From axioms to synthetic inference rules via focusing

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Joint work with Sonia Marin, Elaine Pimentel, and Marco Volpe
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References appear at the end. These slides are available on my web page.

The view from 30,000 feet[1]

Gentzen's Sequent calculi LJ/LK [1935]

- Lots of tiny, micro rules
- **Good** for proving cut-elimination and consistency for both logics
- **Bad** for uses in computer science because proof structure is chaotic and slippery (witness the many rule permutations)

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- introduced linear logic and polarity
- focused sequent proofs for linear logic
- yields macro rules built from Gentzen-style micro rules

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Today's talk

- Apply polarity and focusing to classical and intuitionistic logic . . .
- . . . in order to systematically build synthetic inference rules

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The axioms-as-rules problem

How to incorporate inference rules encoding axioms into existing proof systems for classical and intuitionistic logics?

Projective geometry (Negri & von Plato [NvP11]) – Uniqueness :

$$a \in l \quad \wedge \quad a \in m \quad \wedge \quad b \in l \quad \wedge \quad b \in m \quad \supset \quad a = b \quad \vee \quad l = m$$

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Sur la formalisation des fondements de la géométrie (Boutry [Bou18]) – Congruence :

- $\forall x, y. cong\ x\ y\ y\ x.$
- $\forall x, y, z, w, r, s. cong\ x\ y\ z\ w \supset cong\ x\ y\ r\ s \supset cong\ z\ w\ r\ s.$

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A fresh view to an old problem:
The combination of bipolars and focusing provides simple inference rules based only on atomic formulas.

Motivation

Object

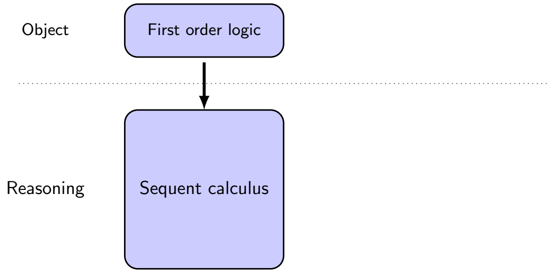
Reasoning

Motivation

Object | First order logic

Reasoning

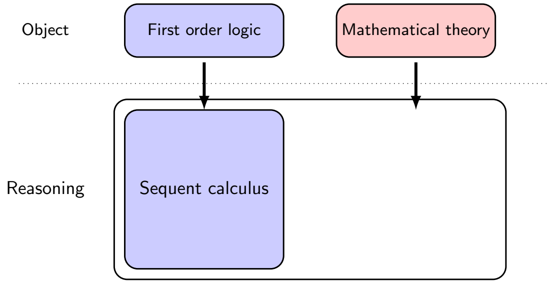
Motivation



Advantages of sequent systems [Gen35] as frameworks

- simple calculi;
- good proof theoretical properties (cut-elimination, consistency);
- can be easily implemented (λProlog, rewriting).

Motivation



Nice idea:

Add mathematical theories to first order logics and reason about them using all the machinery already built for the sequent framework.

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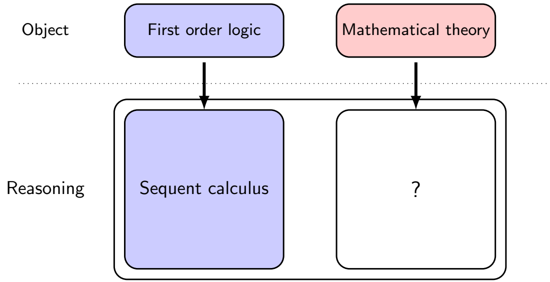
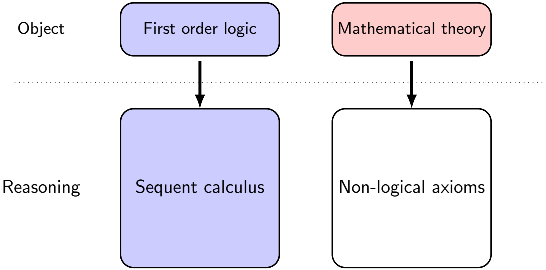


Nice idea:

Add mathematical theories to first order logics and reason about them using all the machinery already built for the sequent framework.

- ⋆ Sara Negri, Jan von Plato, and Roy Dyckhoff, in first-order logic [NvP98, DN15];
- ⋆ as well as, Alex Simpson [Sim94], Luca Viganò [Vig00], Agata Ciabattoni [CGT08], in fragments of first-order logic such as modal and substructural logics;
- ⋆ and Gilles Dowek [DW05, BDEG+21], in Deduction Modulo Theories/Axioms for Math.

Motivation



Add non-logical axioms [NvP98]: assume $\vdash P \supset Q$ and $\vdash P$. Then

$$\dfrac{\dfrac{}{\vdash P} \quad \dfrac{\dfrac{}{\vdash P \supset Q} \quad \dfrac{\dfrac{}{P \vdash P} \quad \dfrac{}{Q \vdash Q}}{P, P \supset Q \vdash Q}\,\supset_l}{P \vdash Q}\,cut}{\vdash Q}\,cut$$

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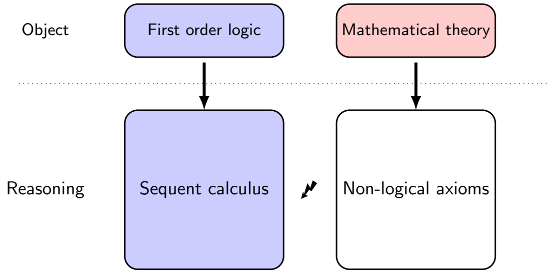

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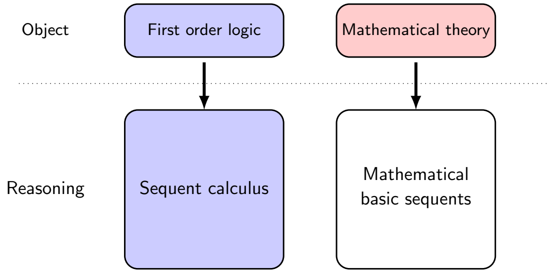
The *Hauptsatz* fails for systems with proper axioms.

Motivation



Add mathematical basic sequents [NvP98]: assume $P \vdash Q$ and $\vdash P$. Then

$$\dfrac{\overline{\vdash P} \quad \overline{P \vdash Q}}{\vdash Q}\ cut$$

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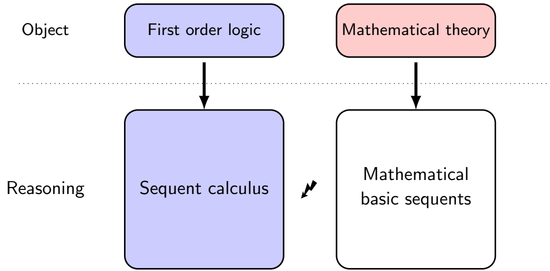

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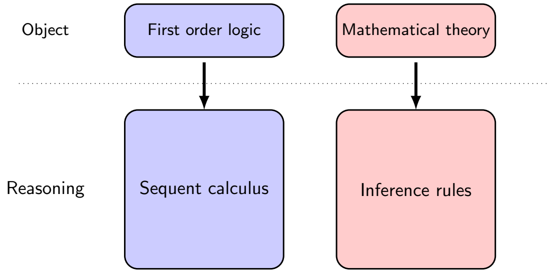
Gentzen: *Hauptsatz* doesn't extend to basic sequents as premises. [Gen38]

Motivation



Add non-logical rules of inference [Sim94, NvP98]:

$$\frac{\Gamma, Q \vdash C}{\Gamma, P \vdash C}\ P \supset Q \qquad \frac{\Gamma, P \vdash C}{\Gamma \vdash C}\ P$$

