitionistic system that exhibits an important feature towards this end: the usage of atoms of mixed polarity.

5 Mixed Polarities and λRCC

The system λRCC [JNS05] was developed as a proof-theoretic foundation for studying concurrency, constraints, and higher-order features in programming languages. We present the λRCC proof system and provide a soundness and completeness theorem for it via a translation to linear logic. We then explore ways to strengthen its focusing features.

5.1 λRCC

λRCC can be represented as an intuitionistic proof system. It defines three distinct set of atoms – c, E, and A – and reduces their distinct down to their head predicates symbols. The language excludes disjunctions on the left, and is specifically designed to treat left and right formulas in an asymmetrical manner similar to that of hereditary Harrop formulas. The *definite clauses* and *goal* formulas of the language are as follows.

$$D ::= true \mid c \mid E \mid D \wedge D \mid G \supset A \mid G \supset D \mid E \supset D \mid \exists x D \mid \forall x D$$
$$G ::= true \mid c \mid A \mid G \wedge G \mid D \supset G \mid G \vee G \mid \exists x G \mid \forall x G$$

A proof theoretic semantics was given for λRCC that is sound and complete with respect to provability in intuitionistic logic. A proof of a sequent of the form $\Gamma \vdash_O G$ is a proof in the intuitionistic sequent calculus with restrictions in three special cases of $\supset Left$:

$$\frac{\Gamma, G \supset A \vdash_O G \quad \overline{\Gamma, A \vdash_O A}\ ID}{\Gamma, G \supset A \vdash_O A} \supset L \qquad \frac{\overline{\Gamma, E \vdash_O E}\ ID \quad \Gamma, E, D \vdash_O G}{\Gamma, E, E \supset D \vdash_O G} \supset L$$

$$\frac{\overline{\Gamma, c \supset D \vdash_O c}\ Const \quad \Gamma, D \vdash_O G}{\Gamma, c \supset D \vdash_O G} \supset L$$

The *Const* rule consults some external constraint system $\mathcal{C}$ to determine if constraint c is valid. Specifically this external relation is written $\Gamma \vdash_{\mathcal{C}} c$. To properly capture this relation we would have to extend LLF with non-logical axioms of the form

$$\overline{\vdash \Gamma^{\perp} :\Downarrow c}\ Const$$

which raises questions concerning cut elimination that are orthogonal to the main subject of this paper. Another approach would be to also capture the $\vdash_{\mathcal{C}}$ relation as focused provability, but $\mathcal{C}$ is generally unknown. However, the principal characteristic of λRCC, from a proof-search perspective, is that of mixed polarities. This feature is already present in the coexistence of the A and E atoms.

There is another subtlety in these rules that we shall not address immediately. Note that the formula $E \supset D$ is not copied to the right premise of the second inference (Γ is a multiset). This is an important refinement for forward-chaining: there is no reason to add redundant copies of D to the specification. The refinement is valid in intuitionistic logic. As our translation into linear logic will be faithful to intuitionistic provability, one can expect that the same kind of refinement would also be admissible, and indeed it will be.

Provability in the sense of $\vdash_O$ is mostly the same as general intuitionistic provability. The specialization is only on implications on the left. Focusing is allowed only on atoms. The interesting characteristics is that it admits atoms of mixed polarity to enable mixed forward/backward-chaining. Specifically the E (and c) atoms must be translated as positives and the A atoms as negatives (this assignment of polarity is actually predicate-base as described in Section 2.2).

The translation for λRCC is therefore identical to the 0/1 translation of general intuitionistic proofs except for implications in definite clauses. There are four different types of D-formulas that involve $\supset$ in λRCC. Besides the three cases displayed above, there is also $c \supset D$, which is considered a special subcase of $G \supset D$.

According to this classification, $(c_1 \wedge c_2) \supset A$ is treated distinctly from $c_1 \supset (c_2 \supset A)$. The latter is treated as a forward-chaining rule, which adds the new rule $c_2 \supset A$ to the specification, whereas the former is to be used in a goal-directed manner. However, this does not affect our soundness and completeness results because, except in the case of atoms, we never consider the situation $D_1^l \vdash_{LL} D_2^l$. We use l and r to denote the translations of D and G formulas respectively.

- $(G \supset D)^l = !\,G^r \multimap !\,D^l = (!\,G^r \otimes ?(D^l)^\perp)^\perp$, where $G \neq c$.
- $(c \supset D)^l = c \multimap !\,D^l = (c \otimes ?(D^l)^\perp)^\perp$
- $(E \supset D)^l = E \multimap !\,D^l = (E \otimes ?(D^l)^\perp)^\perp$
- $(G \supset A)^l = !\,G^r \multimap A = (!\,G^r \otimes A^\perp)^\perp$

The rest of the translation is identical to the 0/1-translation.

A result analogous to Proposition 2 holds for relating the 0/1-translation and the l/r-translation: since it is standard, we do not present this here. The following result can be proved for λRCC in the same manner.

Theorem 8 *Let Γ be a multiset of D-formulas and A-atoms and R be a G-formula or E atom. Let $(\Gamma^l)^\perp = \{?(D^l)^\perp \mid D \in \Gamma'\}$. Let A atoms be assigned negative polarity and E and c atoms positive polarity. Then $\vdash (\Gamma^l)^\perp :\Uparrow R^r$ is provable if and only if $\Gamma \vdash_O R$ is provable. Furthermore, there is a one-to-one mapping between focused proofs under the l/r translation and $\vdash_O$ proofs.*

The proof of the result uses the same mapping between focused proof and $\vdash_I$ proofs except in the following cases, all of which are exceptions to the τBl procedure.

1. $(D'^l)^\perp = ((G \supset A)^l)^\perp = (!\,G^r \multimap A)^\perp = !\,G^r \otimes A^\perp$:

$$\dfrac{\dfrac{\overset{\Pi}{\vdash (\Gamma^l)^\perp :\Uparrow G^r}}{\vdash (\Gamma^l)^\perp :\Downarrow !\,G^r}\,! \quad \dfrac{}{\vdash (\Gamma^l)^\perp : A \Downarrow A^\perp}\,I1}{\vdash (\Gamma^l)^\perp : A \Downarrow \; !\,G^r \otimes A^\perp}\,\otimes \quad \longrightarrow \quad \dfrac{\overset{\tau A(\Pi)}{\Gamma \vdash_O G} \quad \dfrac{}{\Gamma, A \vdash_O A}\,ID}{\Gamma, G \supset A \vdash_O A}\supset L$$

   Here, focusing on $A^\perp$, which has positive polarity because A is assigned negative polarity, can only result in the identity rule, which means that $R^r = A$. This is backchaining.

2. $(D'^l)^\perp = ((E \supset D)^l)^\perp = (E \multimap !\,D^l)^\perp = E \otimes ?(D^l)^\perp$:

$$\dfrac{\dfrac{}{\vdash (\Gamma^l)^\perp :\Downarrow E}\,I_2 \quad \dfrac{\dfrac{\overset{\Pi}{\vdash (\Gamma^l)^\perp, (D^l)^\perp : R^r \Uparrow}}{\vdash (\Gamma^l)^\perp : R^r \Uparrow ?(D^l)^\perp}\,?}{\vdash (\Gamma^l)^\perp : R^r \Downarrow ?(D^l)^\perp}\,R\Downarrow}{\vdash (\Gamma^l)^\perp : R^r \Downarrow E \otimes ?(D^l)^\perp}\,\otimes \quad \longrightarrow \quad \dfrac{\dfrac{}{\Gamma \vdash_O E}\,ID \quad \overset{\tau B(\Pi)}{\Gamma, D \vdash_O R}}{\Gamma, E \supset D \vdash_O R}\supset L$$

   Notice that $E^\perp \in (\Gamma^l)^\perp$. The focus arrow $\Downarrow$ on positive atom E forces that the bounded singleton R^r is split to the right subproof, since R^r cannot be a negated atom, and that $E^\perp \in (\Gamma^l)^\perp$. This is forward-chaining.

The case for $(c \supset D)$ is similar to the case of $(E \supset D)$, with caveats on the $\vdash_C$ relation.

5.2 Enhanced Forward-Chaining

The element missing from our linear-logic embedding of λRCC is the optimization of $\supset L$ rules concerning $E \supset D$ (and $c \supset D$) formulas discussed earlier. We wish to apply a forward-chaining rule only once in each branch of a proof.

An important observation: *in LLF, the unbounded context of the premises of each inference rule is a superset of the unbounded context of its conclusion.*

From this fact it is also clear that multiple occurrences of the same element in an unbounded context are redundant (even though the context is technically a multiset). A formula in an unbounded context is selectable by a $[D_2]$ rule multiple times, since it remains in the context.

Now assume that $(\Gamma^l)^\perp$ contains $(D^l)^\perp$, $E^\perp$, and $(E \multimap {!}D^-)^\perp$, which is $(E \otimes ?(D^l)^\perp)$. Consider the following inference where $(E \otimes ?(D^l)^\perp)$ is selected for focus by a decision rule:

$$\dfrac{\dfrac{}{\vdash (\Gamma^l)^\perp :\Downarrow E}\,I_2 \qquad \dfrac{\dfrac{\dfrac{\Pi}{\vdash (\Gamma^l)^\perp, (D^l)^\perp : R^+ \Uparrow}}{\vdash (\Gamma^l)^\perp : R^+ \Uparrow ?(D^l)^\perp}\,?}{\vdash (\Gamma^l)^\perp : R^+ \Downarrow ?(D^l)^\perp}\,R\Downarrow}{\dfrac{\vdash (\Gamma^l)^\perp : R^+ \Downarrow E \otimes ?(D^l)^\perp}{\vdash (\Gamma^l)^\perp : R^+ \Uparrow}\,[D_2]}\,\otimes$$

Since $(\Gamma^l)^\perp$ already contains $(D^l)^\perp$, the entire inference can be replaced by the subproof Π after modifying it by deleting the redundant occurrence of $(D^l)^\perp$. Thus this enhancement remains admissible in our derived calculus.

5.3 Towards a Strongly Focused System with Mixed Polarities

The given proof-theoretic semantics of λRCC exhibits characteristics of focusing in a rather conservative manner. We wish to use the same techniques as in previous sections to design a new version of λRCC, one that seeks to aggressively take advantage of the possibilities of focusing.

What makes the mixing of polarities particularly difficult in such a context? In other words, why can we not combine LJT and LJQ in a straightforward way? A critical problem appears when a literal of the "wrong" polarity comes under focus. For example, assume that positive atoms are allowed in the LJT' calculus. The situation $\Gamma \xrightarrow{P} [R]$ where P is a positive atom, can result from focusing. The corresponding linear logic proof will result in a reaction rule, giving us $\vdash \Gamma^{\circ\perp} : R^*, P^\perp \Uparrow$. That is, an extra *linear* item has now been introduced into the context. Furthermore, multiple such atoms can accumulate as the proof continues. Such sequents do not correspond to intuitionistic sequents. The liberal use of ! in the 0/1 translation means that "focus" is always fleeting: focusing always stops after one level and the above situations never occurs. The translation of λRCC, which conservatively modifies the 0/1 translation, also disallows left-focus on a positive atom by a careful categorization of formulas. The strengthening of a system such as λRCC with more focusing features is thus not a trivial exercise (our first few attempts were wrong). The construction of a focused system that admits mixed polarities requires a closer examination of the concept of *polarity* and its connection to focusing.

6 Unified Focusing for Intuitionistic Logic

We have examined several variants of intuitionistic proof systems that exhibit, to one degree or another, characteristics of focusing. It has been shown that these systems can be derived naturally from focused proofs in linear logic given the appropriate translation of intuitionistic logic. It is therefore reasonable to search for the existence of a translation that encompasses all such systems in one setting: a unified focusing calculus for intuitionistic logic. To this end we take essential ideas from Girard's *Logic of Unity* [Gir93] (LU). The final translations developed here are essentially compatible with those of LU. To Girard, LU's treatment of intuitionistic logic is only partially satisfactory in that in does not yield certain expected isomorphisms between denotations of formulas. Our primary interest in developing a new intuitionistic sequent calculus, however, is in proof *search.* We show, in fact, that the polarized translations of intuitionistic logic can be motivated entirely within the context of taking maximal advantage of focused proofs.

In the following we shall use the terminology *left* asynchronous to refer to formulas A such that $A^\perp$ is synchronous, and analogously for *left* synchronous.

6.1 Optimizing Asynchronous Decomposition

One subject that we have not yet satisfactorily explored is how to take maximum advantage of asynchronous connectives. A key characteristic of focused proofs is that these connectives can be decomposed eagerly (via "don't care" non-determinism). In the SRCC calculus of the previous section, for example, we see that $\forall R$ and $\supset R$ are processed in such a manner. However, these surely are not the only possible cases. Consider the sequent $\exists x A \vdash_{LL} \exists x A$. A focused proof will immediately recognize the left-instance of $\exists$ as asynchronous and decompose it immediately. The same observation can also be made for intuitionistic logic: that $\exists L$ is invertible. However, we have so far translated intuitionistic sequents $\Gamma \vdash_I R$ strictly to the form $!\Gamma' \vdash_{LL} R'$. All left-formulas are in the scope of ! operators, disallowing their immediate decomposition. In a focused proof of

$$!\exists x!\exists y!A \vdash_{LL} \exists x \exists y A$$

the left formula must wait for a $[D_2]$ rule before being processed. Even then, the decomposition will only be one-level deep, being stopped by the exponential operator in the subformula. The same observation holds for $!A \oplus !B$ and $!A \otimes !B$. If it is valid for an intuitionistic formula to be decomposed eagerly, then why should its linear logic translation not be?

There exists formulas A such that $A \equiv !A$ holds. Girard terms these formulas *left-permeable.* That is, they can move freely between the linear and unbounded contexts on the left. These formulas are closed under the following inductive definition.

- The formulas 0, 1, and any formula of the form $!A$ are left-permeable.
- If A is left-permeable then so is $\exists x A$.
- If A and B are left-permeable then so are $A \oplus B$ and $A \otimes B$.

There are also *right-permeable* formulas where $A \equiv ?A$. Because of our interest here in intuitionistic logic, we shall only make use of left-permeables and their one-sided duals, which are naturally right-permeable. We will thus refer to them as simply *permeable.* Non-permeable formulas are also referred to as *neutral.* All permeable formulas are left-asynchronous.

The permeability analysis clearly suggests that in translating intuitionistic sequents, we should leave all permeables in the linear context. Let our new translation (for full intuitionistic logic) be labeled "–1/ +1." An intuitionistic sequent $\Gamma' \vdash_I R$ should be translated into the form

$$\vdash (\Gamma^{-1})^\perp :\Uparrow R, (\Theta^{-1})^\perp$$

where $(\Theta^{-1})^\perp$ contains all permeable formulas in $(\Gamma'^{-1})^\perp$ and $(\Gamma^{-1})^\perp$ contains the rest. Note that we are not defining a new kind of sequent, only making better use of focused sequents.

The translation should also reflect the new classification. It is easy to see that the translation of $(A \supset B)^{+1}$ should be:

- $(A \supset B)^{+1} = A^{-1} \multimap B^{+1}$ if A^{-1} is permeable
- $(A \supset B)^{+1} = !A^{-1} \multimap B^{+1}$ if A^{-1} is not permeable

The left-side translations of $\exists$, $\vee$ and $\wedge$ should also be refined:

- $(\exists x A)^{-1} = \exists x A^{-1}$ if A^{-1} is permeable
- $(\exists x A)^{-1} = \exists x !A^{-1}$ if A^{-1} is not permeable
- $(A \vee B)^{-1} = A^{-1} \oplus B^{-1}$ if A^{-1} and B^{-1} are both permeable
- $(A \vee B)^{-1} = A^{-1} \oplus !B^{-1}$ if only A^{-1} is permeable
- $(A \vee B)^{-1} = !A^{-1} \oplus B^{-1}$ if only B^{-1} is permeable
- $(A \vee B)^{-1} = !A^{-1} \oplus !B^{-1}$ if neither A^{-1} nor B^{-1} is permeable
- $(A \wedge B)^{-1} = A^{-1} \otimes B^{-1}$ if A^{-1} and B^{-1} are both permeable
- $(A \wedge B)^{-1} = A^{-1} \otimes !B^{-1}$ if only A^{-1} is permeable
- $(A \wedge B)^{-1} = !A^{-1} \otimes B^{-1}$ if only B^{-1} is permeable
- $(A \wedge B)^{-1} = A^{-1} \& B^{-1}$ if neither A^{-1} nor B^{-1} is permeable.

We can also translate the last case, for $(A \wedge B)$ where none of the subformulas are permeable, as $!A^{-1} \otimes !B^{-1}$. However, leaving this formula in left-asynchronous form derives little benefit; its decomposition would be stopped after one level. Moreover, the left-synchronous alternative (&) will allow us to extend backchaining to clauses with this conjunction of negative atoms at the head. We shall make the same alternative available on the right side, giving $\wedge R$ both synchronous and asynchronous interpretations as well. Unfortunately, no analogous alternative exists for $!A^{-1} \oplus !B^{-1}$, at least not for intuitionistic logic.

With this translation, permeable formulas are easily identified: *they are all the left-asynchronous formulas.* All left-asynchronous formulas are placed in the Θ^{-1} portion of the sequent. Each will be decomposed maximally until a subformula of the form 0, 1, or $!A$ is encountered. The following proposition states that $(\Theta^{-1})^{\perp}$ will be completely absorbed into the unbounded context after such decomposition.

Proposition 9 *Every focused proof of* $\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}$ *has the following form:*

$$\dfrac{\begin{matrix} \dfrac{\begin{matrix}\Pi_1 \\ \vdash (\Gamma_1^{-1})^{\perp} :\Uparrow R^{+1}\end{matrix}}{\vdots} & \cdots & \dfrac{\begin{matrix}\Pi_m \\ \vdash (\Gamma_m^{-1})^{\perp} :\Uparrow R^{+1}\end{matrix}}{\vdots} \end{matrix}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}}$$

where every $(\Gamma_i^{-1})^{\perp}$ *is a superset of* $(\Gamma^{-1})^{\perp}$ *which contains only synchronous and atomic formulas.*

Proof It is possible that $m = 0$ in case false $(0^{\perp})$ is in the context. This proposition is proved by induction on the sum of the sizes of all formulas in $(\Theta^{-1})^{\perp}$. Every subformula of a left-asynchronous formula of the translation is another left-asynchronous formula. □

There will a large number of corresponding focused proof variants, one for each form of the translation. For example, there will be five distinct focused proofs that correspond to $\wedge L$, two alone for the & interpretation. They will be focused versions of the five rules found in [Gir93].

Before the rest of the translation is given we first discuss the critical case of atoms.

