Focusing and Polarization in Intuitionistic Logic

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Outline

1. Motivate focusing proof systems
2. A comprehensive approach to focusing for intuitionistic logic
3. LJF: a focusing proof system for intuitionistic logic
4. LKF: a focusing proof system for classical logic
5. Future work and conclusions

Invertible rules and the asynchronous phase

Some inference rules in the sequent calculus are invertible, *e.g.*,

$$\frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow A \supset B} \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \qquad \frac{\Gamma \longrightarrow B[y/x]}{\Gamma \longrightarrow \forall x.B}$$

First focusing principle: when proving a sequent, apply invertible rules exhaustively and in any order.

This is the *asynchronous phase* of proof search: if formulas are "processes" in an "environment," then these formulas "evolve" without communications with the environment.

Non-invertible rules and the synchronous phase

Some inference rules in the sequent calculus are not generally invertible, *e.g.*,

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \wedge B} \qquad \frac{\Gamma \longrightarrow B[t/x]}{\Gamma \longrightarrow \exists x.B}$$

Some *backtracking* is generally necessary within proof search using these inference rules.

Second focusing principle: non-invertible rules are applied in a "chain-like" fashion, focusing on a formula and its synchronous subformulas.

This is the *synchronous phase* of proof search: we will not know if our use of the inference rule is successful without checking with the formula's environment.

Extending the asyn/syn distinction to atoms

Focusing proof systems generally extend the asyn/syn distinction to atoms.

We shall assume that somehow all atoms are given a *bias*, that is, they are either positive (syn-like) or negative (asyn-like).

A *positive formula* is either a positive atom or has a top-level synchronous connective.

A *negative formula* is either a negative atom or has a top-level asynchronous connective.

Some "focusing" behavior

In a sequent calculus presentation of logic programming, "backchaining" is described as "focused application of left-rules."

$$\frac{\Gamma \longrightarrow G \qquad \Gamma, D \overset{\Xi}{\longrightarrow} A}{\Gamma, G \supset D \longrightarrow A} \supset L$$

What is the last inference rule in Ξ?

If formulas are over only $\supset$, $\forall$, and if A is atomic, the following restriction is complete: If D is atomic, then $D = A$ and Ξ is initial; otherwise, Ξ ends with an introduction rule for D.

If one selects the left-hand formula

$$\forall \bar{x}_1 (G_1 \supset \forall \bar{x}_2 (G_2 \supset \cdots \forall \bar{x}_n (G_n \supset A') \ldots))$$

to prove the atom A on the right, then there is a θ such that $A = A'\theta$ and $\Gamma \longrightarrow G_i\theta$ are provable $(i = 1, \ldots, n)$.

Various focusing-like proof system

LLF: Andreoli's original focusing proof system

LKT/LKQ/LK^η: Focusing systems for classical logic [Danos, Joinet, Schellinx]

Uniform proofs [Miller, Nadathur, Scedrov] and *LJT* [Herbelin] permits backward chaining proof.

LJQ [Herbelin] permits forward-chaining proof. *LJQ′* [Dyckhoff, Lengrand] extends it.

λRCC [Jagadeesan, Nadathur, Saraswat] allows mixing forward chaining and backward chaining (in a subset of intuitionistic logic).

LJF (following slides) allows forward and backward proof in all of intuitionistic logic. LJT, LJQ, λRCC, and LJ are subsystems.

LKF (derived from LJF) provides focusing for all of classical logic.

Backward and Forward Chaining

$$\frac{\Gamma \longrightarrow a \qquad \Gamma, b \longrightarrow G}{\Gamma, a \supset b \longrightarrow G} \; a, b \text{ are atoms, focus on } a \supset b$$

Negative atoms: The right branch is trivial; i.e., $b = G$. Continue with $\Gamma \longrightarrow a$ (backward chaining).

Positive atoms: The left branch is trivial; i.e., $\Gamma = \Gamma', a$. Continue with $\Gamma', a, b \longrightarrow G$ (forward chaining).

Let G be $\mathit{fib}(n, f)$ and let Γ contain $\mathit{fib}(0,0)$, $\mathit{fib}(1,1)$, and

$$\forall n \forall f \forall f' [\mathit{fib}(n, f) \supset \mathit{fib}(n+1, f') \supset \mathit{fib}(n+2, f+f')].$$

The nth Fibonacci number is f iff $\Gamma \vdash G$.

If all $\mathit{fib}(\cdot, \cdot)$ are negative then the shortest proof is *exponential* in n.

If all $\mathit{fib}(\cdot, \cdot)$ are positive then the shortest proof is *linear* in n.