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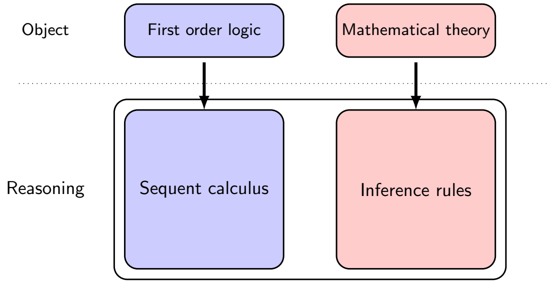

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The sequent $\vdash Q$ now has the (cut-free) proof

$$\cfrac{\cfrac{\overline{Q \vdash Q}}{P \vdash Q}\ P \supset Q}{\vdash Q}\ P$$

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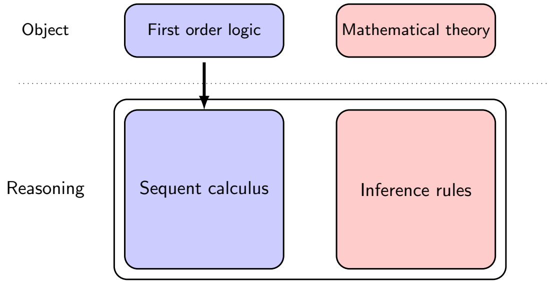
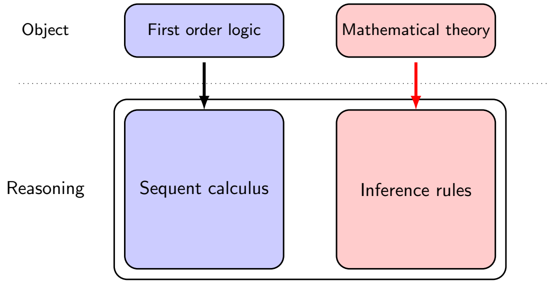


A fresh view to an old problem:

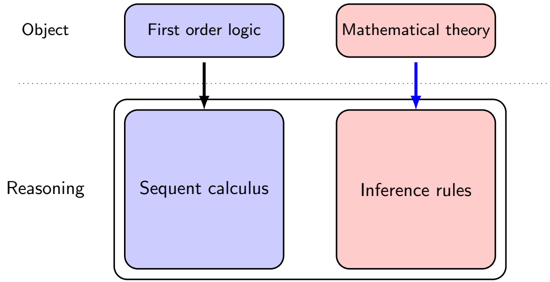
Motivation



Which ones and why?

A fresh view to an old problem:

In this talk



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bipolars + focusing
=
synthetic inference rules
(only atoms)

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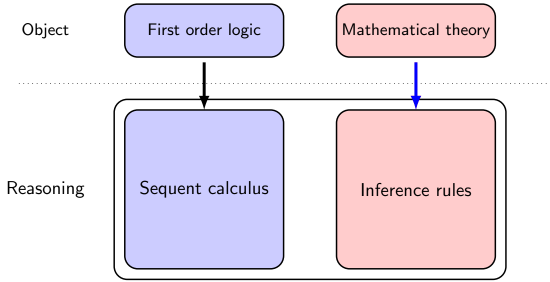

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A fresh view to an old problem:

Classify axioms into a polarities' hierarchy (inspired by [CGT08])

Move focusing [And92] from linear to intuitionistic and classical logic [LM07, LM09]

Identify synthetic inference rules with bipoles for bipolar axioms.

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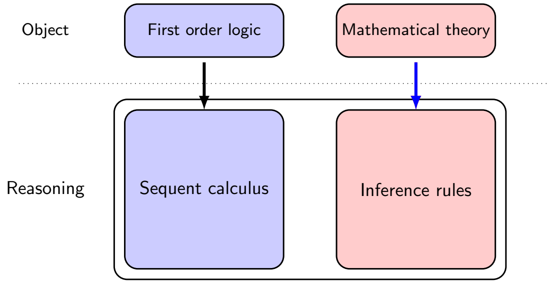

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Identify synthetic inference rules with bipoles for bipolar axioms.

- Systematically compute inference rules from bipolar axioms (λProlog prototype);
- Uniform presentation for classical and intuitionistic first order systems;
- Generalization of the literature (e.g. on geometric theories [Neg03, NvP11, Neg16, CMS13] and [Vig00]);
- Cut-elimination guaranteed for when such synthetic inferences rules are added.

Outline

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Gentzen: sequent calculus

Some *locality*: sequents keep track of open assumptions

where $\Gamma = A_1, \ldots, A_n$ is the context.

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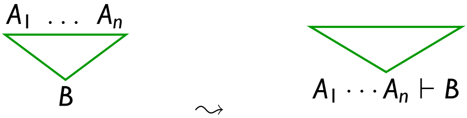

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- Rules: right = introduction rules; left = re-reading elimination rules.

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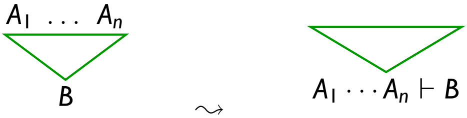

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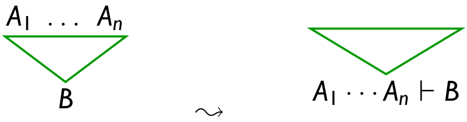

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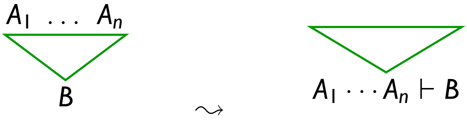

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- Analyticity = cut-elimination.

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\ \text{cut}$$

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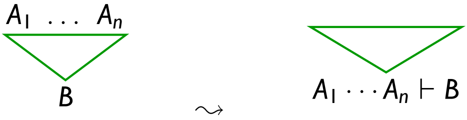

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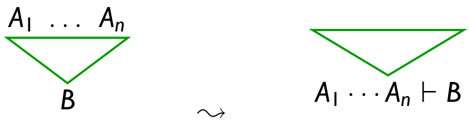

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- Analyticity $\rightsquigarrow$ sub-formula property: induces a structure on the proofs (in terms of the end formula).
- Thus, proof structure can be exploited to formalize reasoning, investigate meta-logical properties of the logic e.g. consistency, decidability, complexity and interpolation, and develop automated deduction procedures.

Outline

Polarization [DJS95]

Let A_0, A_1, and B be atomic, and let Γ be a multiset of formulas.

$$\frac{\Gamma \vdash A_1 \quad \Gamma, A_0 \vdash B}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset$$

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Back-chaining!

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Forward-chaining!

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Mixed protocol:

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Mixed protocol:

Mixing them, e.g., A_i positive for i odd and A_i negative for i even:

$$\dfrac{\dfrac{A_1 \in \Gamma}{\Gamma \vdash A_1} \quad \dfrac{\Gamma \vdash A_2 \quad \dfrac{\dfrac{A_3 \in \Gamma}{\Gamma \vdash A_3} \quad \dfrac{\Gamma \vdash A_4 \quad \dfrac{A_0 = B}{\Gamma, A_0 \vdash B}}{\Gamma, A_4 \supset A_0 \vdash B}}{\Gamma, A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\ L\supset$$

Example: Fibonacci

Let

$$\Delta = \{\mathrm{fib}(0,0), \mathrm{fib}(1,1), \forall n, x, y. [\mathrm{fib}(n,x) \wedge \mathrm{fib}(n+1,y) \supset \mathrm{fib}(n+2, x+y)]\}$$

$\mathrm{fib}(n, N) = N$ is the nth Fibonacci number.