6.2 Permeable Atoms and Their Polarity

The translations in the previous section unifies (left) permeable formulas with (left) asynchronous formulas, but stops at the level of atoms. Should "asynchronous decomposition" extend to atoms? As observed in Section 5, the following is not a desirable situation, where multiple *positive* atoms have accumulated in the linear context:

$$\vdash (\Gamma^{-1})^{\perp} : R^{+1}, (P_1^{-1})^{\perp}, \ldots, (P_m^{-1})^{\perp} \Uparrow$$

Such sequents are not provable unless 0 is derivable from Γ^{-1} (as will become clear below from the full definition of the translation). The decomposition of asynchronous formulas on the left should be *complete,* including atoms that may be substituted with asynchronous formulas. The translations examined in the previous sections have all avoided the above situation carefully. For deriving LJQ, implications appear in the form $A\multimap !B$ on the left and $!A\multimap B$ on the right. The !s are strategically placed so that positive atoms added to the context will be absorbed into the unbounded context. In Girard's original translation, $(A \supset B)^\circ$ is translated as $!A^\circ\multimap B^\circ$, but this is not a problem since all atoms can be assigned negative polarity, and indeed must be for that translation. Focusing ensures that when a negative atom is added to the context by an $\supset L$ rule, the proof must terminate in identity. The translation for λRCC likewise allows formulas of the form $!G\multimap A$ on the left only for negative atoms A.

But how do we generalize such special treatments in a universal setting? One would like to associate positive atoms with left-asynchronous formulas and negative atoms with left-synchronous formulas. Such an association would also be consistent with their intended uses in controlling proof search. In LU, atoms are divided into three classes: left-permeable, right-permeable and neutral[4]. A permeable atom is assumed capable of freely passing between the linear and unbounded contexts without help from exponentials. Two special structural rules exist in LU:

$$\frac{\Gamma; P, \Gamma' \vdash \Delta'; \Delta}{P, \Gamma; \Gamma' \vdash \Delta'; \Delta} \text{ for left-permeable } P \qquad\qquad \frac{\Gamma; \Gamma' \vdash \Delta', N; \Delta}{\Gamma; \Gamma' \vdash \Delta'; \Delta, N} \text{ for right-permeable } N$$

where the meaning of sequent $\Gamma; \Gamma' \vdash \Delta'; \Delta$ in linear logic is $\Gamma, !\Gamma' \vdash_{LL} ?\Delta', \Delta$. In other words, *permeable atoms are also decomposable* in the sense that they can be absorbed into the unbounded context. In the context of focusing, we wish to also achieve the effect that once absorbed into the unbounded context they stay *inert;* they cannot be selected for focus[5]. In other words, permeation occurs in only one direction.

At the outset, it may appear that Girard's classification of atoms in LU is incompatible with that of Andreoli for focused proofs. Surely there is no proof of $A \vdash_{LL} !A$ for an atom A of any focusing polarity. One observes, however, that LU's special version of linear logic can be embedded into "regular" linear logic by translating left-permeable atoms P as $!P$ and right-permeable atoms N as $?N$. Provability in LU is then preserved as provability in linear logic without using the two special rules above. This translation of atoms, however, still does not reveal any meaningful relationship between the left/right permeable atoms and positive/negative atoms in the context of focused proofs. Focus is lost on formulas with an exponential operator in front. In particular, one would not be able to derive the forward/backward-chaining behavior that arise naturally from focusing.

Using the following *asymmetrical* translation for cut-free proofs, we can provide a link between the two classifications of atoms (cut-elimination for LU was proved in [Vau93]; see also [PM05]):

1. $P^{-1} = !P, \quad P^{+1} = P,$ for left-permeable atom P.
2. $N^{-1} = N, \quad N^{+1} = ?N,$ for right-permeable atom N.

[4]Girard also uses the terms "positive" or "+1" for left-permeables and "negative" or "–1" for right permeables (and 0 for neutrals). The –1/ +1 label of our translation is consistent with this terminology in the context of one-sided sequents.

[5]The situation

$$\dfrac{\overline{\vdash \Delta, p : p^{\perp} \Downarrow p}\; I1}{\vdash \Delta, p : p^{\perp} \Uparrow}\; [D_2]$$

cannot appear for embeddings of intuitionistic logic. The presence of linear negation, or the lack thereof, unambiguously places an atom on the left or right-hand side of the intuitionistic sequent.

3. $B^{-1} = B^{+1} = B$, for neutral atom B.

This translation can be extended to embed all cut-free LU proofs into cut-free linear logic proofs, and thus to focused proofs. One can easily show the admissibility of the two special rules for permeables under the translation.

Left-permeable atoms can now be identified as positive atoms and right-permeables as negative atoms in a way that preserves their focusing characteristics. In particular, consider the forward-chaining situation involving a left-permeable atom P:

$$\frac{!P \vdash_{LL} P^{\Downarrow} \quad D^{-1} \vdash_{LL} \Delta}{P^{-1}, P^{+1} \multimap D^{-1} \vdash_{LL} \Delta} \multimap L$$

The $\Downarrow$ indicates the point of focus. The asymmetrical translation is able to preserve focusing on right-side occurrences of P. Likewise for backchaining with a right-permeable N, consider for example

$$!\Gamma, G^{+1} \multimap N^{-1} \vdash_{LL} N^{+1}$$

which can have the following focused proof:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{\vdash \Gamma^{\perp}, N :\Downarrow G^{+1}} \quad \cfrac{}{\vdash \Gamma^{\perp}, N :\Downarrow N^{\perp}}\,[I_2]}{\vdash \Gamma^{\perp}, N :\Downarrow G^{+1} \otimes N^{\perp}}\,[\otimes]}{\vdash \Gamma^{\perp}, N : G^{+1} \otimes N^{\perp} \Uparrow}\,[D_1]}{\vdash \Gamma^{\perp} : G^{+1} \otimes N^{\perp} \Uparrow ?N}\,[?]$$

Conversely, backchaining is not possible on $G \multimap !P$ nor is forward-chaining on $?N \multimap D$. We can now classify left-permeable (positive) atoms alongside synchronous formulas ($\exists$, $\oplus$, $\otimes$, 1 and 0), and right-permeable (negative) atoms alongside the asynchronous formulas ($\forall, \&, \parr, \top, \bot$). The neutral atoms can be assigned any polarity, as before. It can be said that before the current section we have only considered neutral atoms.

Intuitionistic logic in LU contains only positive (left-permeable) and neutral atoms. Right-permeable formulas, which admit structural rules on the right-hand side, are not allowed in intuitionist logic (except for ones that are notational duals of left-permeables). Formulas that are not left-permeable (asynchronous formulas) therefore behave neutrally. Informally speaking, the recognition of the existence of neutral atoms in intuitionistic logic leads us to linear logic, in which all atoms are neutral by default. We naturally incorporate the above translation of left-permeable and neutral atoms for our present task of embedding intuitionistic logic. It is also essential to uniformly *assign all neutral atoms negative focusing polarity.* Neutral atoms assigned to behave positively will again lead to the undesirable situation described above: in Girard's terminology, a sequent with multiple *stoups.* Left-permeable atoms already take the place of positive atoms. From now on we will use the terms "positive," "left-permeable" and "permeable" interchangeably. Whereas previously we avoided using Girard's terminology of positive and negative for reasons of clarity, we now see them as stronger yet compatible forms of focusing polarity. We will also use "negative" and "neutral" interchangeably, at least until we discuss classical logic. (Recall the discussion at the end of Section 2.3 regarding the use of the term "polarity" in linear, intuitionistic, and classical logics.)

As another way to link permeable atoms with polarized focusing atoms, we can also slightly modify LLF to allow the following as an initial rule:

$$\frac{}{\vdash \Gamma, P^{\perp} :\Downarrow !P}\ \text{, for positive atom } P.$$

In such case, there will be no need to use an asymmetrical translation of permeable atoms in order to reveal their roles in focused proofs. It would also mean that our translation of intuitionistic logic can be made *completely symmetrical.* Since it is not our intent in this paper to modify the linear focusing calculus, we will continue to use an asymmetrical translation.

The value of the above analysis is that we are now able to interpret LU, in particular its intuitionistic fragment, in focused linear logic. It now becomes possible to give a unified focusing calculus for intuitionistic logic, *one that is sensitive to the polarity of atoms.*

6.3 Completing the Translation

We now complete the translation began in Section 6.1.

- $false^{-1} = false^{+1} \;=\; 0$
- $true^{-1} = true^{+1} \;=\; 1$
- $(\forall xA)^{-1} = \forall xA^{-1}$
- $(A \supset B)^{-1} = A^{+1} \multimap B^{-1}$
- $(A \wedge B)^{+1} = A^{+1} \& B^{+1}$, if both A^{+1} and B^{+1} are negative polarity (in the linear logic sense)
- $(A \wedge B)^{+1} = A^{+1} \otimes B^{+1}$, otherwise
- $(A \vee B)^{+1} = A^{+1} \oplus B^{+1}$
- $(\forall xA)^{+1} = \forall xA^{+1}$
- $(\exists xA)^{+1} = \exists xA^{+1}$

The term *negative* now includes both asynchronous formulas as well as negative (neutral) atoms. The right-side translations need not be further divided into cases: the !'s are not needed. The *reaction rule* $R \Downarrow$ terminates the focusing mode. We will need to mimic $R \Downarrow$ when writing the corresponding intuitionistic calculus.

An explanation of the translation for $(A \supset B)^{-1}$ is in order. At issue is that we wish to support both forward and backward reasoning. The former seems to call for $(A^{+1} \multimap !B^{-1})$ and the latter requires $(A^{+1} \multimap B^{-1})$. However, we note that the issue is moot if B^{-1} is positive (permeable). Similarly, forward chaining is not possible if A^{+1} is negative. So the only case where $(A^{+1} \multimap !B^{-1})$ is a reasonable alternative is when A^{+1} is positive and B^{-1} is negative. However, in this case it is possible to manufacture a "false positive" using a formula (in intuitionistic logic) of the form $A \supset (B \wedge true)$. The linear logic translation of this formula is $A^{+1} \multimap (!B^{-1} \otimes 1)$ since 1 is positive and B^{-1} is negative, thus inducing the desired exponential[6]. If B^{-1} is positive, an exponential can still be induced, if necessary, using a "false negative" formula of the form $(A \supset (true \supset B) \wedge true)$.

We are thus able to avoid arbitrary solutions such as adding a special connective, or restricting the forms of clauses as in λRCC.

The completeness of this translation is established in the same manner as in previous sections: by appealing to the correctness of the 0/1 translation. Formulas are among the forms

$$true \mid false \mid Atom \mid P \wedge Q \mid P \vee Q \mid P \supset Q \mid \exists xP \mid \forall xP$$

Proposition 10 *Let Γ, R consist of a multiset of intuitionistic formulas. Let Γ_2^{-1} consist of all permeable formulas in Γ^{-1} and Γ_1^{-1} contain all other formulas in Γ^{-1}. If $!\Gamma^0 \vdash_{LL} R^1$ is provable then $!\Gamma_1^{-1}, \Gamma_2^{-1} \vdash_{LL} R^{+1}$ is also provable.*

Proof By the permeability assumption every formula O in Γ_2^{-1} is equivalent to $!O$. It can be shown that for every formula F, $F^1 \vdash_{LL} F^{+1}$ holds and that $!F^{-1} \vdash_{LL} !F^0$ holds for every possible case of F^{-1}. For example, in showing that

$$!(A^{-1} \oplus !B^{-1}) \;\vdash_{LL}\; !(!A^0 \oplus !B^0)$$

we must invoke the permeability of A^{-1}. The cases are otherwise similar to those already argued. □

Another crucial result is the analog of Lemma 4 of Section 4, which handles the presence of 0. The proof of Lemma 4 can be directly lifted to the current setting (the only argument of Lemma 4 that was dependent on the j/q translation was that the translated formulas do not contain negated atoms, which still applies here).

[6] $B \vee false$ and $\exists uB$ where u is vacuous can also be used.

Lemma 11 *If there is a proof of* $\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A^{+1}$ *then there is also a proof of* $\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1}$.

Proof The proof is analogous to those of Lemmas 3 and 4. □

6.4 LJF$_0$

In the derivation of an intuitionistic-level calculus we shall make use of the following forms of sequents:

1. $[\Gamma], \Theta \longrightarrow R$ (top-level) corresponds to $\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp$
2. $[\Gamma] \longrightarrow [R]$ corresponds to $\vdash (\Gamma^{-1})^\perp : R^{+1} \Uparrow$
3. $[\Gamma] \xrightarrow{A} [R]$ corresponds to $\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow (A^{-1})^\perp$
4. $[\Gamma] -_A\!\!\rightarrow$ corresponds to $\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1}$

The brackets [] are used on the left-hand side to separate left-asynchronous formulas (Θ) from others (Γ).

We now re-define, within intuitionistic logic, the notion of the polarity of atoms and formulas.

Definition 12 Assume that atoms are arbitrarily partitioned into two disjoint classes, *positive* and *neutral.* A *positive formula* is closed under the following inductive cases:

1. any positive atom Q.
2. $true$ and $false$.
3. any formula of the form $\exists x A$
4. any formula of the form $A \vee B$
5. any formula $A \wedge B$ where at least one of A and B is positive.

Any formula that is not positive is *neutral.*

In the following development of the "LJF$_0$" calculus, we first give separate rules for each case of the $-1/+1$ translation. This means that there will be a large number of variants for the $\wedge L$, $\vee L$, $\exists L$ and $\supset R$ cases. This choice simplifies the task of defining the mapping between proofs and the corresponding proof of correctness. However, the resulting system will still be in the style of LU. In the following section we shall show how many of these rules can be combined by adding more reaction-type rules.

We shall use the symbols P, Q to represent positive formulas and N, M for negative formulas. Γ represents a multiset of neutral formulas and positive atoms and Θ represents a multiset of positive formulas. Other symbols are arbitrary unless noted.

Rules for left-asynchronous decomposition:

1. $(false^{-1})^\perp = 0^\perp = \top$:

$$\frac{}{[\Gamma], \Theta, false \longrightarrow \mathcal{R}}\ falseL \qquad \longmapsto \qquad \frac{}{\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp, \top}\ \top$$

2. $(true^{-1})^\perp = 1^\perp = \perp$:

$$\frac{[\Gamma], \Theta \longrightarrow R}{[\Gamma], \Theta, true \longrightarrow R}\ trueL \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp}{\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp, \perp}\ \perp$$

3. for positive atom Q, $(Q^{-1})^{\perp} = (!Q)^{\perp} = ?Q^{\perp}$:

$$\frac{[Q,\Gamma],\Theta \longrightarrow R}{[\Gamma],\Theta,Q \longrightarrow R}\,[]^{+} \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp}, Q^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, ?Q^{\perp}}\,?$$

The "decomposition" of a positive atom allows the atom to permeate the unbounded context. Technically, it should be $!Q$ that enters Γ^{-1}, but Γ^{-1} is implicitly $!\Gamma^{-1}$ and $!!Q \equiv !Q$. This slight discrepancy is due to the fact that our "permeable" atom requires a ! to assist in the permeation. Once it enters the [ ], however, the atom stays "locked in:" it cannot be selected for focus. This is a key characteristic of LLF: the permeation of atoms is one-way only.

4. (a) $((\exists x P)^{-1})^{\perp} = (\exists x P^{-1})^{\perp} = \forall x (P^{-1})^{\perp}$:

$$\frac{[\Gamma],\Theta,P \longrightarrow R}{[\Gamma],\Theta,\exists y P \longrightarrow R}\,\exists L^{+} \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, (P^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, \forall y (P^{-1})^{\perp}}\,\forall$$

   (b) $((\exists x N)^{-1})^{\perp} = (\exists x !N^{-1})^{\perp} = \forall x ?(N^{-1})^{\perp}$:

$$\frac{[N,\Gamma],\Theta \longrightarrow R}{[\Gamma],\Theta,\exists y N \longrightarrow R}\,\exists L^{-} \qquad \longmapsto \qquad \frac{\dfrac{\vdash (\Gamma^{-1})^{\perp}, (N^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, ?(N^{-1})^{\perp}}\,?}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, \forall y ?(N^{-1})^{\perp}}\,\forall$$

5. (a) $((P \vee Q)^{-1})^{\perp} = (P^{-1} \oplus Q^{-1})^{\perp} = (P^{-1})^{\perp} \& (Q^{-1})^{\perp}$:

$$\frac{[\Gamma],\Theta,P \longrightarrow R \qquad [\Gamma],\Theta,Q \longrightarrow R}{[\Gamma],\Theta,P \vee Q \longrightarrow R}\,\vee L^{++}$$

$$\longmapsto$$

$$\frac{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, (P^{-1})^{\perp} \quad \vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, (Q^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, (P^{-1})^{\perp} \& (Q^{-1})^{\perp}}\,\&$$

   (b) $((P \vee N)^{-1})^{\perp} = (P^{-1} \oplus !N^{-1})^{\perp} = (P^{-1})^{\perp} \& ?(N^{-1})^{\perp}$:

$$\frac{[\Gamma],\Theta,P \longrightarrow R \qquad [N,\Gamma],\Theta \longrightarrow R}{[\Gamma],\Theta,P \vee N \longrightarrow R}\,\vee L^{+-}$$

$$\longmapsto$$

$$\frac{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, (P^{-1})^{\perp} \quad \dfrac{\vdash (\Gamma^{-1})^{\perp}, (N^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, ?(N^{-1})^{\perp}}\,?}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, (P^{-1})^{\perp} \& ?(N^{-1})^{\perp}}\,\&$$

   (c) $((N \vee P)^{-1})^{\perp} = (!N^{-1} \oplus P^{-1})^{\perp} = ?(N^{-1})^{\perp} \& (P^{-1})^{\perp}$:

$$\frac{[N,\Gamma],\Theta \longrightarrow R \qquad [\Gamma],\Theta,P \longrightarrow R}{[\Gamma],\Theta,N \vee P \longrightarrow R}\,\vee L^{-+}$$

$$\longmapsto$$

$$\frac{\dfrac{\vdash (\Gamma^{-1})^{\perp}, (N^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, ?(N^{-1})^{\perp}}\,? \quad \vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, (P^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1}, (\Theta^{-1})^{\perp}, ?(N^{-1})^{\perp} \& (P^{-1})^{\perp}}\,\&$$

(d) $((N \vee M)^{-1})^{\perp} = (!N^{-1} \oplus !M^{-1})^{\perp} = (N^{-1})^{\perp} \& (M^{-1})^{\perp}$:

$$\frac{[N,\Gamma],\Theta \longrightarrow R \qquad [M,\Gamma],\Theta \longrightarrow R}{[\Gamma],\Theta,N \vee M \longrightarrow R} \vee L^{--}$$

$$\longmapsto$$

$$\frac{\dfrac{\vdash (\Gamma^{-1})^{\perp},(N^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},?(N^{-1})^{\perp}}\,? \qquad \dfrac{\vdash (\Gamma^{-1})^{\perp},(M^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},?(M^{-1})^{\perp}}\,?}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},?(N^{-1})^{\perp}\&?(M^{-1})^{\perp}}\,\&$$

6. (a) $((P \wedge Q)^{-1})^{\perp} = (P^{-1} \otimes Q^{-1})^{\perp} = (P^{-1})^{\perp} \parr (Q^{-1})^{\perp}$:

$$\frac{[\Gamma],\Theta,P,Q \longrightarrow R}{[\Gamma],\Theta,P \wedge Q \longrightarrow R} \wedge L^{++} \quad \longmapsto \quad \frac{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},(P^{-1})^{\perp},(Q^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},(P^{-1})^{\perp} \parr (Q^{-1})^{\perp}}\,\parr$$

(b) $((P \wedge N)^{-1})^{\perp} = (P^{-1} \otimes !N^{-1})^{\perp} = (P^{-1})^{\perp} \parr ?(Q^{-1})^{\perp}$:

$$\frac{[N,\Gamma],\Theta,P \longrightarrow R}{[\Gamma],\Theta,P \wedge N \longrightarrow R} \wedge L^{+-} \quad \longmapsto \quad \frac{\dfrac{\vdash (\Gamma^{-1})^{\perp},(N^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},(P^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},(P^{-1})^{\perp},?(N^{-1})^{\perp}}\,?}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},(P^{-1})^{\perp} \parr ?(N^{-1})^{\perp}}\,\parr$$

(c) $((N \wedge P)^{-1})^{\perp} = (!N^{-1} \otimes P^{-1})^{\perp} = ?(N^{-1})^{\perp} \parr (P^{-1})^{\perp}$:

$$\frac{[N,\Gamma],\Theta,P \longrightarrow R}{[\Gamma],\Theta,N \wedge P \longrightarrow R} \wedge L^{-+} \quad \longmapsto \quad \frac{\dfrac{\vdash (\Gamma^{-1})^{\perp},(N^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},(P^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},(P^{-1})^{\perp},?(N^{-1})^{\perp}}\,?}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow R^{+1},(\Theta^{-1})^{\perp},(P^{-1})^{\perp} \parr ?(N^{-1})^{\perp}}\,\parr$$

One may say that we have "cheated" in the above focused proof by switching the arguments. Technically the portion of a focusing sequent to the right of $\Uparrow$ is an ordered list. However, provability is retained by reordering the list (see the *inversion lemma* "$\mathcal{L}\equiv$" in [And92]), thus the mapping is justified. The remaining case of $\wedge L^{--}$ is found in the section on left-focus rules.