The full picture behind focusing

The complete picture of focusing in *linear logic* is remarkably elegant and is given by Andreoli's focusing proof system.

The complete picture of focusing in *intuitionistic logic* is more complex: we propose LJF as a good solution.

Complexity arises from the interplay of the exponential ! (left-permeability) and polarity: in principle, a positive formula B will behave as if $B \equiv\, !\, B$; negative formulas do not act this way.

LJF: Annotations

Assign *bias* to all atoms: they are either negative or positive.

Annotate every conjunction $\wedge$ as either $\wedge^+$ or $\wedge^-$.

Annotations do not effect provability, although the structure of proofs can vary greatly as annotations change.

Positive formulas are among positive atoms and

$$true,\ \ false,\ \ A \wedge^+ B,\ \ A \vee B, \exists x A.$$

Negative formulas are among negative atoms and

$$A \wedge^- B,\ \ A \supset B,\ \ \forall x A.$$

LJF: The four different sequents

1. $[\Gamma], \Theta \longrightarrow \mathcal{R}$: an *unfocused sequent,* Γ contains negative formulas and positive atoms and $\mathcal{R}$ represents either a formula R or $[R]$.

2. $[\Gamma] \longrightarrow [R]$: all asynchronous formulas have been decomposed: focus is ready for selection.

3. $[\Gamma] \xrightarrow{B} [R]$: *left-focusing* (the focus is B). Means $\Gamma, B \vdash R$.

4. $[\Gamma] -_B\!\!\rightarrow$: *right-focusing* (the focus is B). Means $\Gamma \vdash B$.

You get a "regular" sequent if you drop the brackets and move the focused formula to either the left or right.

Structural Rules: Decision and Reaction

$$\frac{[N, \Gamma] \xrightarrow{N} [R]}{[N, \Gamma] \longrightarrow [R]}\ Lf \qquad \frac{[\Gamma] -_P\!\!\rightarrow}{[\Gamma] \longrightarrow [P]}\ Rf$$

$$\frac{[\Gamma] \longrightarrow N}{[\Gamma] -_N\!\!\rightarrow}\ R_r \qquad \frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]}\ R_l$$

$$\frac{[C, \Gamma], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, C \longrightarrow \mathcal{R}}\ []_l \qquad \frac{[\Gamma], \Theta \longrightarrow [D]}{[\Gamma], \Theta \longrightarrow D}\ []_r$$

Identities

$$\frac{}{[P, \Gamma] -_P\!\!\rightarrow}\ I_r,\ \text{atomic } P \qquad \frac{}{[\Gamma] \xrightarrow{N} [N]}\ I_l,\ \text{atomic } N$$

P is positive; N is negative; C is negative or a positive atom; and D is positive or a negative atom.

Introduction Rules

$$\frac{[\Gamma] \xrightarrow{A_i} [R]}{[\Gamma] \xrightarrow{A_1 \wedge^- A_2} [R]} \wedge^- L \qquad \frac{[\Gamma], \Theta \longrightarrow A \quad [\Gamma], \Theta \longrightarrow B}{[\Gamma], \Theta \longrightarrow A \ \wedge^- B} \wedge^- R$$

$$\frac{[\Gamma], \Theta, A, B \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A \ \wedge^+ B \longrightarrow \mathcal{R}} \wedge^+ L \qquad \frac{[\Gamma] -_A\!\!\rightarrow \quad [\Gamma] -_B\!\!\rightarrow}{[\Gamma] -_{A \ \wedge^+ B}\!\!\rightarrow} \wedge^+ R$$

$$\frac{[\Gamma], \Theta, A \longrightarrow \mathcal{R} \quad [\Gamma], \Theta, B \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A \vee B \longrightarrow \mathcal{R}} \vee L \qquad \frac{[\Gamma] -_{A_i}\!\!\rightarrow}{[\Gamma] -_{A_1 \vee A_2}\!\!\rightarrow} \vee R$$

Each connective has an asynchronous and a synchronous introduction rule.

Introduction Rules (cont.)

$$\frac{[\Gamma] \ -_{A}\!\!\longrightarrow \qquad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]} \supset L \qquad \frac{[\Gamma], \Theta, A \longrightarrow B}{[\Gamma], \Theta \longrightarrow A \supset B} \supset R$$

$$\frac{[\Gamma], \Theta, A \longrightarrow \mathcal{R}}{[\Gamma], \Theta, \exists y A \longrightarrow \mathcal{R}} \exists L^{\dagger} \qquad \frac{[\Gamma] \ -_{A[t/x]}\!\!\longrightarrow}{[\Gamma] \ -_{\exists x A}\!\!\longrightarrow} \exists R$$

$$\frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]} \forall L \qquad \frac{[\Gamma], \Theta \longrightarrow A}{[\Gamma], \Theta \longrightarrow \forall y A} \forall R^{\dagger}$$

(†) As usual, y is not free in the lower sequent.