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Example: Fibonacci

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$$\Delta = \{\mathrm{fib}(0,0), \mathrm{fib}(1,1), \forall n, x, y.[\mathrm{fib}(n,x) \wedge \mathrm{fib}(n+1,y) \supset \mathrm{fib}(n+2, x+y)]\}$$

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Unique proof – exponential in size!

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Many proofs – the smallest is linear in size!

Polarities of connectives

First-order classical and intuitionistic language:

$$A ::= P(x) \mid A \wedge A \mid t \mid A \vee A \mid f \mid A \supset A \mid \exists x\, A \mid \forall x\, A$$

Polarized connectives:

- In classical logic
  - positive and negative versions of the logical connectives and constants:

    $$\wedge^-, \wedge^+, t^-, t^+, \vee^-, \vee^+, f^-, f^+$$

  - first-order quantifiers: $\forall$ negative and $\exists$ positive.
- In intuitionistic logic
  - use polarized classical constants, connectives, and quantifiers, except
  - drop $f^-, \vee^-$, and
  - add negative implication: $\supset$.

How to polarize a classical formula

- atomic formulas are labeled either positive or negative;
- replace all occurrences of true with either t^+ or t^-, of false with either f^+ or f^-, of conjunction with either $\wedge^+$ or $\wedge^-$ or of disjunction with either $\vee^+$ or $\vee^-$. (If there are n occurrences of truth, false, conjunction and disjunction, there are 2^n ways to do this replacement.)

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A formula is positive if it is a positive atom or has a top-level positive connective.
A formula is negative if it is a negative atom or has a top-level negative connective.

Polarity-based hierarchy

Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and

$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$

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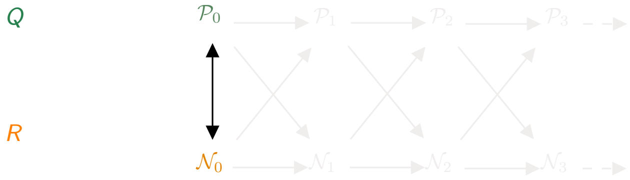

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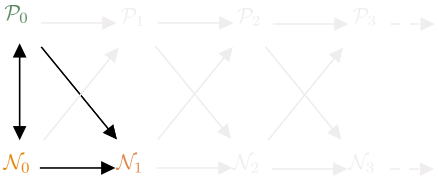
$Q_1 \wedge^- Q_2$



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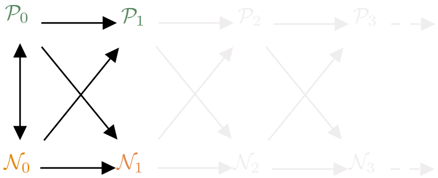

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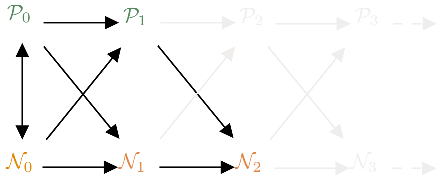

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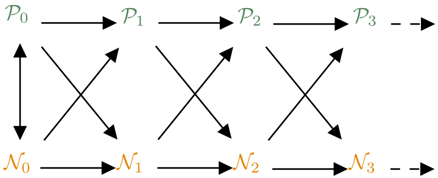

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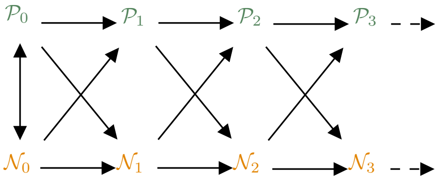

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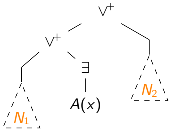

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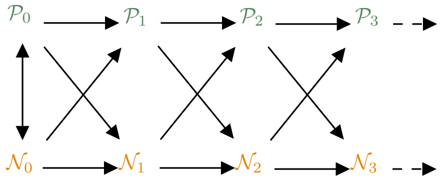

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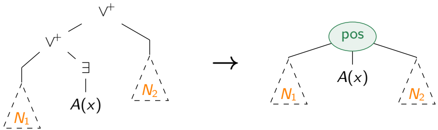

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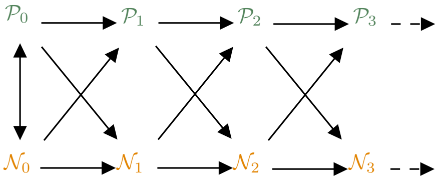

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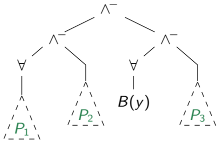

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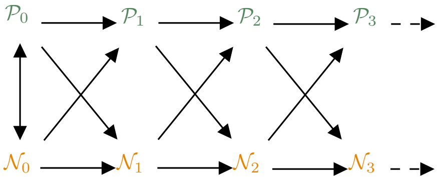

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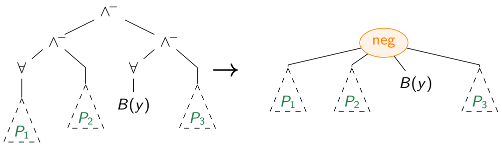

Bipolar formulas

The hierarchy can be specified for intuitionistic or classical formulas.

Any formula in the class $\mathcal{N}_2^C$ / $\mathcal{N}_2^I$ is a classical/ intuitionistic bipolar formula.

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Example. $(P_1 \supset P_2) \vee (Q_1 \supset Q_2)$

- $(P_1 \supset P_2) \vee^- (Q_1 \supset Q_2) \rightsquigarrow$ classical bipolar.
- No polarization yields an intuitionistic bipolar formula.

Outline

What is focusing?

Consider again the sequent

$$\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B$$

with A_i atomic, B a formula and Γ a multiset of formulas.

How to prove it?

Many ways to proceed!

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Focused rule application [And92]:
commit to repeat the $L\supset$ rule on the right premise until the atomic formula A_0 results:

$$\dfrac{\Gamma \vdash A_1 \quad \dfrac{\Gamma \vdash A_2 \quad \dfrac{\Gamma \vdash A_3 \quad \dfrac{\Gamma \vdash A_4 \quad \Gamma, A_0 \vdash B}{\Gamma, A_4 \supset A_0 \vdash B}\ L\supset}{\Gamma, A_3 \supset \cdots \supset A_n \supset A_0 \vdash B}\ L\supset}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\ L\supset}{\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\ L\supset$$

An organizational tool

Focusing provides a way to restrict the proof search space while remaining complete.

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Unfocused

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Unfocused $\longrightarrow$ Focused

LJF and LKF (Liang & M [LM07, LM09])

Two kinds of focused sequents

- $\Downarrow$ sequents to decompose the formula under focus

$$\Gamma \Downarrow B \vdash \Delta \text{ with a left focus on } B$$
$$\Gamma \vdash B \Downarrow \Delta \text{ with a right focus on } B$$

When the conclusion of an introduction rule, then that rule introduced B.