By Proposition 9, Θ will be completely absorbed after the left-asynchronous decomposition phase.

Rules for right asynchronous decomposition:

1. $(N \wedge M)^{+1} = N^{+1} \& M^{+1}$:

$$\frac{[\Gamma] \longrightarrow N \quad [\Gamma] \longrightarrow M}{[\Gamma] \longrightarrow N \wedge M} \wedge R^{+} \quad \longmapsto \quad \frac{\vdash (\Gamma^{-1})^{\perp} :\Uparrow N^{+1} \quad \vdash (\Gamma^{-1})^{\perp} :\Uparrow M^{+1}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow N^{+1} \& M^{+1}}\,\&$$

2. (a) $(N \supset A)^{+1} = !N^{-1} \multimap A^{+1} = A^{+1} \parr ?(N^{-1})^{\perp}$:

$$\frac{[N,\Gamma] \longrightarrow A}{[\Gamma] \longrightarrow N \supset A} \supset R^{-} \quad \longmapsto \quad \frac{\dfrac{\vdash (\Gamma^{-1})^{\perp},(N^{-1})^{\perp} :\Uparrow A^{+1}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow A^{+1},?(N^{-1})^{\perp}}\,?}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow A^{+1} \parr ?(N^{-1})^{\perp}}\,\parr$$

(b) $(P \supset A)^{+1} = P^{-1} \multimap A^{+1} = A^{+1} \parr (P^{-1})^{\perp}$:

$$\frac{[\Gamma],P \supset A}{[\Gamma] \longrightarrow P \supset A} \supset R^{+} \quad \longmapsto \quad \frac{\vdash (\Gamma^{-1})^{\perp} :\Uparrow A^{+1},(P^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow A^{+1} \parr (P^{-1})^{\perp}}\,\parr$$

3. $(\forall y A)^{+1} = \forall y A^{+1}$:

$$\frac{[\Gamma] \longrightarrow A}{[\Gamma] \longrightarrow \forall y A} \; \forall R \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp} :\Uparrow A^{+1}}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow \forall y A^{+1}} \; \forall$$

The usual restrictions on y applies.

Before developing the focusing rules, a number of reaction and decision rules also need to be mimicked in the intuitionistic calculus. They form the bridge between the asynchronous decomposition and focusing phases. They also allow the right-side translations to be more compact. These rules can also be classified as "structural" rules because they re-organize sequents.

Rules of decision and reaction:

1. Decision to focus left:

$$\frac{[N,\Gamma] \xrightarrow{N} [R]}{[N,\Gamma] \longrightarrow [R]} \; Lf \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp}, (N^{-1})^{\perp} : R^{+1} \Downarrow (N^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp}, (N^{-1})^{\perp} : R^{+1} \Uparrow} \; [D_2]$$

2. Decision to focus right:

$$\frac{[\Gamma] -_{P}\!\!\rightarrow}{[\Gamma] \longrightarrow [P]} \; Rf \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp} :\Downarrow P^{+1}}{\vdash (\Gamma^{-1})^{\perp} : P^{+1} \Uparrow} \; [D_1]$$

3. Reaction Left; terminates left focus:

$$\frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]} \; R_l \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp} : R^{+1} \Uparrow (P^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} : R^{+1} \Downarrow (P^{-1})^{\perp}} \; R\Downarrow$$

4. Reaction Right; terminates right focus:

$$\frac{[\Gamma] \longrightarrow N}{[\Gamma] -_{N}\!\!\rightarrow} \; R_r \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp} :\Uparrow N^{+1}}{\vdash (\Gamma^{-1})^{\perp} :\Downarrow N^{+1}} \; R\Downarrow$$

5. Suspend Right; terminates right-asynchronous decomposition:

$$\frac{[\Gamma] \longrightarrow [D]}{[\Gamma] \longrightarrow D} \; []_r \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp} : D^{+1} \Uparrow}{\vdash (\Gamma^{-1})^{\perp} :\Uparrow D^{+1}} \; R\Uparrow$$

Here, D is either a positive formula or neutral atom (not asynchronous). We delay defining a *suspend left* rule because it does *not* correspond to $R\Uparrow$.

Left focus rules:

1. Neutral identity:

$$\frac{}{[\Gamma] \xrightarrow{N} [N]} \; I_l, \text{ atomic } N \qquad \longmapsto \qquad \frac{}{(\Gamma^{-1})^{\perp} : N \Downarrow N^{\perp}} \; I_1$$

2. $((N_1 \wedge N_2)^{-1})^{\perp} = (N_1^{-1} \& N_2^{-1})^{\perp} = (N_1^{-1})^{\perp} \oplus (N_2^{-1})^{\perp}$:

$$\frac{[\Gamma] \xrightarrow{N_i} [R]}{[\Gamma] \xrightarrow{N_1 \wedge N_2} [R]} \; \wedge L^{--} \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^{\perp} : R^{+1} \Downarrow (N_i^{-1})^{\perp}}{\vdash (\Gamma^{-1})^{\perp} : R^{+1} \Downarrow (N_1^{-1})^{\perp} \oplus (N_2^{-1})^{\perp}} \; \oplus$$

3. $((\forall x A)^{-1})^\perp = (\forall x A^{-1})^\perp = \exists x (A^{-1})^\perp$:

$$\frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]} \forall L \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow (A^{-1})^\perp[t/x]}{\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow \exists x (A^{-1})^\perp} \exists$$

4. $((A \supset B)^{-1})^\perp = (A^{+1} \multimap B^{-1})^\perp = A^{+1} \otimes (B^{-1})^\perp$:

$$\frac{[\Gamma] -_A\!\!\rightarrow \qquad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]} \supset L \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1} \quad \vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow (B^{-1})^\perp}{\vdash (\Gamma^{-1})^\perp : R^{+1} \Downarrow A^{+1} \otimes (B^{-1})^\perp} \otimes$$

The validity of this mapping is justified by Lemma 11, which is a restatement of Lemma 4 for the $-1/+1$ translation. That is, if the there is a proof where the $\otimes$ rule splits the context differently due to the presence of 0, then there is also a proof where it is split as above.

Right focus rules:

1. Positive identity; $(P^{-1})^\perp = (!P)^\perp = ?P^\perp$, $P^{+1} = P$:

$$\frac{}{[P, \Gamma] -_P\!\!\rightarrow} I_r, \text{ atomic } P \qquad \longmapsto \qquad \frac{}{\vdash (\Gamma^{-1})^\perp, P^\perp :\Downarrow P} I_2$$

2. $(A \wedge B)^{+1} = A^{+1} \otimes B^{+1}$, either A or B is positive:

$$\frac{[\Gamma] -_A\!\!\rightarrow \quad [\Gamma] -_B\!\!\rightarrow}{[\Gamma] -_{A \wedge B}\!\!\rightarrow} \wedge R^+ \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1} \quad \vdash (\Gamma^{-1})^\perp :\Downarrow B^{+1}}{\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1} \otimes B^{+1}} \otimes$$

3. $(A_1 \vee A_2)^{+1} = A_1^{+1} \oplus A_2^{+1}$:

$$\frac{[\Gamma] -_{A_i}\!\!\rightarrow}{[\Gamma] -_{A_1 \vee A_2}\!\!\rightarrow} \vee R \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Downarrow A_i^{+1}}{\vdash (\Gamma^{-1})^\perp :\Downarrow A_1^{+1} \oplus A_2^{+1}} \oplus$$

4. $(\exists x A)^{+1} = \exists x A^{+1}$:

$$\frac{[\Gamma] -_{A[t/x]}\!\!\rightarrow}{[\Gamma] -_{\exists x A}\!\!\rightarrow} \exists R \qquad \longmapsto \qquad \frac{\vdash (\Gamma^{-1})^\perp :\Downarrow A^{+1}[t/x]}{\vdash (\Gamma^{-1})^\perp :\Downarrow \exists x A^{+1}} \exists$$

The rules enumerated above forms the LJF_0 sequent calculus. As in previous sections, the implicitly recursive mapping is reversible, and proves the following theorem:

Theorem 13 *Let Γ be a multiset of neutral formulas and positive atoms, Θ a multiset of positive formulas, and R any formula. $[\Gamma], \Theta \longrightarrow R$ is provable in LJF_0 if and only if $\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp$ is provable.*

Combining this theorem with the trivial soundness of LJF_0 and earlier results, the completeness of the new calculus is also established.

Corollary 14 *LJF_0 is sound and complete with respect to intuitionistic logic.*

6.5 LJF

In this section, several enhancements are made to LJF_0 to form the final system LJF.

The set of rules presented for LJF_0, although large, are still more compact than those of LU. The LU sequent calculus must, for example, define three versions of $\wedge R^+$ because there is no guarantee that a structural rule would be applied immediately when a subformula of neutral polarity is encountered. The *reaction* rules focusing give us the option of a smaller set of rules. There is one more reaction-type rule we can define:

$$\frac{[C, \Gamma], \Theta \longrightarrow R}{[\Gamma], \Theta, C \longrightarrow R}\ []_l$$

where C is a neutral formula or positive atom. It is important to note that unlike $[]_r$, this rule does not correspond to $R \Uparrow$, but to

$$\frac{\vdash (\Gamma^{-1})^\perp, (C^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp}{\vdash (\Gamma^{-1})^\perp :\Uparrow R^{+1}, (\Theta^{-1})^\perp, ?(C^{-1})^\perp}\ ?$$

We have not included this rule so that the mapping between proofs shown above is precise, and parallels each case of the translation to linear logic. Adding $[]_l$, however, means that many of the rules can now be defined in a way that's independent of polarity. The handling of polarity is confined to the structural rules. All the cases for $\vee L$, $\exists L$, $\supset R$, as well as three cases of $\wedge L$ can now be combined. Focusing ensures that $[]_l$ will be applied immediately above each generic rule. The "permeation rule" $[]^+$ is also subsumed by $[]_l$. The form of the endsequent is now

$$[], \Delta \longrightarrow R$$

where Δ may contain any formula. What distinctly remains, however, are the dual interpretations of $\wedge$ as both $\otimes$ and &.

There is reason for concern that in the LJF_0 calculus, the rule to be applied depends on more than just the top-level connective of a formula in the case of $\wedge$. With the $[]_l$ rule added, the rules for conjunction becomes:

$$\frac{[\Gamma] \xrightarrow{N_i} [R]}{[\Gamma] \xrightarrow{N_1 \wedge N_2} [R]}\ \wedge L^- \qquad \frac{[\Gamma], \Theta, A, B \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A \wedge B \longrightarrow \mathcal{R}}\ \wedge L^+, \text{ either } A \text{ or } B \text{ is positive}$$

$$\frac{[\Gamma], \Theta \longrightarrow N \quad [\Gamma], \Theta \longrightarrow M}{[\Gamma], \Theta \longrightarrow N \wedge M}\ \wedge R^- \qquad \frac{[\Gamma] -_A\!\!\rightarrow \quad [\Gamma] -_B\!\!\rightarrow}{[\Gamma] -_{A \wedge B}\!\!\rightarrow}\ \wedge R^+, \text{ either } A \text{ or } B \text{ is positive}$$

In the worst case it may be necessary to recursively check the polarities of all subformulas before the correct $\wedge$ rule is determined. This is also a characteristic of the LU sequent calculus. An alternative is to separate the $\wedge$ connective into two versions, $\wedge^+$ and $\wedge^-$, and translate each as follows:

- $(P \wedge^+ Q)^{-1} = P^{-1} \otimes Q^{-1}$
- $(P \wedge^+ N)^{-1} = P^{-1} \otimes !N^{-1}$
- $(N \wedge^+ P)^{-1} = !N^{-1} \otimes P^{-1}$
- $(N \wedge^+ M)^{-1} = !N^{-1} \otimes !M^{-1}$
- $(A \wedge^- B)^{-1} = A^{-1} \& B^{-1}$
- $(A \wedge^+ B)^{+1} = A^{+1} \otimes B^{+1}$
- $(A \wedge^- B)^{+1} = A^{+1} \& B^{+1}$

It remains that all left-asynchronous formulas are permeable. The rules for conjunction are now dependent only on the polarity of the top-level connective. The definition of positive formulas is also modified:

all formulas of the form $A \wedge^+ B$ are positive.

No rule now needs to examine more than the top-level structure of a formula to determine its polarity.

The new connectives and rules remain consistent with intuitionistic logic. Soundness is preserved in that if $\wedge^+$ and $\wedge^-$ are both replaced by $\wedge$, then the resulting proofs are valid intuitionistic proofs. Completeness is retained since Proposition 10 can be extended to the new case of the translation: $!(!N^{-1} \otimes !M^{-1}) \vdash_{LL} !(!N^0 \& !M^0)$ can be shown to be provable as well. Of course it is also well-known that $\wedge^+$ and $\wedge^-$ are equivalent given the admissibility of contraction and weakening. Given a provable LJ sequent, one can arbitrarily replace some occurrences of $\wedge$ by $\wedge^+$ and others by $\wedge^-$, and the resulting sequent will remain provable in the focused calculus. The choice of connective depends only on the desired proof-search behavior.

The new connectives can also be seen as new synthetic connectives defined within the LU framework, which was created with the intent of supporting just such inventions.

Of course little is gained, for example, if one right-focuses on $(A \wedge^+ B)$ when both A and B are neutral. Asynchronous decomposition is clearly the better alternative in such a case. This variant of the calculus must rely on a "user" to provide the appropriate version of the connective. The labels $+$ and $-$ on $\wedge$ can be seen as devices that enable user-guided proof search.

The reader may notice that, since there are now two versions of conjunction, there should also be two versions of *true*. Indeed it is also possible to use the linear constant $\top$ to represent the intutionistic *true*. However, in the LC/LU scheme of polarization, $\top$ is a *genuinely negative* element in the sense that $\top \equiv ?\top$. We therefore opt to leave out $\top$ from the formulation of intuitionistic logic, although it will certainly be used in classical logic (see Section 9). In any case we can use *false* $\supset$ *false* to regain the proof-search behavior of $\top$.

We summarize the rules of the modified calculus, which we shall simply call LJF, in Figure 7. Along the way, we also make another, slight modification. The LJF_0 rules are written so that no right-asynchronous decompositions can be performed before the left ones are complete. The reason again being that the portion to the right of $\Uparrow$ in a focusing sequent is technically an ordered list. But there is in fact no reason why the right formula cannot be decomposed first. We use the symbol $\mathcal{R}$ to indicate a right-side formula that can either be of the form R or $[R]$.

6.6 The Neutral Fragment

Positivity in intuitionistic logic is a much stronger attribute than in linear logic. A formula being positive means much more than that it has a top-level synchronous connective. The application of focusing, therefore, clarifies the structure of intuitionistic logic even more than it does for linear logic. In LJF, the role of positive formulas, including their position in a sequent, is clearly identified. The analysis and application of intuitionistic logic will no doubt benefit from this elucidation.

The neutral fragment of intuitionistic logic has received much attention due to its connection to lambda term reduction. We present the neutral fragment of LJF (in fact of LJF_0) in Figure 8. Notice that in this fragment, there is no Θ, which is wholly positive. The box ($[\cdot]$) on the left-hand side becomes meaningless. Neither is there any form of right-side focusing. The distinguished *stoup* formula is found above the sequent arrow $\longrightarrow$.

One can check that the rules of neutral LJF are precisely those of LJF_0 restricted to neutral formulas, with the pertinent reaction rules folded in. Neutralized LJF is in fact what one would expect, plus perhaps additional focusing features. The simple extraction of this important fragment is possible due to the precise identification within LJF of the position and role of positive formulas.

Another important characteristic of the neutral fragment is that all formulas are (either atomic or) asynchronous on the right and synchronous on the left. Thus the right-hand side will be decomposed down to an atom before a left rule can be applied, yielding what are known as *uniform proofs* [MNPS91].

P, Q positive, N, M neutral, other symbols arbitrary unless noted.

Structural Rules (Decision and Reaction):

$$\frac{[N,\Gamma] \xrightarrow{N} [R]}{[N,\Gamma] \longrightarrow [R]}\ Lf \qquad \frac{[\Gamma] -_{P}\!\!\rightarrow}{[\Gamma] \longrightarrow [P]}\ Rf \qquad \frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]}\ R_l \qquad \frac{[\Gamma] \longrightarrow N}{[\Gamma] -_{N}\!\!\rightarrow}\ R_r$$

$$\frac{[C,\Gamma], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, C \longrightarrow \mathcal{R}}\ []_l \qquad \frac{[\Gamma], \Theta \longrightarrow [D]}{[\Gamma], \Theta \longrightarrow D}\ []_r$$

C a neutral formula or positive atom, D a positive formula or neutral atom.