Soundness and Completeness of LJF

Theorem. LJF is sound and complete with respect to intuitionistic logic.

Proof. Soundness is easy: an LJF immediately yields an LJ proof.

Completeness is more difficult: map intuitionistic logic into linear logic using polarities to insert the exponential !: for example,

$$(P \supset B)^{+1} = P^{-1} \multimap B^{+1} \qquad (N \supset B)^{+1} = !\, N^{-1} \multimap B^{+1}$$

$$(A \supset B)^{-1} = A^{+1} \multimap B^{-1}$$

This translation is inspired by Girard's analysis behind LU.

Applying the completeness of focusing proofs in linear logic (due to Andreoli, 1992) completes this proof.

Cut rules

The cut rule for *LJF* takes many forms:

$$\frac{[\Gamma], \Theta \longrightarrow P \quad [\Gamma'], \Theta', P \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}} \qquad \frac{[\Gamma], \Theta \longrightarrow C \quad [C, \Gamma'], \Theta' \longrightarrow \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \longrightarrow \mathcal{R}}$$

$$\frac{[\Gamma] \xrightarrow{B} [P] \quad [\Gamma'], P \longrightarrow [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]} \qquad \frac{[\Gamma] \longrightarrow N \quad [N, \Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]}$$

$$\frac{[\Gamma] -_{C}\!\!\rightarrow \quad [C, \Gamma'] -_{R}\!\!\rightarrow}{[\Gamma\Gamma'] -_{R}\!\!\rightarrow}$$

As before, P is positive, N is negative, and C is negative or a positive atom.

Notice that the last three cut rules retain focus in the conclusion. These rules are admissible.

Size of Connectives

Connectives are small. Forget the focusing result. A great deal of interleaving/parallelism of introduction rules takes place.

Connectives are big. Connectives are maximal collections of async or sync connectives.

By inserting "delays" into formulas, the "big connective" view yields the "small connective" view.

Delays: $\partial^-(B) = \mathit{true} \supset B$ and $\partial^+(B) = \mathit{true} \wedge^+ B$. Clearly, B, $\partial^-(B)$, and $\partial^+(B)$ are logically equivalent, but $\partial^-(B)$ is always negative and $\partial^+(B)$ is always positive.

For example, LJQ' is embedded into LJF by inserting some delays: $B^l = B^r = B$ (atom B), $(A \wedge B)^l = \partial^-(A^l \wedge^+ B^l)$, $(A \wedge B)^r = A^r \wedge^+ B^r$, $(A \vee B)^l = \partial^-(A^l \vee B^l)$, $(A \vee B)^r = A^r \vee B^r$, $(A \supset B)^l = A^r \supset \partial^+(B^l)$, $(A \supset B)^r = \partial^+(A^l \supset B^r)$.

LKF: Focusing for Classical Logic

Classical logic can be mapped into intuitionistic using the well-known double-negation translation: the usual approach yields poor focusing behavior.

Girard provides a polarized version of the double negation to derive LC: we follow that approach.

LKF is a focused, one-sided sequent calculus:

1. atoms are assigned bias and
2. $\wedge$, $\vee$, $\supset$, *true*, and *false* are polarized.

Soundness and completeness of LKF: Use a polarized, double negation translation of classical formulas into intuitionistic formulas and the results for LJF.

Possible Applications

Oracles as proofs: when there is no choice in searching for a proof, just continue; when there is a choice, the oracle provides information to resolve the choice. Oracles can be small but fragile certificates. Focusing should help to develop a more declarative and robust version of oracles.

Tables of lemma (see talk by Nigam): polarities can be used to enforce *re-use* instead of *re-prove*.

There are close links between *games semantics* and logic provided by focused proofs.

Mixing polarities might relate to *mixing evaluation strategies* (call-by-name, call-by-value) in functional programming languages.

Conclusions

Focusing is of fundamental importance whenever one moves from *provability* to *proofs*. There should be a number of applications of focusing in CS.

Getting focusing proof systems for intuitionistic and classical logics involves addressing the issues of bias and permeability.

LJF is a setting focusing intuitionistic logic, allowing for mixed biased atoms.

LJF contains a number of known focusing proof system as subsystems.

LKF is a focusing proof system for classical logic that allows for atoms of mixed bias.