- $\Uparrow$ sequents for invertible introduction rules

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Example of rules:

$$\frac{\Gamma \vdash B_1 \Downarrow \Delta \quad \Gamma \Downarrow B_2 \vdash \Delta}{\Gamma \Downarrow B_1 \supset B_2 \vdash \Delta} \qquad \frac{\Gamma_1 \Uparrow \Gamma_2, B_1 \vdash B_2 \Uparrow \Delta}{\Gamma_1 \Uparrow \Gamma_2 \vdash B_1 \supset B_2 \Uparrow \Delta}$$

non-invertible — invertible

LJF and LKF (Liang & M [LM07, LM09])

The dynamic of proof search:

- A formula is put under focus (the only instance of contraction)

Decide: $\dfrac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l \qquad \dfrac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\ D_r$

LJF and LKF (Liang & M [LM07, LM09])

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- Focus is transferred from conclusion to premises until

LJF and LKF (Liang & M [LM07, LM09])

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- Focus is transferred from conclusion to premises until
  - either the focused phase ends

  Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\ R_r$$

LJF and LKF (Liang & M [LM07, LM09])

The dynamic of proof search:

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  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\ D_r$$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\ R_r$$

  - or the derivation ends

    Initial: $$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N, \Delta}\ I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow \Delta}\ I_r$$

LJF and LKF (Liang & M [LM07, LM09])

The dynamic of proof search:

- A formula is put under focus (the only instance of contraction)

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\ D_r$$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\ R_r$$

  - or the derivation ends

    Initial: $$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N, \Delta}\ I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow \Delta}\ I_r$$

- Once the focus is released, invertible rules eagerly decompose the formula into subformulas, which are ultimately stored in the context.

  Store: $$\frac{\Gamma_1, P \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, P \vdash \Delta_1 \Uparrow \Delta_2}\ S_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \Delta_1 \Uparrow N, \Delta_2}{\Gamma \Uparrow \cdot \vdash N, \Delta_1 \Uparrow \Delta_2}\ S_r$$

LJF and LKF (Liang & M [LM07, LM09])

The dynamic of proof search:

- A formula is put under focus (the only instance of contraction)

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\ D_r$$

- Focus is transferred from conclusion to premises until
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    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\ R_r$$

  - or the derivation ends

    Initial: $$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N, \Delta}\ I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow \Delta}\ I_r$$

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$\Rightarrow$ Sequent derivations are organized into $\Uparrow$ and $\Downarrow$ phases

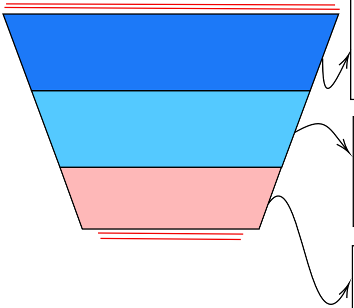
$\Rightarrow$ Synthetic rules result from looking only at border sequents: $\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta$

Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

1. starting with a decide on B;
2. in which no $\Downarrow$ rule occurs above an $\Uparrow$ rule;
3. and only atomic formulas are stored.



Atomic storage

Atoms are stored

$$\frac{C, \Gamma \Uparrow \Theta \vdash \Omega \Uparrow \Delta}{\Gamma \Uparrow C, \Theta \vdash \Omega \Uparrow \Delta}\ \mathsf{s}_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \Omega \Uparrow D, \Delta}{\Gamma \Uparrow \cdot \vdash D, \Omega \Uparrow \Delta}\ \mathsf{s}_r$$

$\Uparrow$ phase

Invertible rules are applied eagerly

$$\frac{\Gamma \Uparrow\ \vdash A, \Omega \Uparrow \Delta \quad \Gamma \Uparrow\ \vdash B, \Omega \Uparrow \Delta}{\Gamma \Uparrow\ \vdash A \wedge^- B, \Omega \Uparrow \Delta}\ \wedge_r^-$$

$\Downarrow$ phase

Focusing persists

$$\frac{\Gamma \vdash A \Downarrow \Delta \quad \Gamma \vdash B \Downarrow \Delta}{\Gamma \vdash A \wedge^+ B \Downarrow \Delta}\ \wedge_r^+$$

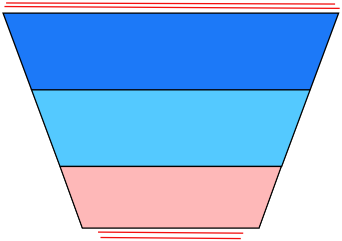
Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

1. starting with a decide on B;
2. in which no $\Downarrow$ rule occurs above an $\Uparrow$ rule;
3. and only atomic formulas are stored.

$$\Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow \Delta_1 \quad \ldots \quad \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow \Delta_n$$



$$\frac{\Gamma, B \Downarrow B \vdash \Delta}{\Gamma, B \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l$$

Corresponding synthetic rule

(in LK or LJ)

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

Outline

1st result: Bipolar $\longleftrightarrow$ Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

Theorem:

- If B is bipolar, then any synthetic inference rule for B is a bipole.
- If every synthetic inference rule for B is a bipole then B is bipolar.

Prototype implementation:
λProlog [MN12, NM88] executable specification of a predicate that relates a bipolar formula to its various bipoles.

$\Rightarrow$ compact given the nature of λProlog

$\Rightarrow$ explicit about the scope of bindings for schematic variables and eigenvariables.

$\Rightarrow$ unproblematic treatment of unification and eigenvariables

2nd result: Cut admissibility

Let $\mathcal{T}$ be a set of bipolar formulas.

$\mathsf{LK}\langle\mathcal{T}\rangle/\mathsf{LJ}\langle\mathcal{T}\rangle$ denotes the extension of LK/LJ with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

Theorem: The cut rule is admissible for the proof systems $\mathsf{LK}\langle\mathcal{T}\rangle/\mathsf{LJ}\langle\mathcal{T}\rangle$.

Note: the proof is simple!

It is a direct consequence of (polarized) cut admissibility in LKF/LJF.

$$\frac{\Gamma \Uparrow \cdot \vdash B \Uparrow \Delta \qquad \Gamma \Uparrow B \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ Cut$$