Identities:

$$\frac{}{[P,\Gamma] -_{P}\!\!\rightarrow}\ I_r,\ \text{atomic } P \qquad \frac{}{[\Gamma] \xrightarrow{N} [N]}\ I_l,\ \text{atomic } N$$

True and False:

$$\frac{}{[\Gamma], \Theta, false \longrightarrow \mathcal{R}}\ falseL \qquad \frac{[\Gamma], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, true \longrightarrow \mathcal{R}}\ trueL \qquad \frac{}{[\Gamma] -_{true}\!\!\rightarrow}\ trueR$$

Conjunctions:

$$\frac{[\Gamma] \xrightarrow{A_i} [R]}{[\Gamma] \xrightarrow{A_1 \wedge^- A_2} [R]}\ \wedge^- L \qquad \frac{[\Gamma], \Theta, A, B \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A\ \wedge^+ B \longrightarrow \mathcal{R}}\ \wedge^+ L$$

$$\frac{[\Gamma], \Theta \longrightarrow A \quad [\Gamma], \Theta \longrightarrow B}{[\Gamma], \Theta \longrightarrow A\ \wedge^- B}\ \wedge^- R \qquad \frac{[\Gamma] -_{A}\!\!\rightarrow \quad [\Gamma] -_{B}\!\!\rightarrow}{[\Gamma] -_{A\ \wedge^+ B}\!\!\rightarrow}\ \wedge^+ R$$

Disjunction

$$\frac{[\Gamma], \Theta, A \longrightarrow \mathcal{R} \quad [\Gamma], \Theta, B \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A \vee B \longrightarrow \mathcal{R}}\ \vee L \qquad \frac{[\Gamma] -_{A_i}\!\!\rightarrow}{[\Gamma] -_{A_1 \vee A_2}\!\!\rightarrow}\ \vee R$$

Implication

$$\frac{[\Gamma] -_{A}\!\!\rightarrow \qquad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]}\ \supset L \qquad \frac{[\Gamma], \Theta, A \longrightarrow B}{[\Gamma], \Theta \longrightarrow A \supset B}\ \supset R$$

Quantifiers:

$$\frac{[\Gamma], \Theta, A \longrightarrow \mathcal{R}}{[\Gamma], \Theta, \exists y A \longrightarrow \mathcal{R}}\ \exists L \qquad \frac{[\Gamma] -_{A[t/x]}\!\!\rightarrow}{[\Gamma] -_{\exists x A}\!\!\rightarrow}\ \exists R \qquad \frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]}\ \forall L \qquad \frac{[\Gamma], \Theta \longrightarrow A}{[\Gamma], \Theta \longrightarrow \forall y A}\ \forall R$$

provided that y is not free in Γ, Θ, or R.

Figure 7: *The Sequent Calculus LJF*

$$\frac{N,\Gamma \xrightarrow{N} [R]}{N,\Gamma \longrightarrow [R]}\ Lf \qquad \frac{\Gamma \longrightarrow [M]}{\Gamma \longrightarrow M}\ []_r \text{ atomic } M \qquad \frac{}{\Gamma \xrightarrow{N} [N]}\ I_l, \text{ atomic } N$$

$$\frac{\Gamma \xrightarrow{N_i} [R]}{\Gamma \xrightarrow{N_1 \wedge N_2} [R]}\ \wedge L^- \qquad \frac{\Gamma \longrightarrow N \quad \Gamma \longrightarrow M}{\Gamma \longrightarrow N \wedge M}\ \wedge R^- \qquad \frac{\Gamma \xrightarrow{N[t/x]} [R]}{\Gamma \xrightarrow{\forall x N} [R]}\ \forall L$$

$$\frac{\Gamma \longrightarrow N}{\Gamma \longrightarrow \forall y N}\ \forall R \qquad \frac{\Gamma \longrightarrow N \qquad \Gamma \xrightarrow{M} [R]}{\Gamma \xrightarrow{N \supset M} [R]}\ \supset L \qquad \frac{N,\Gamma \longrightarrow M}{\Gamma \longrightarrow N \supset M}\ \supset R$$

Figure 8: The Neutral Fragment of LJF

6.7 Decorating Intuitionistic Conjunction

We have justified the inclusion of the dual versions of conjunction in LJF, but have not indicated how to construct a general strategy for choosing between $\wedge^+$ and $\wedge^-$. One possible strategy is derived from an interesting translation from intuitionistic to linear logic explored by Chaudhuri in [Cha06]. Aspects of this translation bare similarities to ours, but it is based on a dimension we have not yet considered: conjunction is translated differently depending on whether or not it is under focus. That is, when we focus on a conjunction it is translated synchronously and when we decompose one it becomes the asynchronous alternative, which is & or $\otimes$ depending on the side of the sequent. We can adopt this strategy in the context of LJF by defining the following transformation from intuitionistic formulas to LJF formulas, i.e., to formulas with $\wedge^+$ and $\wedge^-$. The translation is defined by two mutually recursive functions a and f (for "active" and "focal" respectively), and each will have a left (l) and right (r) version.

for all atoms B as well as the constants $true$ and $false$, $a(B)^l = a(B)^r = f(B)^l = f(B)^r \ = \ B$.
$a(A \wedge B)^r = a(A)^r \wedge^- a(B)^r$
$f(A \wedge B)^r = f(A)^r \wedge^+ f(B)^r$
$a(A \vee B)^r = f(A \vee B)^r = f(A)^r \vee f(B)^r$
$f(A \supset B)^r = a(A \supset B)^r = a(A)^l \supset a(B)^r$
$a(A \wedge B)^l = a(A)^l \wedge^+ a(B)^l$
$f(A \wedge B)^l = f(A)^l \wedge^- f(B)^l$
$a(A \supset B)^l = f(A \supset B)^l = f(A)^r \supset f(B)^l$
$f(A \vee B)^l = a(A \vee B)^l = a(A)^l \vee a(B)^l$

Intuitionistic sequents $\Delta \vdash_I R$ is transformed to $[], a(\Delta)^l \longrightarrow a(R)^r$. This transformation mirrors the one in [Cha06] except that the translation here is defined only at the level of intuitionistic formulas. That is to say, the extra dimension of basing the translation on the focusing phases of proofs can also be considered before we define a translation to linear logic. By adopting the f/a transformation, we can embed Chaudhuri's proposed intuitionistic focusing system within LJF[7].

Under the f/a decorating strategy above, the sequent

$$A,\ A \supset B,\ (A \wedge B) \supset (C \wedge D)\ \vdash_I\ B \wedge C$$

becomes the LJF sequent

$$[],\ A,\ A \supset B,\ (A \wedge^+ B) \supset (C \wedge^- D)\ \longrightarrow\ B \wedge^- C$$

[7]The system as stated in [Cha06] does not allow for atoms of both polarity. However, Frank Pfenning is developing, in unpublished notes, another system based on this work, and which clearly has the intent of allowing mixed polarities. We hesitate stating any formal connections until that system is finalized.

assuming that A, B, C and D are atoms.

Note however, that this strategy is not always optimal in terms of proof search. The atoms can be given positive polarity, which enables forward-chaining. The "subgoals" generated from proving $B \wedge C$ can then share the forward derivation of B. That is, the following decoration may be preferable:

$$[],\ A,\ A \supset B,\ (A \wedge^{+} B) \supset (C \wedge^{+} D) \ \longrightarrow\ B \wedge^{+} C$$

In particular, the conclusion of the sequent is made synchronous to prevent immediate decomposition. A formal strategy along lines of polarity will again lead us back to the tables of LU:

$g(A \wedge B) = g(A) \wedge^{-} g(B)$ if both $g(A)$ and $g(B)$ are negative
$g(A \wedge B) = g(A) \wedge^{+} g(B)$ if at least one of $g(A)$ and $g(B)$ is positive
etc ...

The trade-off in using such a strategy along with LJF, compared to using LJF_0, is that we would not have to check the polarities of subformulas during proof search: they would be decided beforehand and appropriately decorated with $\wedge^{-}$ or $\wedge^{+}$. Clearly if one adopts such a strategy then LJF proofs become identified with LJF_0 proofs.

7 Cut Elimination

Given the various forms of sequents, a large number of cut rules can be written for LJF (and even more for LJF_0). They all reduce to two essential forms:

$$\frac{[\Gamma], \Theta \longrightarrow P \quad [\Gamma'], \Theta', P \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}}\ Cut^+$$

$$\frac{[\Gamma], \Theta \longrightarrow C \quad [C, \Gamma'], \Theta' \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}}\ Cut^-$$

However, to provide a direct cut-elimination procedure, several other cut rules are needed simultaneously. That is, the cut-elimination procedure is mutually recursive among seven different forms of cut: the two above plus the following:

$$\frac{[\Gamma] -_A\!\!\rightarrow \quad [\Gamma'], \Theta', A \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}} \Downarrow Cut^+$$

$$\frac{[\Gamma], \Theta \longrightarrow B \quad [\Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'], \Theta \longrightarrow [R]} \Downarrow Cut^-$$

$$\frac{[\Gamma] -_C\!\!\rightarrow \quad [C, \Gamma'] -_R\!\!\rightarrow}{[\Gamma\Gamma'] -_R\!\!\rightarrow}\ Cut^{\rightarrow}$$

$$\frac{[\Gamma] \xrightarrow{B} [P] \quad [\Gamma'], P \longrightarrow [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\ Cut_1^{\leftarrow}$$

$$\frac{[\Gamma] \longrightarrow N \quad [N, \Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\ Cut_2^{\leftarrow}$$

In all rules, P is a positive formula, N a neutral formula, and C can be a neutral formula or a positive atom.

Although the number of cuts is large, during the proof of cut-elimination it will become apparent that even more variants may be needed. For example, the following variant of Cut^+ allows the cut formula to be boxed:

$$\frac{[\Gamma], \Theta \longrightarrow [P] \quad [\Gamma'], \Theta', P \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}}\ Cut^{+'}$$

This rule clearly reduces to Cut^+ by applying $[]_r$ to the left premise. However, the resulting *height* of the cut then increases. Another example is the following variant of $Cut^{\rightarrow}$:

$$\frac{[\Gamma] \longrightarrow N \quad [N, \Gamma'] -_R\!\!\rightarrow}{[\Gamma\Gamma'] -_R\!\!\rightarrow}\ Cut^{\rightarrow'}$$

which reduces to $Cut^{\rightarrow}$ by applying the reaction rule R_r to the left premise, but which also increases the height of the left subproof.

To avoid the proliferation of such obviously redundant cuts we employ a modified inductive measure for the *height* of cut rules.

Definition: The *height* of a cut rule of the form

$$\frac{\overset{\Pi}{Left\_Premise} \quad \overset{\Pi'}{Right\_Premise}}{Conclusion}$$

is the height of the subproof Π plus the height of Π' *except* in the following cases:

1.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ [\Gamma],\Theta \longrightarrow [P]\end{matrix}}{[\Gamma],\Theta \longrightarrow P}\,[]_r \qquad \begin{matrix}\Pi'\\ [\Gamma'],\Theta',P \longrightarrow \mathcal{R}\end{matrix}}{[\Gamma\Gamma'],\Theta\Theta' \longrightarrow \mathcal{R}}\,Cut^+$$

2.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ [\Gamma],\Theta \longrightarrow [C]\end{matrix}}{[\Gamma],\Theta \longrightarrow C}\,[]_r \qquad \begin{matrix}\Pi'\\ [C,\Gamma'],\Theta' \longrightarrow \mathcal{R}\end{matrix}}{[\Gamma\Gamma'],\Theta\Theta' \longrightarrow \mathcal{R}}\,Cut^-, \text{ atom } C$$

3.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ [\Gamma] \longrightarrow N\end{matrix}}{[\Gamma]-_N\!\!\rightarrow}\,R_r \qquad \begin{matrix}\Pi'\\ [N,\Gamma']-_R\!\!\rightarrow\end{matrix}}{[\Gamma\Gamma']-_R\!\!\rightarrow}\,Cut^{\rightarrow}$$

4.

$$\dfrac{\begin{matrix}\Pi_1\\ [\Gamma] \xrightarrow{B} [P]\end{matrix} \qquad \dfrac{\begin{matrix}\Pi'\\ [P,\Gamma'] \longrightarrow [R]\end{matrix}}{[\Gamma'],P \longrightarrow [R]}\,[]_l}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\,Cut_1^{\leftarrow}, \text{ for positive atom } P$$

In these cases, the height of the cuts is defined as the height of Π_1 plus the height of Π'. That is, we regard these inferences as single steps.

The height of proofs is defined in the usual way except for these special cases. We can also assume without loss of generality that the subproofs Π, Π_1 and Π' are free of cuts, since a cut-elimination procedure can start at the "top" of the proof tree. □.

In the same manner, it is in fact possible to absorb $\Downarrow Cut^+$ and $\Downarrow Cut^-$ into Cut^+ and Cut^- respectively. However, since these cuts correspond precisely to the so called "key cases" of cut-elimination, their independent treatment may help clarify the structure of cut-elimination.

7.1 Admissibility of Cuts

For the formal arguments we first state a general lemma.

Lemma 15 *The following holds for LJF:*

(contraction): *if there is a cut-free proof of* $[A,A,\Gamma] \longrightarrow \mathcal{R}$ *then there is also a cut-free proof of* $[A,\Gamma] \longrightarrow \mathcal{R}$ *of the same height.*

(weakening 1): *if there is a cut-free proof of* $[\Gamma]-_R\!\!\rightarrow$ *then there is also a cut-free proof of* $[D,\Gamma]-_R\!\!\rightarrow$ *of the same height.*

(weakening 2): *if there is a cut-free proof of* $[\Gamma] \xrightarrow{N} [R]$ *then there is also a cut-free proof of* $[D,\Gamma] \xrightarrow{N} [R]$ *of the same height.*

Proof All cases of the lemma follow from the fact that in a LJF proof, the multiset inside [.] on the left (i.e. the unbounded context in linear sequents) never decreases from conclusion to premise. The proofs are straightforward inductions. In fact, not only provability but the exact structure of proofs is preserved. □

The following first-order substitution lemma is also required for permuting the cuts on quantified formulas.

Lemma 16 *If* $[\Gamma], \Theta \longrightarrow \mathcal{R}$ *is provable in LJF then there is also a proof of* $[\Gamma[t/x]], \Theta[t/x] \longrightarrow \mathcal{R}[t/x]$ *of the same height.*

Proof By induction on the height of LJF proofs. □

The lemma assumes the usual restriction on the capture of the free variables of t.

If one is only concerned with the *admissibility* of cuts, then both Cut^+ and Cut^- are admitted by virtue of completeness with respect to LJ. For the admissibility of the other rules, $\Downarrow Cut^+$ is reducible to

$$\dfrac{\dfrac{\dfrac{[\Gamma] -_{P}\!\!\rightarrow}{[\Gamma] \longrightarrow [P]}\ Rf}{[\Gamma] \longrightarrow P}\ []_r \qquad [\Gamma'], \Theta, P \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta \longrightarrow \mathcal{R}}\ Cut^+$$

If P where neutral, we also get that $[\Gamma] \longrightarrow P$ is provable by virtue of the R_r rule, which must be the last rule of the left subproof. The cut then reduces to a Cut^- instead of Cut^+.

By virtue of the lemma above, $\Downarrow Cut^-$ can be replaced by:

$$\dfrac{[\Gamma], \Theta \longrightarrow N \qquad \dfrac{[N, \Gamma'] \xrightarrow{N} [R]}{[N, \Gamma'] \longrightarrow [R]}\ Lf}{[\Gamma\Gamma'], \Theta \longrightarrow R}\ Cut^-$$

If N is positive, then via R_l the cut reduces to Cut^+.

Still other forms of cut can be derived in similar fashion, such as:

$$\dfrac{[\Gamma] \xrightarrow{N} [P] \qquad [\Gamma'], \Theta', P \longrightarrow \mathcal{R}}{[N, \Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}}\ Cut$$

But such cuts are non-essential.

The following lemma establishes the admissibility of $Cut^{\rightarrow}$ and its detailed arguments are also used in the overall cut elimination proof.

Lemma 17 $Cut^{\rightarrow}$ *is admissible.*

Proof We must show that, if $[\Gamma] -_{C}\!\!\rightarrow$ and $[C, \Gamma'] -_{R}\!\!\rightarrow$ are provable, then $[\Gamma\Gamma'] -_{R}\!\!\rightarrow$ is also provable. The arguments are arranged into the following cases:

1. C is a positive atom and $C = R$. In this case, the result follows immediately from the assumption $[\Gamma] -_{C}\!\!\rightarrow$ and Lemma 15 above (weakening 1).

2. C is a positive atom such that $C \neq R$, then $C \in \Gamma$, and $[\Gamma\Gamma'] -_{R}\!\!\rightarrow$ follows by weakening on the second premise.

3. C is a neutral (negative) formula. The proof of $[\Gamma] -_{C}\!\!\rightarrow$ must end in the reaction rule R_r, which means that $[\Gamma] \longrightarrow C$ is provable. Under this assumption, we now examine the structure of R.

(a) R is neutral. With the same argument as above, this means we must have

$$\frac{[C,\Gamma'] \longrightarrow R}{[C,\Gamma']-_{R}\rightarrow}\ R_r$$

And so the $Cut^{\rightarrow}$ rule can be replaced by Cut^{-}, which is already admissible:

$$\frac{\dfrac{[\Gamma] \longrightarrow C \quad [C,\Gamma'] \longrightarrow R}{[\Gamma\Gamma'] \longrightarrow R}\ Cut^{-}}{[\Gamma\Gamma']-_{R}\rightarrow}\ R_r$$

The rest of the cases are for positive formula R, and is by induction on the form of R (implicitly, also on the height of the $Cut^{\rightarrow}$).

(b) $R = true$. Nothing interesting here.

(c) R is a positive atom. It has already been ruled out that $R = C$. Thus if $[C,\Gamma']-_{R}\rightarrow$ is provable then it must be that $R \in \Gamma'$. Thus $[\Gamma\Gamma']-_{R}\rightarrow$ is also provable by identity.

(d) $R = A_1 \vee A_2$. Focusing forces $[C,\Gamma']-_{A_i}\rightarrow$ to be provable. By inductive hypothesis, we have that $[\Gamma\Gamma']-_{A_i}\rightarrow$ is provable, and so we can derive:

$$\frac{[\Gamma\Gamma']-_{A_i}\rightarrow}{[\Gamma\Gamma']-_{A_1\vee A_2}\rightarrow}\ \vee R$$

(e) $R = A \wedge^{+} B$. We must have:

$$\frac{[C,\Gamma']-_{A}\rightarrow \quad [C,\Gamma']-_{B}\rightarrow}{[C,\Gamma']-_{A\wedge^{+}B}\rightarrow}\ \wedge^{+}R$$

And so by inductive hypothesis, we can derive

$$\frac{[\Gamma\Gamma']-_{A}\rightarrow \quad [\Gamma\Gamma']-_{B}\rightarrow}{[\Gamma\Gamma']-_{A\wedge^{+}B}\rightarrow}\ \wedge^{+}R$$

(f) The case of the first-order quantifier $R = \exists x A$ is similar to the above.

Implicitly the lemma's proof shows how $Cut^{\rightarrow}$ ultimately reduces to multiple Cut^{-} rules at lesser heights. It also shows how $Cut^{\rightarrow}$ is permuted upwards while preserving right-side focus (in the cases of $R = A_1 \vee A_2$ for example). □

Lemma 18 *$Cut_1^{\leftarrow}$ and $Cut_2^{\leftarrow}$ are admissible.*

Proof

Part I: $Cut_1^{\leftarrow}$.

The proof is by induction on B (implicitly on the total height of the two premises of $Cut_1^{\leftarrow}$).

1. If B is positive, then the proof of the left premise must end in a R_l reaction rule, which means that $[\Gamma], B \longrightarrow [P]$ is provable. The cut is then replaced by a Cut^{+}, which is already admissible:

$$\frac{\dfrac{\dfrac{[\Gamma], B \longrightarrow [P]}{[\Gamma], B \longrightarrow P}\ []_r \quad [\Gamma'], P \longrightarrow [R]}{[\Gamma\Gamma'], B \longrightarrow [R]}\ Cut^{+}}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\ R_l$$

Note that according to the modified measure for the *height* of cuts, the new $Cut^{=}$ rule now has lower height than the original $Cut_1^{\leftarrow}$ rule. This observation is not needed for the admissibility argument but is necessary for proving the termination of simultaneous cut-elimination (see next section).

2. B cannot be an atom, since P is positive.

3. $B = D \supset E$. The left subproof must end in:

$$\frac{[\Gamma] -_D\!\!\rightarrow \quad [\Gamma] \xrightarrow{E} [P]}{[\Gamma] \xrightarrow{D \supset E} [P]} \supset L$$

By inductive hypothesis on E, we have that $[\Gamma\Gamma'] \xrightarrow{E} [R]$ is provable. The case then follows from:

$$\frac{[\Gamma\Gamma'] -_D\!\!\rightarrow \quad [\Gamma\Gamma'] \xrightarrow{E} [R]}{[\Gamma\Gamma'] \xrightarrow{D \supset E} [R]} \supset L$$

Weakening was applied to $[\Gamma] -_D\!\!\rightarrow$ by virtue of Lemma 15. Implicitly, the $Cut_1^{\leftarrow}$ has been pushed up one level.

4. The other cases for B, $D \wedge^- E$ and $\forall x D$ are similar.

Part II: $Cut_2^{\leftarrow}$.

This cut is slightly more involved, and requires also the admissibility of $Cut^{\rightarrow}$. The proof is by induction on B.

1. B is positive. By the necessary application of R_l we get that $[N, \Gamma'], B \longrightarrow [R]$ is provable. We then have:

$$\cfrac{\cfrac{[\Gamma] \longrightarrow N \quad [N, \Gamma'], B \longrightarrow [R]}{[\Gamma\Gamma'], B \longrightarrow [R]}\ Cut^-}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\ R_l$$

Here again, the modified height measurement applies to the Cut^- rule. The rest of the cases are for negative (neutral) B.

2. B is a negative atom. Then it must be that $B = R$. But then the conclusion $[\Gamma\Gamma'] \xrightarrow{B} [B]$ also holds by identity.

3. B is of the form $D \supset E$. The right subproof must end in

$$\frac{[N, \Gamma'] -_D\!\!\rightarrow \quad [N, \Gamma'] \xrightarrow{E} [R]}{[N, \Gamma'] \xrightarrow{D \supset E} [R]} \supset L$$

By inductive hypothesis on E (or the right subproof) we have that $[\Gamma\Gamma'] \xrightarrow{E} [R]$ is provable. We can then make the following derivation by appealing to the admissibility of $Cut^{\rightarrow}$:

$$\cfrac{\cfrac{\cfrac{[\Gamma] \longrightarrow N}{[\Gamma] -_N\!\!\rightarrow}\ R_r \quad [N, \Gamma'] -_D\!\!\rightarrow}{[\Gamma\Gamma'] -_D\!\!\rightarrow}\ Cut^{\rightarrow} \quad [\Gamma\Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'] \xrightarrow{D \supset E} [R]} \supset L$$

This is another instance when the modified height measure of cut rules apply.

4. The cases for $B = D \wedge^- E$ and $B = \forall x D$ are similar. In fact, $D \supset E$ is the only case that requires $Cut^{\rightarrow}$.

□

The case analysis of the admissibility proofs of $Cut^{\rightarrow}$, $Cut_1^{\leftarrow}$ and $Cut_2^{\leftarrow}$ already form parts of the overall cut-elimination procedure. Clearly the variant forms of cut are interdependent.

It is worthwhile noting that the forms of $Cut^{\rightarrow}$, $Cut_1^{\leftarrow}$ and $Cut_2^{\leftarrow}$ must be carefully stated. These are the precise conditions under which focus can be kept *after* a cut. If, instead of as stated, the left premise of $Cut^{\rightarrow}$ were of the form $[\Gamma] \longrightarrow C$, then the cut will fail to be admissible if C is a positive atom:

$$\frac{[A, A \supset C] \longrightarrow C \quad [C] -_{C} \!\!\rightarrow}{[A, A \supset C] -_{C}\!\!\rightarrow}$$

is clearly not admissible as the conclusion is not provable[8].

Similarly, consider the following variation on $Cut_1^{\leftarrow}$:

$$\frac{[\Gamma] \xrightarrow{A} [A] \quad [A, B] \longrightarrow [B]}{[B, \Gamma] \xrightarrow{A} [B]}$$

where A and B are distinct neutral atoms. This variant is not admissible: the cut formula must be positive.

7.2 Simultaneous Elimination of Cuts

More involved then the admissibility of the cuts is a full cut-elimination proof and the corresponding cut-elimination procedure it implies. The case-breakdowns of the lemmas for the admissibility of $Cut^{\rightarrow}$, $Cut_1^{\leftarrow}$ and $Cut_2^{\leftarrow}$ already form parts of the procedure.

The inductive measure for the procedures is the lexicographical ordering on the degree of the largest cut formula and the *height* of cuts, with the modified definition of height. The proof remains structurally similar to other cut-elimination proofs *except* when non-key cuts are permuted above focus rules. That is, when the formula under focus is not the cut formula. These situations require the rules $Cut^{\rightarrow}$, $Cut_1^{\leftarrow}$ and $Cut_2^{\leftarrow}$. That is, they require cuts that preserve focus. A better approach would be to argue that each synchronous phase of a focused proof constitute a critical section, i.e., a "derived rule of inference". We can then permute the cuts above the entire focused segment, when focusing switches (via reactions) to asynchronous decompositions. However, currently it is not clear how to formalize such an argument.

We will not enumerate all cases of the proof for the simple reason that there will be too many. We already know that the cuts are in fact admissible. We will only give the structure of the mutually recursive procedures and show representative cases.

Procedure *Elim_Cut*$^{+}$: the elimination of Cut^{+}:

$$\frac{\overset{\Pi}{[\Gamma], \Theta \longrightarrow P} \quad \overset{\Pi'}{[\Gamma'], \Theta', P \longrightarrow \mathcal{R}}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}}\ Cut^{+}$$

The procedure dispatches on the last rules of subproofs Π and Π'.

1. If the last rule of Π or Π' is one of *parametric* asynchronous decomposition on Θ in Π or $\mathcal{R}$ in Π' (but not on P), the cut can be permuted easily upwards. These cases are analogous to the "non-key" cases of simpler cut-elimination proofs. We can even assume that Θ and Θ' are empty and that $\mathcal{R}$ is of the form $[R]$. We show one representative case: for $\mathcal{R} = A \wedge^{-} B$ (that is, when the last rule in Π' is $\wedge^{-}R$), the cut is replaced by

$$\frac{\dfrac{[\Gamma], \Theta \longrightarrow P \quad [\Gamma'], \Theta', P \longrightarrow A}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow A}\ Cut^{+} \quad \dfrac{[\Gamma], \Theta \longrightarrow P \quad [\Gamma'], \Theta', P \longrightarrow B}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow B}\ Cut^{+}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow A \wedge^{-} B}\ \wedge^{-}R$$

where the new Cut^{+} rules are at lesser heights.

[8] A similar example appeared in [DL06].

2. If the last rules of Π are of the form

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ [\Gamma]-_P\!\!\rightarrow\end{array}}{[\Gamma]\longrightarrow[P]}\,Rf}{[\Gamma]\longrightarrow P}\,[]_r$$

   The Cut^+ reduces to

$$\dfrac{[\Gamma]-_P\!\!\rightarrow \quad [\Gamma'],\Theta',P\longrightarrow\mathcal{R}}{[\Gamma\Gamma'],\Theta'\longrightarrow\mathcal{R}}\Downarrow Cut^+$$

   which is of lesser total height: the sum of the heights of Π_1 and Π' is less than the sum of the heights of Π and Π'. Note that by virtue of case 1, we can rule out the situation where P is decomposed before Θ is empty (i.e. before the left subproof is ready for focus).

3. If the last rules of Π are of the form

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ [\Gamma]\stackrel{N}{\longrightarrow}[P]\end{array}}{[\Gamma]\longrightarrow[P]}\,Lf}{[\Gamma]\longrightarrow P}\,[]_r\ ,\ N\in\Gamma$$

   then the Cut^+ is replaced depending on the form of N, which could be $A\supset B$, $A\wedge^- B$ or $\forall xA$. The most interesting case, as usual, is $A\supset B$. The inference

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma]-_A\!\!\rightarrow \quad [\Gamma]\stackrel{B}{\longrightarrow}[P]}{[\Gamma]\stackrel{A\supset B}{\longrightarrow}[P]}\supset L}{[\Gamma]\longrightarrow[P]}\,Lf}{[\Gamma]\longrightarrow P}\,[]_r \qquad [\Gamma'],P\longrightarrow[R]}{[\Gamma\Gamma']\longrightarrow R}\,Cut^+$$

   is replaced by

$$\dfrac{\dfrac{\dfrac{[\Gamma\Gamma']-_A\!\!\rightarrow \qquad \dfrac{[\Gamma]\stackrel{B}{\longrightarrow}[P] \quad [\Gamma'],P\longrightarrow[R]}{[\Gamma\Gamma']\stackrel{B}{\longrightarrow}[R]}\,Cut_1^{\leftarrow}}{[\Gamma\Gamma']\stackrel{A\supset B}{\longrightarrow}[R]}\supset L}{[\Gamma\Gamma']\longrightarrow[R]}\,Lf}{[\Gamma\Gamma']\longrightarrow R}\,[]_r$$

   Lemma 15 is again applied for weakening. The other cases will also use $Cut_1^{\leftarrow}$. Note that although the modified inductive measure applies to the Cut^+ rule, the replacement cut is still at a lower height.

Procedure *Elim_Cut*$^-$: the elimination of Cut^-:

$$\dfrac{\begin{array}{c}\Pi\\ [\Gamma],\Theta\longrightarrow C\end{array} \quad \begin{array}{c}\Pi'\\ [C,\Gamma'],\Theta'\longrightarrow\mathcal{R}\end{array}}{[\Gamma\Gamma'],\Theta\Theta'\longrightarrow\mathcal{R}}\,Cut^-$$

where C is a neutral (negative) formula or positive atom.

1. If the last rule of Π or of Π' is one of parametric asynchronous decomposition on Θ, Θ' or $\mathcal{R}$, permute the Cut^- upwards.

2. if Π is of the form

$$\dfrac{\dfrac{\dfrac{}{[\Gamma]\xrightarrow{C}}\ Ir}{[\Gamma] \longrightarrow [C]}\ Rf}{[\Gamma] \longrightarrow C}\ []_r$$

where C is a positive atom. Then it must be that $C \in \Gamma$, and so the conclusion $[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}$ follows by weakening (Lemma 15) from the right subproof Π'.

3. if Π is of the form

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ [\Gamma] \xrightarrow{N} [C]\end{array}}{[\Gamma] \longrightarrow [C]}\ Lf}{[\Gamma] \longrightarrow C}\ []_r$$

where C is a positive atom and $N \in \Gamma$, the cut replacement depends on the form of N. These cases are similar to the analogous ones in procedure *Elim_Cut*$^+$. Here's another example:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma] \xrightarrow{A_1} [C]}{[\Gamma] \xrightarrow{A_1 \wedge^- A_2} [C]}\ \wedge^- L}{[\Gamma] \longrightarrow [C]}\ Lf}{[\Gamma] \longrightarrow C}\ []_r \qquad [C, \Gamma'] \longrightarrow [R]}{[\Gamma\Gamma'] \longrightarrow [R]}\ Cut^-$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma] \xrightarrow{A_1} [C] \qquad \dfrac{[C, \Gamma'] \longrightarrow [R]}{[\Gamma'], C \longrightarrow [R]}\ []_l}{[\Gamma\Gamma'] \xrightarrow{A_1} [R]}\ Cut_1^{\leftarrow}}{[\Gamma\Gamma'] \xrightarrow{A_1 \wedge^- A_2} [R]}\ \wedge^- L}{[\Gamma\Gamma'] \longrightarrow [R]}\ Lf}{[\Gamma\Gamma'] \longrightarrow R}\ []_r$$

We are assuming that $\mathcal{R}$ has been decomposed to $[R]$. A careful calculation of the modified height measurement shows that the replacement $Cut_1^{\leftarrow}$ rule is at a lower "height".

The rest of the cases can assume that C is negative, as well as that Θ, Θ' are empty and that $\mathcal{R}$ is of the form $[R]$.

4. If Π' is of the form

$$\dfrac{[C, \Gamma] \xrightarrow{C} [R]}{[C, \Gamma'] \longrightarrow [R]}\ Lf$$

Then the cut becomes a $\Downarrow Cut^-$ (key cut) at a lesser height.

5. If Π' is of the form

$$\dfrac{[C, \Gamma] \xrightarrow{N} [R]}{[C, \Gamma'] \longrightarrow [R]}\ Lf$$

where $N \neq C$, then, depending on N, this case reduces to $Cut_2^{\leftarrow}$. Here is one example:

$$\dfrac{[\Gamma] \longrightarrow C \qquad \dfrac{\dfrac{[C, \Gamma'] \xrightarrow{A[t/x]} [R]}{[C, \Gamma'] \xrightarrow{\forall x A} [R]}\ \forall L}{[C, \Gamma'] \longrightarrow [R]}\ Lf}{[\Gamma\Gamma'] \longrightarrow [R]}\ Cut^-$$

is replaced by

$$\dfrac{\dfrac{\dfrac{[\Gamma] \longrightarrow C \quad [C,\Gamma'] \xrightarrow{A[t/x]} [R]}{[\Gamma\Gamma'] \xrightarrow{A[t/x]} [R]}\ Cut_2^{\leftarrow}}{[\Gamma\Gamma'] \xrightarrow{\forall x A} [R]}\ \forall L}{[\Gamma\Gamma'] \longrightarrow [R]}\ Lf$$

and other cases are similar.

6. If Π' is of the form

$$\dfrac{[C,\Gamma']-_{R}\!\!\rightarrow}{[C,\Gamma'] \longrightarrow [R]}\ Rf$$

then we will need the $Cut^{\rightarrow}$ rule, depending on the form of R. For example, if $R = A \wedge^{+} B$, then

$$\dfrac{[\Gamma] \longrightarrow C \qquad \dfrac{\dfrac{[C,\Gamma']-_{A}\!\!\rightarrow \quad [C,\Gamma']-_{B}\!\!\rightarrow}{[C,\Gamma']-_{A\wedge^{+}B}\!\!\rightarrow}\ \wedge^{+}R}{[C,\Gamma'] \longrightarrow [A \wedge^{+} B]}\ Rf}{[\Gamma\Gamma'] \longrightarrow [A \wedge^{+} B]}\ Cut^{-}$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma] \longrightarrow C}{[\Gamma]-_{C}\!\!\rightarrow}\ R_r \qquad [C,\Gamma']-_{A}\!\!\rightarrow}{[\Gamma\Gamma']-_{A}\!\!\rightarrow}\ Cut^{\rightarrow} \qquad \dfrac{\dfrac{[\Gamma] \longrightarrow C}{[\Gamma]-_{C}\!\!\rightarrow}\ R_r \qquad [C,\Gamma']-_{B}\!\!\rightarrow}{[\Gamma\Gamma']-_{B}\!\!\rightarrow}\ Cut^{\rightarrow}}{[\Gamma\Gamma']-_{A\wedge^{+}B}\!\!\rightarrow}\ \wedge^{+}R}{[\Gamma\Gamma'] \longrightarrow [A \wedge^{+} B]}\ Rf$$

The new $Cut^{\rightarrow}$ rules are at lesser heights than the original Cut^{-} rule again because of the modified definition of the height of cuts. Other cases are similar.

Procedure *Elim_Cut*$^{\rightarrow}$**:** See Lemma 17.

Procedure *Elim_Cut*$_1^{\leftarrow}$**:** See Lemma 18 part I.

Procedure *Elim_Cut*$_2^{\leftarrow}$**:** See Lemma 18 part II.

Procedure *Elim_*$\Downarrow Cut^{+}$**:**

Parametric asynchronous decomposition permutes as before. In fact, we can assume that $\Downarrow Cut^{+}$ is not applied until Θ' is empty and $\mathcal{R}$ is of the form $[R]$. Also note that if the cut formula is not positive, then the rules above the cut are necessarily R_r on the left premise and $[]_l$ on the right premise, which reduces the $\Downarrow Cut^{+}$ rule to a Cut^{-} rule.

The other cases correspond to "key" cases where the cut formula is positive. Here is one example:

$$\dfrac{\dfrac{[\Gamma]-_{A}\!\!\rightarrow \quad [\Gamma]-_{B}\!\!\rightarrow}{[\Gamma]-_{A\wedge^{+}B}\!\!\rightarrow}\ \wedge^{+}R \qquad \dfrac{[\Gamma'], \Theta', A, B \longrightarrow \mathcal{R}}{[\Gamma'], \Theta', A \wedge^{+} B \longrightarrow \mathcal{R}}\ \wedge^{+}L}{[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}}\ \Downarrow Cut^{+}$$

is replaced by

$$\dfrac{[\Gamma\Gamma']-_{B}\!\!\rightarrow \qquad \dfrac{[\Gamma]-_{A}\!\!\rightarrow \quad [\Gamma'], \Theta', A, B \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta', B \longrightarrow \mathcal{R}}\ \Downarrow Cut^{+}}{[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}}\ \Downarrow Cut^{+}$$

So one cut is replaced by two cuts, but of lower degree.

The case of the existiential quantifier invokes the first-order substitution lemma:

$$\dfrac{\dfrac{[\Gamma]-_{A[t/x]}\!\!\rightarrow}{[\Gamma]-_{\exists xA}\!\!\rightarrow}\ \exists R \qquad \dfrac{[\Gamma'], \Theta', A \longrightarrow \mathcal{R}}{[\Gamma'], \Theta', \exists x A \longrightarrow \mathcal{R}}\ \exists L}{[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}}\ \Downarrow Cut^+$$

is replaced by

$$\dfrac{[\Gamma]-_{A[t/x]}\!\!\rightarrow \qquad [\Gamma'], \Theta', A[t/x] \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta' \longrightarrow \mathcal{R}}\ \Downarrow Cut^+$$

Since the $\exists L$ rule requires that x is not free in Γ', Θ' or $\mathcal{R}$, Lemma 16 applies, and gives us a cut at a lower height.