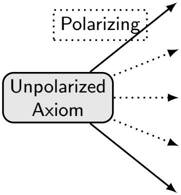
Rules from axioms

Rules from axioms

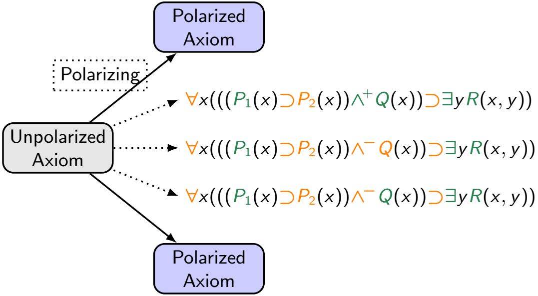
Unpolarized Axiom

$$\forall x(((P_1(x) \supset P_2(x)) \wedge Q(x)) \supset \exists y R(x, y))$$

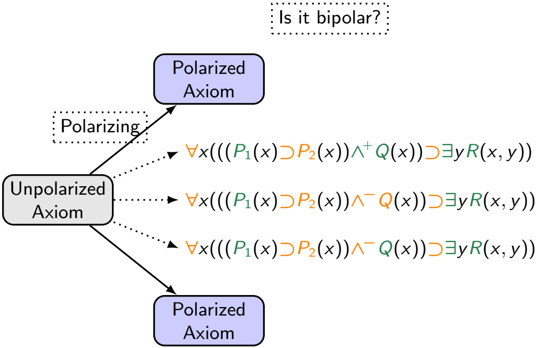
Rules from axioms



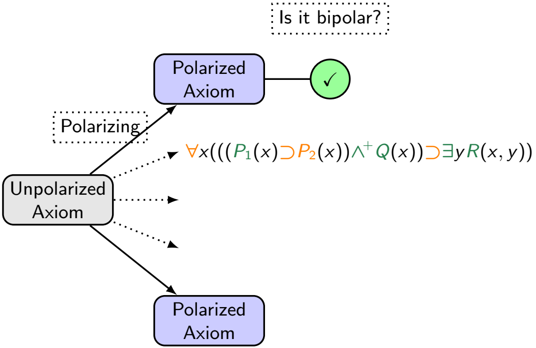
Rules from axioms



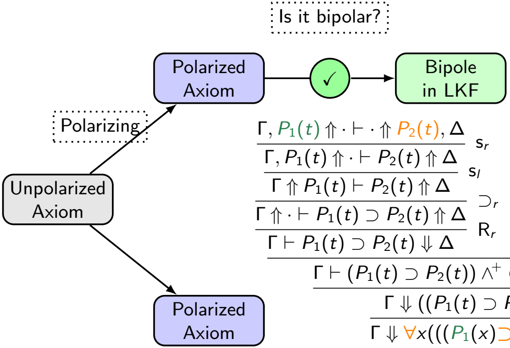
Rules from axioms



Rules from axioms



Rules from axioms



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\dfrac{}{\Gamma \vdash Q(t) \Downarrow \Delta}\ I_r
}{\Gamma \vdash (P_1(t) \supset P_2(t)) \wedge^+ Q(t) \Downarrow \Delta}\ \wedge_r^+
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\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\ s_l}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\ \exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\ R_l
}{\dfrac{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \wedge^+ Q(t)) \supset \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \wedge^+ Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\ \forall_l}\ \supset_l
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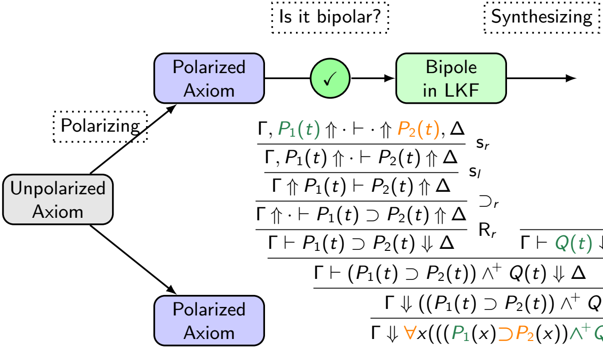
Rules from axioms



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\quad
\dfrac{}{\Gamma \vdash Q(t) \Downarrow \Delta}\, I_r
}{\Gamma \vdash (P_1(t) \supset P_2(t)) \wedge^+ Q(t) \Downarrow \Delta}\, \wedge_r^+
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\dfrac{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \wedge^+ Q(t)) \supset \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow \forall x (((P_1(x) \supset P_2(x)) \wedge^+ Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\, \forall_l
}\, \supset_l
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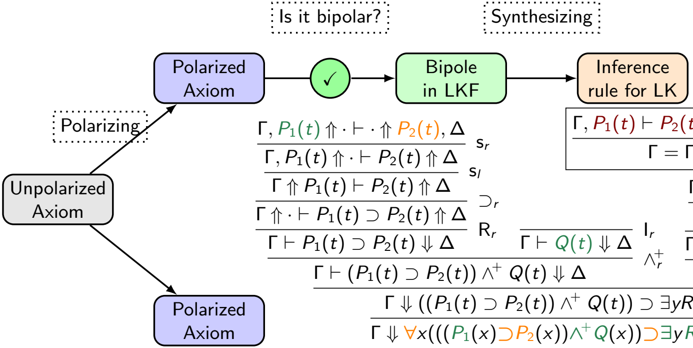
Rules from axioms



$$\dfrac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}$$

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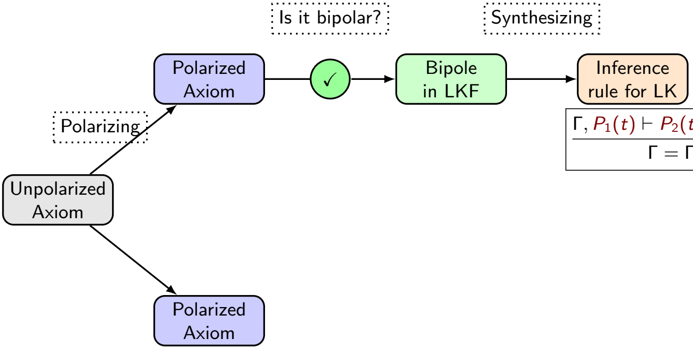
$$\dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\ s_l}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\ \exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\ R_l$$

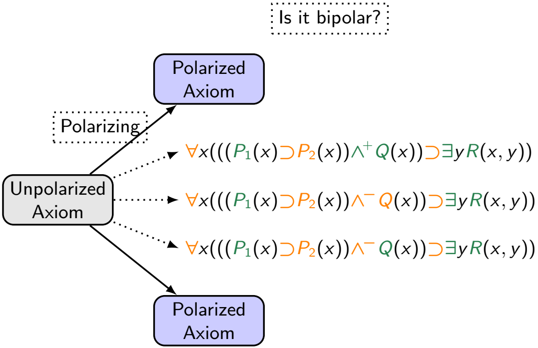
$$\dfrac{\dfrac{\Gamma \vdash (P_1(t) \supset P_2(t)) \wedge^+ Q(t) \Downarrow \Delta \qquad \Gamma \Downarrow \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \wedge^+ Q(t)) \supset \exists y R(t,y) \vdash \Delta}\ \supset_l}{\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \wedge^+ Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\ \forall_l$$

Rules from axioms

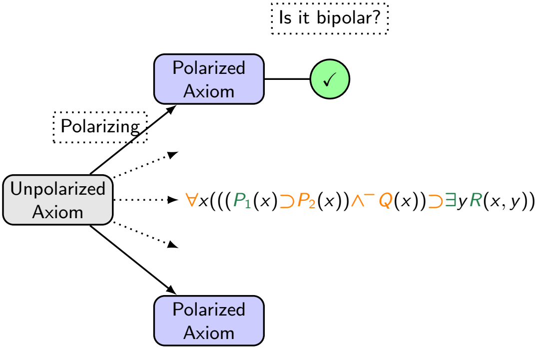


$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}$$

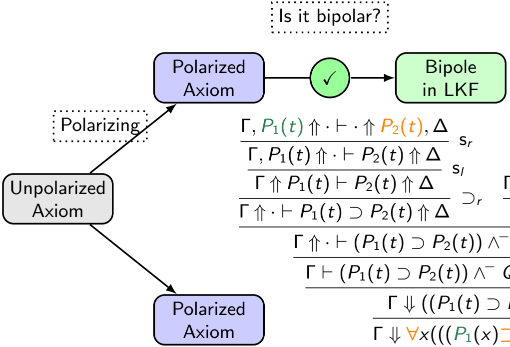
Rules from axioms



Rules from axioms



Rules from axioms



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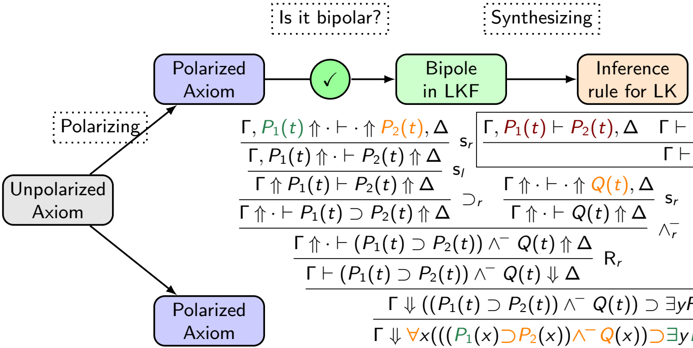
$$