Procedure $Elim\_\Downarrow Cut^-$**:**
Assume that Θ is empty when the rule is applied. These cases are the rest of the "key" cuts, where the cut formula is neutral. Here two more examples:

$$\dfrac{\dfrac{[\Gamma], \Theta \longrightarrow A \quad [\Gamma], \Theta \longrightarrow B}{[\Gamma], \Theta \longrightarrow A \wedge^- B}\ \wedge^- R \qquad \dfrac{[\Gamma'] \stackrel{A}{\longrightarrow} [R]}{[\Gamma'] \stackrel{A\wedge^- B}{\longrightarrow} [R]}\ \wedge^- L}{[\Gamma\Gamma'], \Theta \longrightarrow [R]}\ \Downarrow Cut^-$$

is replaced by

$$\dfrac{[\Gamma], \Theta \longrightarrow A \quad [\Gamma'] \stackrel{A}{\longrightarrow} [R]}{[\Gamma\Gamma'], \Theta \longrightarrow [R]}\ \Downarrow Cut^-$$

And for implication:

$$\dfrac{\dfrac{[\Gamma], \Theta, A \longrightarrow B}{[\Gamma], \Theta \longrightarrow A \supset B}\ \supset R \qquad \dfrac{[\Gamma']-_{A}\!\!\rightarrow \quad [\Gamma'] \stackrel{B}{\longrightarrow} [R]}{[\Gamma'] \stackrel{A\supset B}{\longrightarrow} [R]}\ \supset L}{[\Gamma\Gamma'], \Theta \longrightarrow [R]}\ \Downarrow Cut^-$$

is replaced by

$$\dfrac{\dfrac{[\Gamma']-_{A}\!\!\rightarrow \quad [\Gamma], \Theta, A \longrightarrow B}{[\Gamma\Gamma'], \Theta \longrightarrow B}\ \Downarrow Cut^+ \qquad [\Gamma'] \stackrel{B}{\longrightarrow} [R]}{[\Gamma\Gamma'\Gamma'], \Theta \longrightarrow [R]}\ \Downarrow Cut^-$$

with implicit contractions.
The above procedures independently prove the following:

Theorem 19 *The rules* Cut^+, Cut^-, $\Downarrow Cut^+$, $\Downarrow Cut^-$, $Cut^{\rightarrow}$, $Cut_1^{\leftarrow}$ *and* $Cut_2^{\leftarrow}$ *are all admissible in* LJF.

□

8 Embedding Intuitionistic Systems within LJF

LJF is capable of being a host-logic framework in that it can account for several other intuitionistic proof systems. For example, a system such as LJQ' can be embedded into LJF using techniques similar to its translation to linear logic. We define a translation of intuitionistic to intuitionistic formulas for this effect. The key is to neutralize the additional focusing characteristics of LJF. Let $\partial^-(A)$ and $\partial^+(A)$ denote the following

$$\partial^-(A) = true \supset A \quad \text{and} \quad \partial^+(A) = true \wedge^+ A.$$

That is, $\partial^-(A)$ is always negative/neutral and $\partial^+(A)$ is always positive. The following translation, which is necessarily hereditary, uses these devices to stop asynchronous decomposition on the left as well as the right hand side. It also needs to stop left-side focusing. However, the only left-synchronous formulas in LJQ' are of the form $A \supset B$, thus it suffices to stop focusing on B. The translation uses l and r labels to indicate left and right-side translations.

- atom B: $B^l = B^r = B$
- $false^l = \partial^-(false)$, $false^r = false$
- $(A \wedge B)^l = \partial^-(A^l \wedge^+ B^l)$, $(A \wedge B)^r = A^r \wedge^+ B^r$
- $(A \vee B)^l = \partial^-(A^l \vee B^l)$, $(A \vee B)^r = A^r \vee B^r$
- $(A \supset B)^l = A^r \supset \partial^+(B^l)$, $(A \supset B)^r = \partial^+(A^l \supset B^r)$

Since all formulas A^l are negative and all A^r are positive, we can embed the LJQ' sequent $\Gamma \Rightarrow G$ as $[\Gamma^l] \longrightarrow [G^r]$. LJQ' focusing sequents $\Gamma \rightarrow G$ naturally becomes $[\Gamma^l]-_{G^r}\!\!\rightarrow$. The correspondance between LJQ' rules and LJF derivations is sampled below:

1.

$$\frac{\Gamma \rightarrow R}{\Gamma \Rightarrow R}\ Der \qquad \longrightarrow \qquad \frac{[\Gamma^l]-_{R^r}\!\!\rightarrow}{[\Gamma^l] \longrightarrow [R^r]}\ Rf$$

2. $(A \wedge B)^r = A^r \wedge^+ B^r$:

$$\frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B}\ R\wedge' \qquad \longrightarrow \qquad \frac{[\Gamma^l]-_{A^r}\!\!\rightarrow \quad [\Gamma^l]-_{B^r}\!\!\rightarrow}{[\Gamma^l]-_{A^r \wedge^+ B^r}\!\!\rightarrow}\ \wedge^+ R$$

3. $(A \supset B)^r = \partial^+(A^l \supset B^r) = true \wedge^+ (A^l \supset B^r)$:

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \rightarrow A \supset B}\ R\supset' \qquad \longrightarrow \qquad \dfrac{\dfrac{}{[\Gamma^l]-_{true}\!\!\rightarrow}\ t \quad \dfrac{\dfrac{\dfrac{[\Gamma^l, A^l] \longrightarrow B^r}{[\Gamma^l], A^l \longrightarrow B^r}\ []_l}{[\Gamma^l] \longrightarrow A^l \supset B^r}\ \supset R}{[\Gamma^l]-_{A^l \supset B^r}\!\!\rightarrow}\ R_r}{[\Gamma^l]-_{\partial^+(A^l \supset B^r)}\!\!\rightarrow}\ \wedge^+ R$$

4. All left rules begin with Lf ($[D_2]$). $(A \supset B)^l = A^r \supset \partial^+(B^l) = A^r \supset (true \wedge^+ B^l)$:

$$\frac{\Gamma \rightarrow A \quad \Gamma, B \Rightarrow R}{\Gamma, A \supset B \Rightarrow R}\ L\supset'$$

becomes

$$\dfrac{\dfrac{[\Gamma^l, A^r \supset \partial^+(B^l)]-_{A^r}\!\!\rightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \supset B)^l, B^l] \longrightarrow [R^r]}{[\Gamma^l, A^r \supset \partial^+(B^l), B^l], true \longrightarrow [R^r]}\, trueL}{[\Gamma^l, A^r \supset \partial^+(B^l)], true, B^l \longrightarrow [R^r]}\, []_l}{[\Gamma^l, A^r \supset \partial^+(B^l)], true \wedge^+ B^l \longrightarrow [R^r]}\, \wedge^+ L}{[\Gamma^l, A^r \supset \partial^+(B^l)] \xrightarrow{true \wedge^+ B^l} [R^r]}\, R_l}{[\Gamma^l, A^r \supset \partial^+(B^l)] \xrightarrow{A^r \supset \partial^+(B^l)} [R^r]}\, \supset L}{[\Gamma^l, A^r \supset \partial^+(B^l)] \longrightarrow [R^r]}\, Lf$$

5. $(A \wedge B)^l = \partial^-(A^l \wedge^+ B^l) = true \supset (A^l \wedge^+ B^l)$:

$$\dfrac{\Gamma, A, B \Rightarrow R}{\Gamma, A \wedge B \Rightarrow R}\, L\wedge' \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{}{[\Gamma^l, (A \wedge B)^l]-_{true}\!\!\rightarrow}\, trueR \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \wedge B)^l, A^l, B^l] \longrightarrow [R^r]}{[\Gamma^l, (A \wedge B)^l, A^l], B^l \longrightarrow [R^r]}\, []_l}{[\Gamma^l, (A \wedge B)^l], A^l, B^l \longrightarrow [R^r]}\, []_l}{[\Gamma^l, (A \wedge B)^l], A^l \wedge^+ B^l \longrightarrow [R^r]}\, \wedge^+ L}{[\Gamma^l, (A \wedge B)^l] \xrightarrow{(A^l \wedge^+ B^l)} [R^r]}\, R_l}{[\Gamma^l, (A \wedge B)^l] \xrightarrow{\partial^-(A^l \wedge^+ B^l)} [R^r]}\, \supset L}{[\Gamma^l, (A \wedge B)^l] \longrightarrow [R^r]}\, Lf$$

The fact that the mapping is one-to-one also offers another proof of completeness of LJQ', this time based on the completeness of LJF.

8.1 Embedding LJT in LJF

Other systems can also be embedded into LJF using similar *delaying* transformations. For LJT', focusing should be stopped on the right and asynchronous decomposition stopped on the left. We define the following transformation, reusing the symbols l and r for convenience:

- for atom B: $B^l = B^r = B$. (all atoms negative)
- $\mathit{false}^l = \partial^-(\mathit{false})$, $\mathit{false}^r = \mathit{false}$, $\mathit{true}^l = \partial^-(\mathit{true})$, $\mathit{true}^r = \mathit{true}$
- $(A \supset B)^l = \partial^-(A^r) \supset B^l$, $(A \supset B)^r = A^l \supset B^r$
- $(A \wedge B)^l = A^l \wedge^- B^l$, $(A \wedge B)^r = A^r \wedge^- B^r$
- $(A \vee B)^l = \partial^-(A^l \vee B^l)$, $(A \vee B)^r = \partial^-(A^r) \vee \partial^-(B^r)$
- $(\exists x A)^l = \partial^-(\exists x A^l)$, $(\exists x A)^r = \exists x \partial^-(A^r)$
- $(\forall x A)^l = \forall x A^l$, $(\forall x A)^r = \forall x A^r$

All formulas A^l are negative, so the LJT' sequent $\Gamma \Rightarrow G$ becomes $[\Gamma^l] \longrightarrow G^r$. Sequents $\Gamma \xrightarrow{A} [G]$ naturally becomes $[\Gamma^l] \xrightarrow{A^l} [G^r]$. Samples mappings are given below:

1.

$$\dfrac{\Gamma, B \xrightarrow{B} [R]}{\Gamma, B \Rightarrow [R]}\, Choose \qquad \longrightarrow \qquad \dfrac{[B^l, \Gamma^l] \xrightarrow{B^l} [R^r]}{[B^l, \Gamma^l] \longrightarrow [R^r]}\, Lf$$

2.

$$\dfrac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow [A_1 \vee A_2]}\ \vee R \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l] \longrightarrow A_i^r}{[\Gamma^l], \mathit{true} \longrightarrow A_i^r}\ \mathit{trueL}}{[\Gamma^l] \longrightarrow \mathit{true} \supset A_i^r}\ \supset R}{[\Gamma^l] \xrightarrow{\partial^-(A_i^r)}}\ R_r}{[\Gamma^l] \xrightarrow{\partial^-(A_1^r) \vee \partial^-(A_2^r)}}\ \vee R}{[\Gamma^l] \longrightarrow [\partial^-(A_1^r) \vee \partial^-(A_2^r)]}\ Rf$$

Right-side focus is only one level deep.

3.

$$\dfrac{\Gamma, A \Rightarrow [R]}{\Gamma \xrightarrow{\exists y A} [R]}\ \exists L \qquad \longrightarrow \qquad \dfrac{\dfrac{}{[\Gamma^l, (\exists y A)^l] \xrightarrow{\mathit{true}}}\ \mathit{trueR} \qquad \dfrac{\dfrac{\dfrac{[\Gamma^l, (\exists y A)^l, A^l] \longrightarrow [R^r]}{[\Gamma^l, (\exists y A)^l], A^l \longrightarrow [R^r]}\ []_l}{[\Gamma^l, (\exists y A)^l], \exists y A^l \longrightarrow [R^r]}\ \exists L}{[\Gamma^l, (\exists y A)^l] \xrightarrow{\exists y A^l} [R^r]}\ R_r}{[\Gamma^l, (\exists y A)^l] \xrightarrow{\partial^-(\exists y A^l)} [R^r]}\ \supset L$$

Left-side decomposition is also only one level deep.

4.

$$\dfrac{\Gamma \Rightarrow A \quad \Gamma \xrightarrow{B} [R]}{\Gamma \xrightarrow{A \supset B} [R]}\ \supset L \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \supset B)^l] \longrightarrow A^r}{[\Gamma^l, (A \supset B)^l], \mathit{true} \longrightarrow A^r}\ \mathit{trueL}}{[\Gamma^l, (A \supset B)^l] \longrightarrow \mathit{true} \supset A^r}\ \supset R}{[\Gamma^l, (A \supset B)^l] \xrightarrow{\partial^-(A^r)}}\ R_r \qquad [\Gamma^l, (A \supset B)^l] \xrightarrow{B^l} [R^r]}{[\Gamma^l, (A \supset B)^l] \xrightarrow{\partial^-(A^r) \supset B^l} [R^r]}\ \supset L$$

Other cases are similar.

8.2 Embedding LJ

Given the above encodings of "less focused" proofs, we can device a general strategy for preventing focusing features in general:

1. to stop eager asynchronous decomposition on the left, make sure the embedding A^l is negative (except for atoms).
2. to stop eager asynchronous decomposition on the right, make sure A^r is positive (except for atoms).
3. to stop left-side focus after one level, use subformulas of the form $\partial^+(A^l)$ in embeddings of left-synchronous formulas.
4. to stop right-side focus, use subformulas of the form $\partial^-(A^r)$.

Naturally, it is possible to embed arbitrary LJ proofs (in the sense of $\vdash_I$) as LJF proofs. The embedding is given in Table 3.

Sequents $\Gamma \vdash_I G$ are embedded as $[\Gamma^l] \longrightarrow [G^r]$. The embedding echos the 0/1 translation of $\vdash_I$ proofs into linear logic. One can observe the effects of the embedding in the following sample mappings:

F	F^l (left)	F^r (right)
atom C	C	C
false	$\partial^-(\mathit{false})$	*false*
true	$\partial^-(\mathit{true})$	*true*
$A \wedge B$	$\partial^+(A^l) \wedge^- \partial^+(B^l)$	$\partial^+(A^r \wedge^- B^r)$
$A \vee B$	$\partial^-(A^l \vee B^l)$	$\partial^-(A^r) \vee \partial^-(B^r)$
$A \supset B$	$\partial^-(A^r) \supset \partial^+(B^l)$	$\partial^+(A^l \supset B^r)$
$\exists x A$	$\partial^-(\exists x A^l)$	$\exists x \partial^-(A^r)$
$\forall x A$	$\forall x \partial^+(A^l)$	$\partial^+(\forall x A^r)$

Table 3: Embedding of LJ proofs as LJF proofs

- embedding of $\supset L$:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \supset B)^l] \longrightarrow A^r}{[\Gamma^l, (A \supset B)^l], \mathit{true} \longrightarrow A^r}\ \mathit{trueL}}{[\Gamma^l, (A \supset B)^l] \longrightarrow \mathit{true} \supset A^r}\ \supset R}{[\Gamma^l, (A \supset B)^l] -_{\partial^-(A^r)}\!\!\longrightarrow}\ R_r
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma^l, (A \supset B)^l, B^l] \longrightarrow [R^r]}{[\Gamma^l, \partial^-(A^r) \supset \partial^+(B^l)], B^l \longrightarrow [R^r]}\ []_l}{[\Gamma^l, \partial^-(A^r) \supset \partial^+(B^l)], \mathit{true}, B^l \longrightarrow [R^r]}\ \mathit{trueL}}{[\Gamma^l, \partial^-(A^r) \supset \partial^+(B^l)], \mathit{true} \wedge^+ B^l \longrightarrow [R^r]}\ \wedge^+ L}{[\Gamma^l, \partial^-(A^r) \supset \partial^+(B^l)] \overset{\mathit{true} \wedge^+ B^l}{\longrightarrow} [R^r]}\ R_l
}{
\dfrac{[\Gamma^l, (A \supset B)^l] \overset{\partial^-(A^r) \supset \partial^+(B^l)}{\longrightarrow} [R^r]}{[\Gamma^l, (A \supset B)^l)] \longrightarrow [R^r]}\ Lf
}\ \supset L
$$

- embedding of $\wedge R$:

$$
\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{[\Gamma] \longrightarrow [A^r]}{[\Gamma] \longrightarrow A^r}\ []_r \quad \dfrac{[\Gamma] \longrightarrow [B^r]}{[\Gamma] \longrightarrow B^r}\ []_r}{[\Gamma] \longrightarrow A^r \wedge^- B^r}\ \wedge^- R}{[\Gamma] -_{A^r \wedge^- B^r}\!\!\longrightarrow}\ R_r
\qquad
\dfrac{}{[\Gamma] -_{\mathit{true}}\!\!\longrightarrow}\ \mathit{trueR}
}{[\Gamma] -_{(A^r \wedge^- B^r) \wedge^+ \mathit{true}}\!\!\longrightarrow}\ \wedge^+ R}{[\Gamma] \longrightarrow [\partial^+(A^r \wedge^- B^r)]}\ Rf
$$

Other cases are similar. The embedding also offers a completeness proof for LJF independently of its origins in linear logic. Together with the cut elimination results of Section 7, one can study LJF as purely intuitionistic proof theory.

Proposition 20 *LJF is sound and complete with respect to intuitionistic logic.*

Proof Soundness is trivial. For completeness, the embedding of Table 3 shows that if $\Gamma \vdash_I G$ is provable then $[\Gamma^l] \longrightarrow [G^r]$ is provable. Let A^* represent the formula A with all occurrences of $\wedge$ replaced by $\wedge^-$. We can show that for every formula A: $[], A^* \longrightarrow A^l$ and $[], A^r \longrightarrow A^*$ are provable. These follow from simultaneous inductions. Then using the cut rules and cut-elimination results of Section 7, we also have that $[], \Gamma^* \longrightarrow G^*$ is provable. □

9 Embedding Classical Logic in Intuitionistic Logic

In this section we use the newly formed proof system LJF to formulate a *focused* sequent calculus for classical logic that reveals the latter's constructive content in a way similar to the LC system on which it is based. While it is possible to derive such a system again using linear logic, classical logic can also be embedded within intuitionistic logic using the well-known *double-negation* translations of Gödel [Göd33], Gentzen and Kolmogorov, who had the earliest [Kol25]. These translations do not, however, yield significant focusing features: they are not sufficiently sensitive to polarities. The use of double-negation plays the role of a throttle, similar to the role played by the exponential ! when mapping intuitionistic logic into linear logic. Girard's *polarized* version of the double negation translation approaches the problem of capturing duality in a more subtle way. This translation was given as one way to explain LC. Looking at focused proofs in the image of this translation will indeed yield a focused version of LC. However, following the style of LJF, we can also define dual versions of each propositional connective. This leads to a more usable calculus. We thus modify the LC translation in a natural way, which is still consistent with its original intent. Our translation will also take advantage of the separation between $\wedge^+$ and $\wedge^-$ in LJF. The final system we derive will be called *LKF*.

First we define what polarity means in this new setting. From the perspective of linear logic and LU, we must now separate the usage of terms "negative" from "neutral." Negative formulas are right-permeable.

Definition 21 Atoms in LKF are arbitrarily of positive or negative polarity. Positive formulas are among positive atoms, $\mathcal{T}$, $\mathcal{F}$, $A \wedge^+ B$, $A \vee^+ B$, $A \supset^+ B$ and $\exists x A$. Negative formulas are among negative atoms, $\neg\mathcal{T}$, $\neg\mathcal{F}$, $A \wedge^- B$, $A \vee^- B$, $A \supset^- B$ and $\forall x A$. Negation $\neg A$ is defined by de Morgan duals:

1. For literals A, $\neg A$ is the dual of A. In particular, $\neg\neg A = A$
2. $\wedge^+$ is the dual of $\vee^-$
3. $\wedge^-$ is the dual of $\vee^+$
4. $\exists$ is the dual of $\forall$

The dual versions of disjunction naturally give rise to dual versions of implication. Negative implication $A \supset^- B$ is defined as $\neg A \vee^- B$ and $A \supset^+ B$ is defined as $\neg A \vee^+ B$. The negation of a positive is negative and vice versa. Formulas in classical logic can be assumed to be in negation normal form: negation appears only in front of atoms and constants.

The constants $\mathcal{T}$, $\mathcal{F}$, $\neg\mathcal{T}$ and $\neg\mathcal{F}$ are best described, respectively, as 1, 0, $\bot$ and $\top$ in linear logic. Just as we have dual versions of each connective, we also have dual versions of each identity. But this is not linear logic as the formulas are polarized *in the extreme.* There is no "splitting the context." The distinction between $\vee^-$ and $\vee^+$, and between $\wedge^-$ and $\wedge^+$, is not in provability but in the structuring of proofs. We have taken the liberty of using many of the same symbols for the connectives of LKF as in LJF. We shall only derive a one-sided calculus.

Let $\sim A$ represent the intuitionistic formula $A \supset \phi$ where ϕ is *some unspecified positive atom.* The intuitionistic polarity of $\sim A$ is always neutral. The polarized embedding of classical logic into intuitionistic logic is given in Table 4.

In the table, a formula of the form A is meant to represent formulas that are *not* preceded by a $\sim$.

Variations are possible on the embedding. The reader may be concerned in particular as to why the classical $\wedge^-$ is not defined in terms of the intuitionistic $\wedge^-$. The embeddings are selected to enforce the dualities $\wedge^-/\vee^+$ and $\wedge^+/\vee^-$. The alternatives would also work, but will increase the complexity of the mapping between proofs. Here, the cases all follow the pattern P or $\sim P$ where P is a positive formula (in the intuitionistic sense). The neutralized positive formulas correspond to asynchronous formulas while the non-neutralized ones can accept focus. Perhaps surprising is that, in order to maintain the scenario of assigning either polarity to classical atoms, *all classical atoms are assigned positive intuitionistic polarity.* If this were not the case then the negation of a negative atom would not be positive. Neutral intuitionistic

Let $(.)^{\approx}$ *represent the translation.*

$\mathcal{T}^{\approx} = true$, $\mathcal{F}^{\approx} = false$, $(\neg\mathcal{T})^{\approx} = \sim true$, $(\neg\mathcal{F})^{\approx} = \sim false$.

For positive classical atom P, $P^{\approx} = P$, *where* P *is assigned positive intuitionistic polarity.*

For negative classical atom N, $N^{\approx} = \sim N$, *where* N *is assigned positive intuitionistic polarity.*

For compound formulas (A, B *represent formulas not preceded by* $\sim$*):*

$\mathcal{A}^{\approx}$	$\mathcal{B}^{\approx}$	$(\mathcal{A} \wedge^+ \mathcal{B})^{\approx}$	$(\mathcal{A} \wedge^- \mathcal{B})^{\approx}$	$(\mathcal{A} \vee^+ \mathcal{B})^{\approx}$	$(\mathcal{A} \vee^- \mathcal{B})^{\approx}$	$(\neg\mathcal{A})^{\approx}$
A	B	$A \wedge^+ B$	$\sim(\sim A \vee \sim B)$	$A \vee B$	$\sim(\sim A \wedge^+ \sim B)$	$\sim A$
A	$\sim B$	$A \wedge^+ \sim B$	$\sim(\sim A \vee B)$	$A \vee \sim B$	$\sim(\sim A \wedge^+ B)$	.
$\sim A$	B	$\sim A \wedge^+ B$	$\sim(A \vee \sim B)$	$\sim A \vee B$	$\sim(A \wedge^+ \sim B)$	A
$\sim A$	$\sim B$	$\sim A \wedge^+ \sim B$	$\sim(A \vee B)$	$\sim A \vee \sim B$	$\sim(A \wedge^+ B)$	.

$\mathcal{A}^{\approx}$	$\mathcal{B}^{\approx}$	$(\mathcal{A} \supset^+ \mathcal{B})^{\approx}$	$(\mathcal{A} \supset^- \mathcal{B})^{\approx}$	$(\forall x\mathcal{A})^{\approx}$	$(\exists x\mathcal{A})^{\approx}$
A	B	$\sim A \vee B$	$\sim(A \wedge^+ \sim B)$	$\sim(\exists x \sim A)$	$\exists x A$
A	$\sim B$	$\sim A \vee \sim B$	$\sim(A \wedge^+ B)$	.	.
$\sim A$	B	$A \vee B$	$\sim(\sim A \wedge^+ \sim B)$	$\sim(\exists x A)$	$\exists x \sim A$
$\sim A$	$\sim B$	$A \vee \sim B$	$\sim(\sim A \wedge^+ B)$	.	.

Table 4: Embedding of LKF into LJF

atoms are not used in the translation. Note also that the classical $\wedge^+$, $\vee^+$ and $\exists$ correspond exactly to the intuitionistic $\wedge^+$, $\vee$ and $\exists$.

When encoding the translated formulas into the left-hand sides of intuitionistic sequents, positive formulas will need to be neutralized (preceded by $\sim$) while neutralized formulas will become positive (they will loose the $\sim$).

9.1 LKF

Translating a classical sequent $\vdash \Delta$ into LJF involves the following steps:

1. Divide Δ into the form $[\Gamma'], \Theta'$, where Γ' contains all positive formulas $C_1 \dots C_j$ of Δ and Θ' contains all negative formulas $D_1 \dots D_k$ of Δ.
2. Translate Γ' and Θ' into $\Gamma'^{\approx}$ and $\Theta'^{\approx}$. We have that $\Gamma'^{\approx} = \{P_1, \dots, P_j\}$ for positive intuitionistic formulas $P_1 \dots P_j$ and that $\Theta'^{\approx} = \{\sim Q_1, \dots, \sim Q_k\}$ for positive intuitionistic formulas $Q_1 \dots Q_k$.
3. Form the LJF sequent $[\Gamma], \Theta \longrightarrow [\phi]$, where $\Gamma = \{\sim P_1, \dots \sim P_j\}$ and $\Theta = \{Q_1, \dots Q_k\}$.

There will be two types of LKF sequents, which will correspond to the following LJF sequents:

unfocused sequent: $\vdash [\Gamma'], \Theta'$ corresponds to $[\Gamma], \Theta \longrightarrow [\phi]$.

focused sequent: $\mapsto [\Gamma'], \mathcal{P}$ corresponds to $[\Gamma] \xrightarrow{\mathcal{P}^{\approx}}$

It is assumed that the intuitionistic positive atom ϕ is distinct from all classical atoms, so there is no possibility of conflict.

Mapping between LJF proofs and LKF proofs

We will not enumerate the mapping between proofs for every case of the translation, as we have done for LJF. The complete mapping can be deduced from the following examples. To help clarify the presentation let us summarize how to interpret formulas in an LJF sequent of the form

$$[\Gamma, \sim P, N], \Theta, Q \longrightarrow [\phi]$$

- $\sim P$ is the image of a positive classical formula $\mathcal{P}$. P is a positive intuitionistic formula such that $P = \mathcal{P}^{\approx}$.
- Q is the image of a negative classical formula $\mathcal{Q}$. In fact, $\mathcal{Q}^{\approx} = \sim Q$. Q is also a positive intuitionistic formula.
- N is the image of a negative classical literal $\mathcal{N}$ which has permeated the box $[]$. N is a positive intuitionistic atom.

In the following presentation we sometimes use the notation $\cdot A$ to refer to an intuitionistic formula that can be either A or $\sim A$.

1. For an example of an asynchronous decomposition rule, consider $(\mathcal{A} \wedge^- \mathcal{B})^{\approx} \;=\; \sim(\sim A \vee B)$ where $\mathcal{A}$ is positive and $\mathcal{B}$ is negative (classically). The proof translation is:

$$\dfrac{\dfrac{[\Gamma, \sim A], \Theta \longrightarrow [\phi]}{[\Gamma], \Theta, \sim A \longrightarrow [\phi]}\,[]_l \qquad [\Gamma], \Theta, B \longrightarrow [\phi]}{[\Gamma], \Theta, \sim A \vee B \longrightarrow [\phi]}\,\vee L \qquad \longmapsto \qquad \dfrac{\vdash [\Gamma', \mathcal{A}], \Theta' \quad \vdash [\Gamma'], \Theta', \mathcal{B}}{\vdash [\Gamma'], \Theta', \mathcal{A} \wedge^- \mathcal{B}}\,\wedge^-$$

   Three other variants can be displayed for the different combinations of polarities. Let us quickly mimic the $[]_l$ reaction rule before the calculus explodes in size.

2. The $[]$ rule:

$$\dfrac{[\Gamma, \sim C], \Theta \longrightarrow [\phi]}{[\Gamma], \Theta, \sim C \longrightarrow [\phi]}\,[]_l \qquad \longmapsto \qquad \dfrac{\vdash [\Gamma', \mathcal{C}], \Theta'}{\vdash [\Gamma'], \Theta', \mathcal{C}}\,[]$$

   Here, $\sim C$ is the image of a classical positive formula in an LJF sequent. That is, $\mathcal{C}^{\approx} = C$.

   However, there is another case for this rule: the *permeation* case. That is, if $\mathcal{C}$ is a classically negative literal, then $\mathcal{C}^{\approx} = \sim Q$ where atom Q is assigned *positive* intuitionistic polarity. To allow complete asynchronous decomposition, this positive intuitionistic atom is absorbed into the boxed context. In other words, the permeation of a positive atom in LJF is mirrored by the permeation of negative literal in LKF:

$$\dfrac{[\Gamma, Q], \Theta \longrightarrow [\phi]}{[\Gamma], \Theta, Q \longrightarrow [\phi]}\,[]^+ \qquad \longmapsto \qquad \dfrac{[\Gamma', \mathcal{C}], \Theta'}{[\Gamma'], \Theta', \mathcal{C}}\,[]$$

   Thus the side condition for the LKF $[]$ rule is that $\mathcal{C}$ *is either a positive formula or a negative literal.* The LKF $[]$ rule therefore corresponds precisely to the full LJF $[]_l$ rule.

3. With the $[]$ rule, here is the typical form of asynchronous rules. For $(\mathcal{A} \vee^- \mathcal{B})^{\approx} = \sim(\cdot A \wedge^+ \cdot B)$:

$$\dfrac{[\Gamma], \Theta, \cdot A, \cdot B \longrightarrow [\phi]}{[\Gamma], \Theta, \cdot A \wedge^+ \cdot B \longrightarrow [\phi]}\,\wedge^+ L \qquad \longmapsto \qquad \dfrac{\vdash [\Gamma'], \Theta', \mathcal{A}, \mathcal{B}}{\vdash [\Gamma'], \Theta', \mathcal{A} \vee^- \mathcal{B}}\,\vee^-$$

4. For an example of a focused sequence, consider the case $(\mathcal{A} \vee^+ \mathcal{B})^{\approx} = A \vee B$, where $\mathcal{A}$ and $\mathcal{B}$ are both positive. Recall that $A \vee B$ will be neutralized when placed on the left-hand side of the intuitionistic sequent:

$$\dfrac{\dfrac{\dfrac{[\Gamma, \sim(A \vee B)] -_{A}\!\!\rightarrow}{[\Gamma, \sim(A \vee B)] -_{A \vee B}\!\!\rightarrow}\,\vee R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{[\Gamma, \sim(A \vee B), \phi] -_{\phi}\!\!\rightarrow}\,Ir}{[\Gamma, \sim(A \vee B), \phi] \longrightarrow [\phi]}\,Rf}{[\Gamma, \sim(A \vee B)], \phi \longrightarrow [\phi]}\,[]_l}{[\Gamma, \sim(A \vee B)] \xrightarrow{\phi} [\phi]}\,R_l}{[\Gamma, \sim(A \vee B)] \xrightarrow{(A \vee B) \supset \phi} [\phi]}\,\supset L}{[\Gamma, \sim(A \vee B)] \longrightarrow [\phi]}\,Lf$$

Focus continues until there is a polarity switch, indicated by a $\sim$. It is clearly seen that all focusing starts with the template:

$$\dfrac{\dfrac{\dfrac{\vdots}{[\Gamma, \sim P] -_{P}\!\rightarrow} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{[\Gamma, \sim P, \phi] -_{\phi}\!\rightarrow}\ Ir}{[\Gamma, \sim P, \phi] \longrightarrow [\phi]}\ Rf}{[\Gamma, \sim P], \phi \longrightarrow [\phi]}\ []_l}{[\Gamma, \sim P] \xrightarrow{\phi} [\phi]}\ R_l}{[\Gamma, \sim P] \xrightarrow{P \supset \phi} [\phi]}\ \supset L}{[\Gamma, \sim P] \longrightarrow [\phi]}\ Lf$$

This entire template corresponds to the classical version of Lf $(D2)$:

$$\dfrac{\mapsto [\Gamma', \mathcal{P}], \mathcal{P}}{\vdash [\Gamma', \mathcal{P}]}\ focus$$

Here, $\mathcal{P}$ must be a positive formula. Negative literals correspond to positive atoms in the intuitionistic sequent and cannot be selected for focus.

5. The focused rules of LKF then directly mimic the corresponding right-focus rules of LJF:

$$\dfrac{[\Gamma] -_{\cdot A_i}\!\rightarrow}{[\Gamma] -_{\cdot A_1 \vee \cdot A_2}\!\rightarrow}\ \vee R \qquad \longmapsto \qquad \dfrac{\mapsto [\Gamma'], \mathcal{A}_i}{\mapsto [\Gamma'], \mathcal{A}_1 \vee^+ \mathcal{A}_2}\ \vee^+$$

$$\dfrac{[\Gamma] -_{\cdot A}\!\rightarrow \quad [\Gamma] -_{\cdot B}\!\rightarrow}{[\Gamma] -_{\cdot A \wedge^+ \cdot B}\!\rightarrow}\ \wedge^+ R \qquad \longmapsto \qquad \dfrac{\mapsto [\Gamma'], \mathcal{A} \quad \mapsto [\Gamma'], \mathcal{B}}{\mapsto [\Gamma'], \mathcal{A} \wedge^+ \mathcal{B}}\ \wedge^+$$

$$\dfrac{[\Gamma] -_{\cdot A[t/x]}\!\rightarrow}{[\Gamma] -_{\exists x \cdot A}\!\rightarrow}\ \exists R \qquad \longmapsto \qquad \dfrac{\mapsto [\Gamma'], \mathcal{A}[t/x]}{\mapsto [\Gamma'], \exists x \mathcal{A}}\ \exists$$

The identity rule is also a focused rule:

$$\dfrac{}{[\Gamma, P] -_{P}\!\rightarrow}\ Ir \qquad \longmapsto \qquad \dfrac{}{\mapsto [\Gamma', \neg P], P}\ ID$$

6. Focusing terminates at points of polarity switch indicated by $\sim$. That is:

$$\dfrac{\dfrac{\dfrac{[\Gamma], A \longrightarrow [\phi]}{[\Gamma], A \longrightarrow \phi}\ []_r}{[\Gamma] \longrightarrow A \supset \phi}\ \supset R}{[\Gamma] -_{\sim A}\!\rightarrow}\ R_r$$

The corresponding LKF reaction rule is therefore:

$$\dfrac{\vdash [\Gamma'], \mathcal{N}}{\mapsto [\Gamma'], \mathcal{N}}\ Release$$

where $\mathcal{N}$ is negative (that is, when $\mathcal{N}^{\approx} = \ \sim A$).

7. Here is one of the rules for the constants:

$$\dfrac{}{[\Gamma], \Theta, \mathit{false} \longrightarrow [\phi]}\ \mathit{falseL} \qquad \longmapsto \qquad \dfrac{}{\vdash [\Gamma'], \Theta', \neg \mathcal{F}}\ \mathit{absurd}$$

The few remaining cases are similar.

The entire classical focusing sequent calculus induced from the $\approx$ embedding is displayed in Figure 9. To avoid confusion with the presentation of LJF, we use Θ to represent the positive context and Γ to represent the negative context. We have chosen to present two versions of the identity rule, although technically they can be combined into one.

P positive, N negative, Θ consists of positive formulas and negative literals.
End-sequents have the form $\vdash [], \Gamma$.

Structural Rules (decision and reaction):

$$\frac{\vdash [\Theta, C], \Gamma}{\vdash [\Theta], \Gamma, C}\ [] \qquad \frac{\mapsto [P, \Theta], P}{\vdash [P, \Theta]}\ \mathit{Focus} \qquad \frac{\vdash [\Theta], N}{\mapsto [\Theta], N}\ \mathit{Release}$$

C is a positive formula or a negative literal.

Identities :

$$\frac{}{\mapsto [\neg P, \Theta], P}\ ID^{+},\ \text{atomic } P \qquad \frac{}{\mapsto [N, \Theta], \neg N}\ ID^{-},\ \text{atomic } N$$

Initials:

$$\frac{}{\mapsto [\Theta], \mathcal{T}}\ \mathit{indeed} \qquad \frac{}{\vdash [\Theta], \Gamma, \neg\mathcal{F}}\ \mathit{absurd} \qquad \frac{\vdash [\Theta], \Gamma}{\vdash [\Theta], \Gamma, \neg\mathcal{T}}\ \mathit{trivial}$$

Asynchronous Connectives:

$$\frac{\vdash [\Theta], \Gamma, A \quad \vdash [\Theta], \Gamma, B}{\vdash [\Theta], \Gamma, A \wedge^{-} B}\ \wedge^{-} \qquad \frac{\vdash [\Theta], \Gamma, A, B}{\vdash [\Theta], \Gamma, A \vee^{-} B}\ \vee^{-}$$

$$\frac{\vdash [\Theta], \Gamma, B, \neg A}{\vdash [\Theta], \Gamma, A \supset^{-} B}\ \supset^{-} \qquad \frac{\vdash [\Theta], \Gamma, A}{\vdash [\Theta], \Gamma, \forall x A}\ \forall$$

x not free in Θ, Γ.