Rules from axioms



$$\dfrac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}$$

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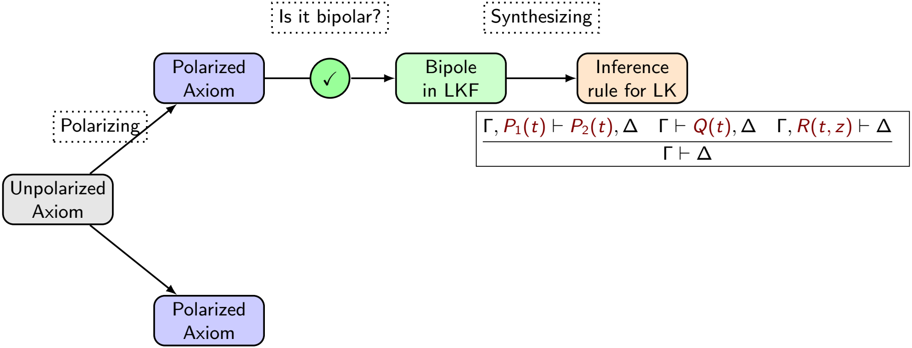
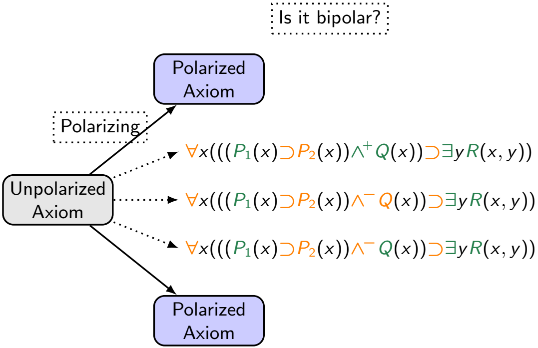
Rules from axioms



Rules from axioms



$$\forall x(((P_1(x) \supset P_2(x)) \wedge^{+} Q(x)) \supset \exists y R(x,y))$$

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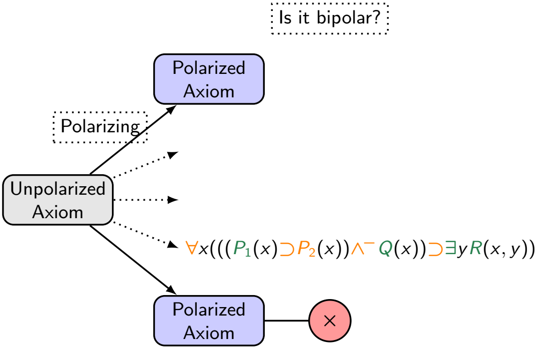
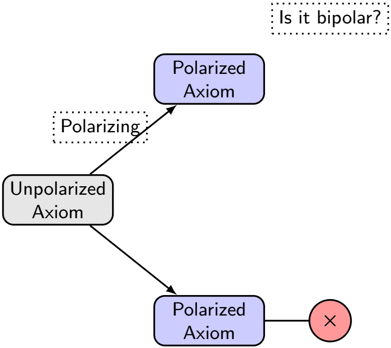
$$\forall x(((P_1(x) \supset P_2(x)) \wedge^{-} Q(x)) \supset \exists y R(x,y))$$

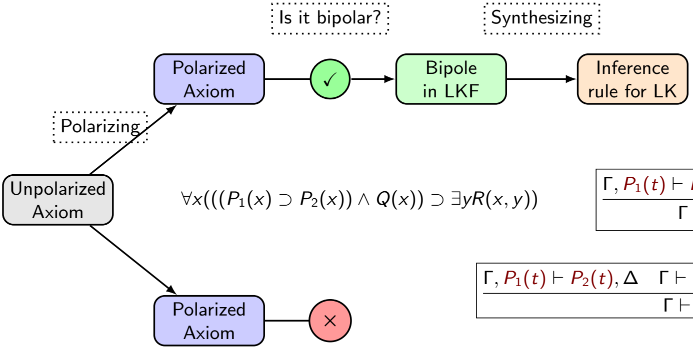
Rules from axioms



Rules from axioms



Rules from axioms

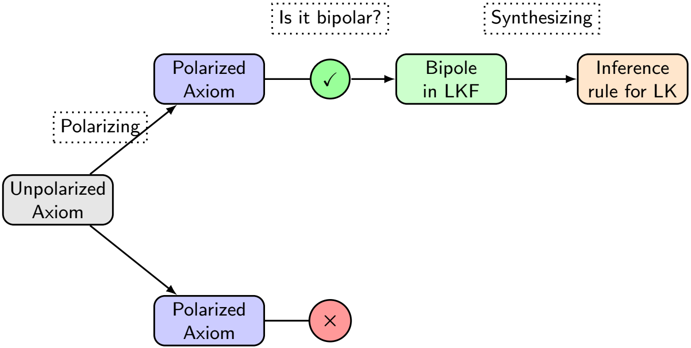


$$\forall x(((P_1(x) \supset P_2(x)) \wedge Q(x)) \supset \exists y R(x, y))$$

$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma, R(t, z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}$$

$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t, z) \vdash \Delta}{\Gamma \vdash \Delta}$$

Rules from axioms



Outline

Geometric axioms as bipoles

Geometric implication:

$$\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \supset \exists \overline{x}_1 M_1 \vee \ldots \vee \exists \overline{x}_n M_n)$$

- P_i atomic;
- $M_j = Q_{j_1} \wedge \ldots \wedge Q_{j_{k_j}}$, Q_{j_k} atomic;
- none of the variables in the vectors $\overline{x}_j$ are free in P_i.

Geometric axioms as bipoles

Polarized geometric implication:

$$\forall \overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \supset \exists \overline{x}_1 \hat{M}_1 \vee^{\pm} \ldots \vee^{\pm} \exists \overline{x}_n \hat{M}_n)$$

- P_i^+, P_i^- atomic;
- $\hat{M}_j = Q_{j_1}^{\pm} \wedge^{+} \ldots \wedge^{+} Q_{j_{k_j}}^{\pm}$, $Q_{j_k}^{\pm}$ atomic;
- none of the variables in the vectors $\overline{x}_j$ are free in P_i.

Geometric axioms as bipoles

Polarized geometric implication:

$$\forall \overline{z}(P_1^+ \wedge^+ \ldots \wedge^+ P_m^+ \supset \exists \overline{x}_1 \hat{M}_1 \vee^\pm \ldots \vee^\pm \exists \overline{x}_n \hat{M}_n)\,,$$

Corresponding bipole:

$$\frac{\overline{Q}_1[\overline{y}_1/\overline{x}_1], \Gamma \Uparrow \vdash \Uparrow \Delta \quad \ldots \quad \overline{Q}_n[\overline{y}_n/\overline{x}_n], \Gamma \Uparrow \vdash \Uparrow \Delta}{\overline{P}, \Gamma' \Uparrow \vdash \Uparrow \Delta}$$

with $\overline{P} = \{P_i^+\}, \overline{Q_j} = \{Q_{j_k}^\pm\}$.

Corresponding LK rule:

$$\frac{\overline{Q}_1[\overline{y}_1/\overline{x}_1], \Gamma \vdash \Delta \quad \ldots \quad \overline{Q}_n[\overline{y}_n/\overline{x}_n], \Gamma \vdash \Delta}{\overline{P}, \Gamma' \vdash \Delta}\ GRS$$

Geometric axioms as bipoles

Polarized geometric implication:

$$\forall \overline{z}(P_1^- \wedge^\pm \ldots \wedge^\pm P_m^- \supset \exists \overline{x}_1 \hat{M}_1 \vee^\pm \ldots \vee^\pm \exists \overline{x}_n \hat{M}_n)\,,$$

Corresponding bipole:

$$\frac{\overline{Q}_j[\overline{y}_j/\overline{x}_j], \Gamma \Uparrow \vdash \Uparrow \Delta \quad \ldots \quad \Gamma \Uparrow \vdash \Uparrow P_i, \Delta}{\Gamma \Uparrow \vdash \Uparrow \Delta}\ m+n \text{ premises}$$

with $\overline{Q_j} = \{Q_{j_k}\}$.