Synchronous Connectives:

$$\frac{\mapsto [\Theta], A \quad \mapsto [\Theta], B}{\mapsto [\Theta], A \wedge^{+} B}\ \wedge^{+} \qquad \frac{\mapsto [\Theta], A_i}{\mapsto [\Theta], A_1 \vee^{+} A_2}\ \vee^{+} \qquad \frac{\mapsto [\Theta], A[t/x]}{\mapsto [\Theta], \exists x A}\ \exists$$

$$\frac{\mapsto [\Theta], \neg A}{\mapsto [\Theta], A \supset^{+} B}\ \supset^{+}_{v} \qquad \frac{\mapsto [\Theta], B}{\mapsto [\Theta], A \supset^{+} B}\ \supset^{+}_{t}$$

Figure 9: LKF: Focused Sequent Calculus for Classical Logic

9.2 Correctness of LKF

The mapping between proofs exhibited in the previous section is implicitly an inductive proof of the following proposition:

Proposition 22 *Let Gamma, Γ', Θ and Θ' be as defined in the forgoing.*

1. $\vdash [\Gamma'], \Theta'$ *is provable if and only if* $[\Gamma], \Theta \longrightarrow [\phi]$ *is provable.*
2. $\mapsto [\Gamma'], \mathcal{P}$ *is provable if and only if* $[\Gamma] {-_{\mathcal{P}\approx}}\!\!\longrightarrow$ *is provable.*

There are many ways in which one might proceed to prove the correctness of LKF. A direct approach would first show cut elimination, then use an argument similar to that of Proposition 20 for LJF. We offer the following proof, which appeals to the correctness of the Gödel-Gentzen translation. That translation is as follows:

$$
\begin{aligned}
g(P) &= \sim\sim P \ \text{ for atom } P \\
g(\neg A) &= \sim g(A) \\
g(A \wedge B) &= g(A) \wedge g(B) \\
g(A \vee B) &= \sim(\sim g(A) \wedge \sim g(B)) \\
g(\forall x A) &= \forall x (g(A)) \\
g(\exists x A) &= \sim \forall x \sim g(A)
\end{aligned}
$$

It is verifiable that this translation is still valid if $\sim$ is interpreted using ϕ instead of *false*. The case for implication is not needed since it is a defined connective of classical logic, and is not used in the $\approx$ translation.

Given a "regular" classical formula, we need to decide how to decorate the propositional connectives with $+$ or $-$. Any decoration would be enough, technically, to show completeness. However, we wish to in fact show that the choice is arbitrary.

Definition 23 Given a classical formula F, let $F^\bullet$ be the formula F with all occurrences of $\wedge$ replaced by $\wedge^-$ or and all $\vee$ replaced by $\vee^+$. Also, classical *false* is replaced by $\mathcal{F}$ and classical *true* is replaced by $\neg\mathcal{F}$[9]. Given an Intuitionistic LJF formula G, let G° represent G with all occurrences of $\wedge^-$ and $\wedge^+$ replaced by $\wedge$.

The "opposite" choice for the $\bullet$ decoration would in fact lead to a simpler proof of Lemma 24 below, since the $\approx$ translation of $\wedge^+$ and $\vee^-$ match up more directly with the Gödel-Gentzen translation. The choice made here is therefore the more interesting combination. For the proof of completeness, we also *assign all classical atoms positive polarity.* Negative atoms are replaced with the negation of positive atoms (this is the original way to define polarized atoms in the presence of involutive negation).

We observe that the following are easily verifiable in intuitionistic logic:

1. $A \vdash_I \sim\sim A$; $\sim\sim\sim A \equiv \sim A$; $\sim\sim\sim\sim A \equiv \sim\sim A$.
2. $(\sim\sim A \vee \sim\sim B) \ \vdash_I \ \sim\sim (A \vee B)$
3. $(\sim\sim A \wedge \sim\sim B) \ \equiv \ \sim\sim (A \wedge B)$ (*because* $\wedge$ *is both positive and negative*)
4. $\exists x (\sim\sim A) \ \vdash_I \ \sim\sim \exists x A$

The following is also easily provable:

> *For formula F such that $F^\bullet$ is* negative, $F^{\bullet\approx\circ} \equiv \sim\sim F^{\bullet\approx\circ}$ *is provable in intuitionistic logic.*

Here, $F^{\bullet\approx\circ}$ is a "regular" formula that has been decorated through $\bullet$ with $\wedge^-$ and $\vee^+$, translated using $\approx$ to an LJF formula, then having the $+/-$ decoration erased through $\circ$.

The following states the correctness of the Gödel-Gentzen translation, where $\vdash_C$ indicates classical derivation:

[9] So that the trivial case is recognized immediately, without requiring focus. However, this choice is also arbitrary.

if $\vdash_C F$ is provable then $\vdash_I g(F)$ is provable.

We are now ready to prove a core lemma for the completeness of LKF.

Lemma 24 *For all classical formulas F:*

1. $g(F) \vdash_I F^{\bullet\approx\circ}$ *if $F^\bullet$ is negative.*
2. $g(F) \vdash_I \sim\sim F^{\bullet\approx\circ}$ *if $F^\bullet$ is positive.*

Proof By induction on F. Each entry of the $\approx$ translation table needs to be verified separately. We provide the following representative cases:

1. For the case of atoms, condition 2 is easily verified since all classical atoms are assigned positive polarity. Also, for formulas in negation normal form, negation appear only before positive atoms, so the proof is also easy for the case of negation (since $\sim\sim\sim P \vdash_I \sim P$).

2. For an example of case 1, consider $F = \mathcal{A} \wedge \mathcal{B}$ such that $\mathcal{A}^\bullet$ is positive and $\mathcal{B}^\bullet$ is negative. We have that $F^{\bullet\approx\circ} = \sim(\sim A^\circ \vee B^\circ)$ where $\mathcal{A}^{\bullet\approx} = A$ and $\mathcal{B}^{\bullet\approx} = \sim B$. Observe the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{g(\mathcal{A}) \vdash_I \sim\sim A^\circ}{g(\mathcal{A}) \wedge g(\mathcal{B}) \vdash_I \sim\sim A^\circ}\wedge L \qquad \sim\sim A^\circ, \sim A^\circ \vdash_I \phi}{g(\mathcal{A}) \wedge g(\mathcal{B}), \sim A^\circ \vdash \phi}\mathit{Cut} \qquad \dfrac{\dfrac{g(\mathcal{B}) \vdash_I \sim B^\circ}{g(\mathcal{A}) \wedge g(\mathcal{B}) \vdash_I \sim B^\circ}\wedge L \qquad \sim B^\circ, B^\circ \vdash_I \phi}{g(\mathcal{A}) \wedge g(\mathcal{B}), B^\circ \vdash_I \phi}\mathit{Cut}}{g(\mathcal{A}) \wedge g(\mathcal{B}), (\sim A^\circ \vee B^\circ) \vdash_I \phi}\vee L}{g(\mathcal{A}) \wedge g(\mathcal{B}) \vdash_I \sim(\sim A^\circ \vee B^\circ)}\supset R$$

   Since $\mathcal{A}^{\bullet\approx} = A$ and $\mathcal{B}^{\bullet\approx} = \sim B$, the premises of the derivation follow immediately from inductive hypothesis: case 2 for the left premise, case 1 for the right premise. We also note that if $\wedge^+$ was used instead of $\wedge^-$ in the $\bullet$ decoration, the above case would match up more directly with the Gödel-Gentzen translation g.

3. For an example of case 2, consider $F = \exists x\mathcal{A}$ where $\mathcal{A}^\bullet$ is positive and $\mathcal{A}^{\bullet\approx} = A$. So $F^{\bullet\approx\circ} = \exists x A^\circ$. $g(F) = \sim\forall x \sim g(\mathcal{A})$. Consider the following[10]:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ldots g(\mathcal{A}) \vdash_I \sim\sim A^\circ}{\ldots g(\mathcal{A}) \vdash_I \exists x \sim\sim A^\circ}\exists R \qquad \ldots \phi \vdash_I \phi}{\ldots g(\mathcal{A}), \sim\exists x \sim\sim A^\circ \vdash_I \phi}\supset L}{\ldots \sim\exists x \sim\sim A^\circ \vdash_I \sim g(\mathcal{A})}\supset R}{\ldots \sim\exists x \sim\sim A^\circ \vdash_I \forall x \sim g(\mathcal{A})}\forall R \qquad \ldots \phi \vdash_I \phi}{\sim\forall x \sim g(\mathcal{A}), \sim\exists x \sim\sim A^\circ \vdash_I \phi}\supset L}{\sim\forall x \sim g(\mathcal{A}) \vdash_I \sim\sim\exists x \sim\sim A^\circ}\supset R \qquad \sim\sim\exists x \sim\sim A^\circ \vdash_I \sim\sim\sim\sim\exists x A^\circ}{\sim\forall x \sim g(\mathcal{A}) \vdash_I \sim\sim\sim\sim\exists x A^\circ}\mathit{Cut} \qquad \sim\sim\sim\sim\exists x A^\circ \vdash_I \sim\sim\exists x A^\circ}{\sim\forall x \sim g(\mathcal{A}) \vdash_I \sim\sim\exists x A^\circ}\mathit{Cut}$$

   The remaining premise follow from inductive hypothesis (case 2). The other cases follow these patterns.

□

The earliest translation of Kolmogorov, which places $\sim\sim$ almost universally, can also be used for the proof. It would make the proof slightly easier for the positive case, but the Gödel-Gentzen translation is better for the negative case.

[10] One can check carefully that the arguments are still valid if we interpret the Gödel-Gentzen translation as using 0 for the definition of $\sim$ instead of ϕ.

Theorem 25 *LKF is sound and complete with respect to classical logic.*

Proof Combining the correctness of the Gödel-Gentzen translation, Lemma 24 and the completeness of LJF, given a classical formula F such that $\vdash_C F$ is provable, we have that:

1. for $F^\bullet$ negative, we know that $F^{\bullet\approx} = \sim Q$ and that $[], Q \longrightarrow [\phi]$ is provable in LJF.
2. for$F^\bullet$ positive, we know that $F^{\bullet\approx} = P$ and that $[\sim P] \longrightarrow [\phi]$ is provable in LJF.

Now by proposition 22, referring to the correspondance between LJF proofs and LKF proofs, we have in both cases $\vdash [], F^\bullet$ is provable in LKF. Soundness is trivial as usual. □

9.3 Deriving LKF from Linear Logic

We have constructed this embedding of classical logic as a further demonstration of the abilities of LJF as a hosting framework. The embedding also revealed interesting relationships between classical and intuitionistic polarity. However, the derivation of LKF from focused linear logic (LLF) is simpler. One need only to define each connective to be either wholly positive or negative. For example, the translation of $\vee^-$ is:

$A^p \parr B^p$, if A and B are both negative
$A^p \parr ?B^p$, if only A is negative
$?A^p \parr B^p$, if only B is negative
$?A^p \parr ?B^p$, if A and B are both positive

This translation is called the *"polaro"* translation in [DJS97]. The +1/−1 translation, the ≈ translation and the polaro translation are all direct derivatives of the LC/LU analysis of polarity.

With the adjustment on the treatment of atoms discussed in Section 6.2, LKF is derivable from the linear focused calculus LLF in the same manner that LJF is derived. In fact, the derivation shows that each LKF rule, *including the structural rules,* are in one-to-one correspondance with instances of inferences rules in LLF.

Given that the same translation is used in [DJS97] and [LQdF05], one can expect close connections between LKF and these systems, which are also formulations of classical logic. The system LK_p^η was defined in [DJS97], and later this was extended to $LK_{pol}^{\eta,\rho}$ in [LQdF05]. These systems essentially reformulate focusing for classical logic. The authors of [DJS97] opted not to present LK_p^η as a sequent calculus because they feared that it will have the cumbersome size of LU. As we have just witnessed, this cumbersome size can be avoided by adopting LLF-style reaction rules.

Given our stated goals, the choice in adopting Andreoli's system is justified in that LKF and LJF have the form of compact sequent calculi ready for implementation. More significantly perhaps, LK_p^η and $LK_{pol}^{\eta,\rho}$ define focusing for classical logic. They map to polarized forms of linear logic (LLP and LL_{pol}). LLF is defined for full classical linear logic. LKF is embedded within LLF in the same way that LC is embedded within LU. LLF is well suited as a generic host for other logics.

10 Future work

There is much that can be done following the derivation of the basic system LJF, not the least of which is to investigate its possible applications. There are also several ways in which our work can be extended. We indicate some works in progress below.

10.1 Cut-Elimination for LKF

To complete the work on LKF, which is in fact derivable from both linear and intuitionistic focusing, we will show cut-elimination.

Three principal cut rules for LKF are proposed:

$$\frac{\vdash [C, \Theta], \Gamma \quad \vdash [\Theta'], \Gamma', \neg C}{\vdash [\Theta\Theta'], \Gamma\Gamma'} \; Cut_p, \; \textit{prime cut}$$

$$\frac{\mapsto [\Theta], B \quad \vdash [\Theta'], \Gamma', \neg B}{\vdash [\Theta\Theta'], \Gamma'} \; Cut_k, \; \textit{key cut}$$

$$\frac{\mapsto [\Theta, P], B \quad \vdash [\Theta'], \neg P}{\mapsto [\Theta\Theta'], B} \; Cut_f, \; \textit{focused cut}$$

Note that in the focused cut, P cannot be a negative literal, since then if $B = \neg P$ then the conclusion likely have no proof. (B can be negative, which will reduce the rule to a prime cut via the *Release* rule).

Other variants of cut such as

$$\frac{\vdash [\Theta, M], \Gamma \quad \vdash [\Theta', \neg M], \Gamma'}{\vdash [\Theta\Theta'], \Gamma\Gamma'} \; cut$$

where M is a negative literal are omitted because they easily reduce to one of the three principal cuts. In this particular case, the rule reduces to a prime cut by applying the [] rule to either premise. The key cut can similarly be reduced, since one of the cut formulas is positive and subject to the [] rule. However, it is displayed separately because it represents the key cases quite precisely.

The admissibility of the three cuts follow either from soundness and completeness with respect to LK, or from a mutually recursive cut elimination procedure. Surely the arguments will be simpler than for LJF. However, we also wish to explore alternative ways to prove cut-elimination for focused proof systems.

10.2 Multi-Succedent Intuitionistic Logic

There are good reasons to consider the admissibility of truly negative elements in intuitionistic logic. For example, if we are to accept that $\otimes$ and & can coexist in an intuitionistic calculus, it is reasonable to consider also $\top$ as an alternative to 1 for representing true. $\top$ is indeed the translation of $\neg\mathcal{F}$ in LC and LKF. Furthermore, multiple conclusion versions of LJ are known [Dra88] and have also been studied as a variant of LJQ' [DL06]. It remains to determine if these variations are still essentially consistent with the polarity analysis of LC and LU.

10.3 Focused LU

In the mappings between the LJ, LJQ and LJT systems and focused proofs, a common phenomenon was that relatively large focused proof segments interpret single proof steps on the intuitionistic side. One cannot help but notice that the mapping that induced LJF no longer exhibit this characteristic. The focused proof segment that correspond to each LJF rule is at most one level higher. The extra level is in fact due solely to the ? rule, which in the focusing calculus is a structural rule. With the $[]_l$ rule, which mimics the ? rule, even this discrepancy disappears. The rules of LJF can therefore be seen as but a fragment of a larger and more universal system. The same can in fact be said of focused classical logic, if one were to derive LKF also from LLF. With a slight adjustment on the treatment of atoms, LLF is in fact the cornerstone of a *focused logic of unity.* In particular, linear, intuitionistic and classical sequents can be classified as follows:

Linear

$$\vdash \Gamma : \Delta \Uparrow \Theta, \qquad\qquad \vdash \Gamma : \Delta \Downarrow D$$

The only restriction is that Δ consists of synchronous formulas and literals[11].

Intuitionistic

$$\vdash \Gamma^\perp :\Uparrow R, \Theta^\perp, \qquad \vdash \Gamma^\perp : R \Uparrow \Theta^\perp, \qquad \vdash \Gamma^\perp :\Downarrow R, \qquad \vdash \Gamma^\perp : R \Downarrow D^\perp$$

The use of $(.)^\perp$ indicates the clear separation between the left and right-hand sides of intuitionistic sequents. $\Gamma^\perp$ consists of only positive formulas and neutral literals. Another important restriction is that there is at most one formula between the colon and the focus arrow ($\Uparrow$ or $\Downarrow$).

Classical

$$\vdash \Gamma :\Uparrow \Theta, \qquad\qquad \vdash \Gamma :\Downarrow D$$

Γ consists of positive formulas and negative literals. The other key restriction is that the space between the colon and the focus arrow is empty.

It is possible to find suitable symbols for the connectives of the three logics, then formulate one large, focused sequent calculus in which all are included. The system might be called LUF. New logics may also be defined in this system. The theorems of LU can be restated, but in a stronger form: not only *can* we stay within each fragment during proofs, but in fact we *will*.

10.4 Extension to Second Order Quantifiers

We naturally want to extend our work to second order. But there are issues that we wish to explore further before determining the appropriate form of the second order quantifiers. It is not difficult to determine the effect of replacing an atom with a formula of the *same polarity* on the structure of a focused proof; that is, the effect of a substitution that, for example, maps a predicate constant (of arty m) with the abstracted formula $\lambda x_1 \ldots \lambda x_m.T$. This suggests that the second-order quantifiers should come in the forms $\forall_+$, $\forall_-$, $\exists_+$ and $\exists_-$.

Such a strategy means, however, that we must necessarily tie polarity to *predicates.* As observed in Section 2, the LLF system is flexible enough to assign polarities to individual closed atoms. There are also practical reasons to consider assigning different polarities to the same predicate on different branches of a proof. This we may need to study the effect of replacing an atom with a formula of the opposite polarity.

We also wish to study cut-elimination at second order, and whether the improved structuring of proofs using polarities will simplify such an effort.

11 Conclusion

The derivation of the LJF sequent calculus shows that the importance of Andreoli's focusing calculus extends far beyond its original intent. Focusing and LU are a near perfect match and their combination seems to clarify each other. For example, the number of inference rules for LJF drop significantly compared to the intuitionistic fragment of LU, which has, indeed, a large number of inference rules. The focusing of intuitionistic logic using LJF provides more structural information about intuitionistic logic than the corresponding focusing result does for linear logic. Intuitionistic sequents are more constrained in how polarities can be distributed.

It remains to examine the impact of our new focusing calculus on logic programming, theorem proving, and in the formulation of term-reduction systems.

[11] although further restrictions are used in other formulations of focusing [Gir01].

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