Corresponding LK rule:

$$\frac{\overline{Q}_j[\overline{y}_j/\overline{x}_j], \Gamma \vdash \Delta \quad \ldots \quad \Gamma \vdash P_i, \Delta}{\Gamma \vdash \Delta}\ m+n \text{ premises}$$

Co-geometric axioms as bipoles

Polarized co-geometric implication:

$$\forall \overline{z}(\forall \overline{x}_1 \hat{M}_1 \wedge^{\pm} \ldots \wedge^{\pm} \forall \overline{x}_n \hat{M}_n \supset P_1^- \vee^- \ldots \vee^- P_m^-)\,,$$

with $\hat{M}_j = Q_{j_1}^{\pm} \vee^- \ldots \vee^- Q_{j_{k_j}}^{\pm}$.

Corresponding bipole:

$$\frac{\Gamma \Uparrow \vdash \Uparrow \overline{Q}_1[\overline{y}_1/\overline{x}_1], \Delta \quad \ldots \quad \Gamma \Uparrow \vdash \Uparrow \overline{Q}_n[\overline{y}_n/\overline{x}_n], \Delta}{\Gamma \Uparrow \vdash \Uparrow \overline{P}, \Delta'}$$

Corresponding LK rule:

$$\frac{\Gamma \vdash \overline{Q}_1[\overline{y}_1/\overline{x}_1], \Delta \quad \ldots \quad \Gamma \vdash \overline{Q}_n[\overline{y}_n/\overline{x}_n], \Delta}{\Gamma \vdash \overline{P}, \Delta'} \; co-GRS_c$$

Co-geometric axioms as bipoles

Polarized co-geometric implication:

$$\forall \overline{z}(\forall \overline{x}_1 \hat{M}_1 \wedge^{\pm} \ldots \wedge^{\pm} \forall \overline{x}_n \hat{M}_n \supset P_1^{+} \vee^{\pm} \ldots \vee^{\pm} P_m^{+}),$$

with $\hat{M}_j = Q_{j_1}^{\pm} \vee^{-} \ldots \vee^{-} Q_{j_{k_j}}^{\pm}$.

Corresponding bipole:

$$\frac{\Gamma \Uparrow \ \vdash \ \Uparrow \overline{Q}_j[\overline{y}_j/\overline{x}_j], \Delta \quad \ldots \quad \Gamma, P_i \Uparrow \ \vdash \ \Uparrow \Delta}{\Gamma \Uparrow \ \vdash \ \Uparrow \Delta} \ m+n \text{ premises}$$

Corresponding LK rule:

$$\frac{\Gamma \vdash \overline{Q}_j[\overline{y}_j/\overline{x}_j], \Delta \quad \ldots \quad \Gamma, P_i \vdash \Delta}{\Gamma \vdash \Delta} \ m+n \text{ premises}$$

Universal axioms as bipoles

$$\forall \bar{z}(P_1 \wedge \ldots \wedge P_m \supset Q_1 \vee \ldots \vee Q_n)$$

Universal axioms as bipoles

$$\forall \overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \supset Q_1^{\pm} \vee^{\pm} \ldots \vee^{\pm} Q_n^{\pm})$$

More choices in the selection of polarities while still remaining bipolar formulas.

Universal axioms as bipoles

$$\forall \bar{z}(P_1^+ \wedge^+ \ldots \wedge^+ P_m^+ \supset Q_1^\pm \vee^+ \ldots \vee^+ Q_n^\pm)$$

More choices in the selection of polarities while still remaining bipolar formulas.

$$\frac{Q_1, \Gamma \Uparrow \vdash \Uparrow \Delta \quad \ldots \quad Q_n, \Gamma \Uparrow \vdash \Uparrow \Delta}{\overline{P}, \Gamma' \Uparrow \vdash \Uparrow \Delta} \; FRL_c$$

Universal axioms as bipoles

$$\forall \overline{z}(P_1^+ \wedge^+ \ldots \wedge^+ P_m^+ \supset Q_1^\pm \vee^+ \ldots \vee^+ Q_n^\pm)$$

More choices in the selection of polarities while still remaining bipolar formulas.

$$\frac{Q_1, \Gamma \Uparrow \vdash \Uparrow \Delta \quad \ldots \quad Q_n, \Gamma \Uparrow \vdash \Uparrow \Delta}{\overline{P}, \Gamma' \Uparrow \vdash \Uparrow \Delta} \; FRL_c$$

$$a \in l \wedge^+ a \in m \wedge^+ b \in l \wedge^+ b \in m \supset a = b \vee^+ l = m$$

$$\frac{\Gamma, a = b \vdash \Delta \quad \Gamma, l = m \vdash \Delta}{\Gamma, a \in l, a \in m, b \in l, b \in m \vdash \Delta} \; Uni_p$$

Universal axioms as bipoles

$$\forall \overline{z}(P_1^{\pm} \wedge^- \ldots \wedge^- P_m^{\pm} \supset Q_1^- \vee^- \ldots \vee^- Q_n^-)$$

More choices in the selection of polarities while still remaining bipolar formulas.

$$\frac{\Gamma \Uparrow \vdash \Uparrow P_1, \Delta \quad \ldots \quad \Gamma \Uparrow \vdash \Uparrow P_m, \Delta}{\Gamma \Uparrow \vdash \Uparrow \overline{Q}, \Delta} \; FRR_c$$

Universal axioms as bipoles

$$\forall \overline{z}(P_1^{\pm} \wedge^- \ldots \wedge^- P_m^{\pm} \supset Q_1^- \vee^- \ldots \vee^- Q_n^-)$$

More choices in the selection of polarities while still remaining bipolar formulas.

$$\frac{\Gamma \Uparrow \vdash \Uparrow P_1, \Delta \quad \ldots \quad \Gamma \Uparrow \vdash \Uparrow P_m, \Delta}{\Gamma \Uparrow \vdash \Uparrow \overline{Q}, \Delta}\ FRR_c$$

$$a \in l \wedge^- a \in m \wedge^- b \in l \wedge^- b \in m \supset a = b \vee^- l = m$$

$$\frac{\Gamma \vdash \Delta, a \in l \quad \Gamma \vdash \Delta, a \in m \quad \Gamma \vdash \Delta, b \in l \quad \Gamma \vdash \Delta, b \in m}{\Gamma \vdash \Delta, a = b, l = m}\ Uni_n$$

Horn clauses as bipoles

$$\forall \bar{z}(P_1 \wedge \ldots \wedge P_m \supset Q)$$

Horn clauses as bipoles

$$\forall \bar{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \supset Q^{\pm})$$

Even more choices in the selection of polarities while still remaining bipolar formulas!

Horn clauses as bipoles

$$\forall \bar{z}(P_1^+ \wedge^+ \ldots \wedge^+ P_m^+ \supset Q^+)$$

Even more choices in the selection of polarities while still remaining bipolar formulas!

$$\frac{Q, \Gamma \vdash \Delta}{\overline{P}, \Gamma' \vdash \Delta}\ FC$$

Forward-chaining
[Sim94, NvP98, CMS13]

Horn clauses as bipoles

$$\forall \bar{z}(P_1^- \wedge^- \ldots \wedge^- P_m^- \supset Q^-)$$

Even more choices in the selection of polarities while still remaining bipolar formulas!

$$\frac{\Gamma \vdash P_1, \Delta \quad \ldots \quad \Gamma \vdash P_m, \Delta}{\Gamma \vdash Q, \Delta'}\ BC$$

Back-chaining
[Vig00]

Implementation – Part I [MMPV22]

Formula

$\forall u \forall v \forall w (adj\ u\ v \supset (path\ v\ w \supset path\ u\ w))$ Positive atoms.

λProlog encoding

```
(all u\ all v\ all w\ imp (atm (adj u v))
                        (imp (atm (path v w)) (atm (path u w)))),
```

Goal

```
reduce (syncL Gamma F (atm B)) Premises.
```

Inference rule

$$\frac{adj\ X\ Z, path\ Z\ Y, path\ X\ Y, L \vdash B}{adj\ X\ Z, path\ Z\ Y, L \vdash B}$$

Implementation – Part I [MMPV22]

Formula

$\forall u \forall v \forall w(adj\ u\ v \supset (path\ v\ w \supset path\ u\ w))$ Negative atoms.

λProlog encoding

```
(all u\ all v\ all w\ imp (atm (adj u v))
                          (imp (atm (path v w)) (atm (path u w)))),
```

Goal

```
reduce (syncL Gamma F (atm B)) Premises.
```

Inference rule

$$\frac{\Gamma \vdash adj\ X\ Y \quad \Gamma \vdash path\ Y\ Z}{\Gamma \vdash path\ X\ Z}$$

Implementation – Part II [MMPV22]

Formula

$\forall u \forall v (\forall w (\texttt{in}\ w\ u \supset \texttt{in}\ w\ v) \supset \texttt{subset}\ u\ v)$ Positive atoms.

λProlog encoding

```
(all u\ all v\ imp (all w\ imp (atm (in w u)) (atm (in w v)))
                   (atm (subset u v))).
```

Goal

```
reduce (syncL Gamma F (atm B)) Premises.
```

Inference rule

$$\frac{\texttt{in}\ w\ X, \Gamma \vdash \texttt{in}\ w\ Y \quad \texttt{subset}\ X\ Y, \Gamma \vdash B}{\Gamma \vdash B}$$

Implementation – Part II [MMPV22]

Formula

$\forall u \forall v (\forall w (\texttt{in}\ w\ u \supset \texttt{in}\ w\ v) \supset \texttt{subset}\ u\ v)$ Negative atoms.

λProlog encoding

```
(all u\ all v\ imp (all w\ imp (atm (in w u)) (atm (in w v)))
                    (atm (subset u v))).
```

Goal

```
reduce (syncL Gamma F (atm B)) Premises.
```

Inference rule

$$\frac{\Gamma,\ \texttt{in}\ w\ X \vdash \texttt{in}\ w\ Y}{\Gamma \vdash \texttt{subset}\ X\ Y}$$

Affine geometry

- Parallel lines.
- Affine geometry = (Euclidean geometry - congruence) $\vee$ (projective geometry + parallels).
- $l \parallel m$, $par(l, a)$.
- General:

$$l \parallel l \qquad l \parallel m \supset m \parallel l \qquad l \parallel m \wedge m \parallel n \supset l \parallel n$$

- Incidency:

$$a \in par(l, a) \qquad par(l, a) \parallel l.$$

- Uniqueness

$$a \in l \wedge a \in m \wedge l \parallel m \supset l = m.$$

- Substitution

$$l \parallel m \wedge m = n \supset l \parallel n.$$

System GA

I. General

$$\frac{}{\Gamma \vdash \Delta, l \parallel l}\ Ref \quad \frac{\Gamma \vdash \Delta, l \parallel m}{\Gamma \vdash \Delta, m \parallel l}\ Sym \quad \frac{\Gamma \vdash \Delta, l \parallel m \quad \Gamma \vdash \Delta, m \parallel n}{\Gamma \vdash \Delta, l \parallel n}\ Tr$$

II. Incidency

$$\frac{}{\Gamma \vdash \Delta, a \in par(l, a)}\ IA \quad \frac{}{\Gamma \vdash \Delta, par(l, a) \parallel l}\ Par$$

III. Uniqueness

$$\frac{\Gamma \vdash \Delta, a \in l \quad \Gamma \vdash \Delta, a \in m \quad \Gamma \vdash \Delta, l \parallel m}{\Gamma \vdash \Delta, l = m}\ Unipar$$

IV. Substitution

$$\frac{\Gamma \vdash \Delta, l \parallel m \quad \Gamma \vdash \Delta, m = n}{\Gamma \vdash \Delta, l \parallel n}\ SA$$

Outline

Connection with hypersequents?

Gödel-Dummett logic: LJ plus the axiom $(P \supset Q) \vee (Q \supset P)$.

Polarize this and make it negative (to store on the left of a sequent):

$$[(P \supset Q) \vee^{+} (Q \supset P)] \wedge^{-} \top^{-}$$

This is not a bipole.

$$\dfrac{\dfrac{\dfrac{\Gamma, P \supset Q \Uparrow \cdot \vdash \cdot \Uparrow C}{\Gamma \Uparrow P \supset Q \vdash \cdot \Uparrow C} \quad \dfrac{\Gamma, Q \supset P \Uparrow \cdot \vdash \cdot \Uparrow C}{\Gamma \Uparrow Q \supset P \vdash \cdot \Uparrow C}}{\Gamma \Uparrow (P \supset Q) \vee^{+} (Q \supset P) \vdash \cdot \Uparrow C}}{\dfrac{\Gamma \Downarrow (P \supset Q) \vee^{+} (Q \supset P) \vdash C}{\Gamma \Downarrow [(P \supset Q) \vee^{+} (Q \supset P)] \wedge^{-} \top^{-} \vdash C}}$$

Connection with hypersequents?

Gödel-Dummett logic: LJ plus the axiom $(P \supset Q) \vee (Q \supset P)$.

Polarize this and make it negative (to store on the left of a sequent):

$$[(P \supset Q) \vee^+ (Q \supset P)] \wedge^- \top^-$$

This is not a bipole.

$$\dfrac{\begin{array}{c}\text{“}P \supset Q\text{''} \\ \vdots \\ \Gamma \vdash C\end{array} \qquad \begin{array}{c}\text{“}Q \supset P\text{''} \\ \vdots \\ \Gamma \vdash C\end{array}}{\Gamma \vdash C}$$

Connection with hypersequents?

Gödel-Dummett logic: LJ plus the axiom $(P \supset Q) \vee (Q \supset P)$.

Polarize this and make it negative (to store on the left of a sequent):

$$[(P \supset Q) \vee^{+} (Q \supset P)] \wedge^{-} \top^{-}$$

This is not a bipole.

$$\dfrac{\begin{array}{c}\text{“}P \supset Q\text{''} \\ \vdots \\ \Gamma \vdash C\end{array} \qquad \begin{array}{c}\text{“}Q \supset P\text{''} \\ \vdots \\ \Gamma \vdash C\end{array}}{\Gamma \vdash C}$$

This rule resembles the communication rule in hypersequents:

$$\dfrac{G \mid \Gamma_1 \vdash P \mid H \qquad G \mid \Gamma_2 \vdash Q \mid H}{G \mid \Gamma_1 \vdash Q \mid \Gamma_2 \vdash P \mid H}$$

To conclude

- ⋆ Synthetic inference rules generated using polarization and focusing provide inference rules that capture certain classes of axioms.
- ⋆ In particular: bipolar formulas correspond to inference rules for atoms.
- ⋆ As geometric formulas are examples of bipolar formulas, polarized versions of such formulas yield well known inference systems derived from geometric formulas.
- ⋆ Polarization of subsets of geometric formulas explain the forward-chaining and backward-chaining variants of their synthetic inference rules.
- ⋆ Direct proof of cut-elimination for the proof systems that arise from incorporating synthetic inference rules based on polarized formulas.
- ⋆ Additionally, all of these results work equally well in both classical and intuitionistic logics using the corresponding LKF and LJF focused proof systems.

Thank you!

Questions?

Art by Nadia Miller

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