Notation for focused proof systems

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Standardized notation can be helpful, especially after a topic has matured. For example, Gentzen used $\longrightarrow$ as an infix symbol to build sequents from lists of formulas. While one also sees $\Rightarrow$ used, it seems that in recent years, $\vdash$ has become the standard, especially when proof theory is applied to type theory (where $\vdash$ is invariably used) and computational logic (where $\longrightarrow$ is often used for other purposes). This notation is overloaded: the expression $\Gamma \vdash B$ is both a syntactic object (a sequent) and the mathematics-level judgment that Γ entails B. Nonetheless, we seem to be able to live with such overloading.

1 The many notations for focused sequents

Since I am working with various focused proof systems, I want a coherent and appealing way to present the sequents and rules involved with building the two focusing phases. This notation should be consistent over various sequent calculus systems, including one-sided and two-sided; single-conclusion and multi-conclusion; single-focus and multi-focus. I have come up with the following notation and terminology. I share these here since others might find this notation convenient.

Let us first list some of the notations used to present focused sequents.

1. Andreoli's original format was only for single-sided sequents for linear logic. His triadic sequent system Σ_3 [1] had sequents of the form $\vdash \Theta\colon \Gamma \Uparrow L$ and $\vdash \Theta\colon \Gamma \Downarrow F$ where F denotes a formula, Γ, Δ, and L are multisets of formulas, weakening and contraction are allowed in Θ (i.e., the classical zone), weakening and contraction are not allowed in Γ (i.e., the linear zone), and introduction rules take place in L and with F.

2. Girard used zones (with one acting as the stoup) in LU and LC [9, 10] with the classical zones closest to the $\vdash$. Herbelin, Curien, and Munch-Maccagnoni used similar notation [5, 6, 11] as did Zeilberger [18]. In their presentation of LKT/LKQ, Danos et al. [7] also used sequents with various zones, one of which denotes the stoup.

3. LJQ by Dyckhoff and Lengrand [8] used the two arrows $\Rightarrow$ and $\longrightarrow$ to denote the sequents used in the two separate phases.

4. Chaudhuri et al. [4] used $\Gamma; \Delta \gg B$ and $\Gamma; \Delta; B \ll C$ in focused linear logic systems.

5. In some early writing of mine [14], I put a formula above or below the sequent arrow to denote the formula in focus (denoting the left or right focus). Chuck Liang and I in [12] continued that practice and added brackets to denote different zones: e.g., $\vdash [\Gamma], \Theta \longrightarrow [R]$, $\vdash [\Gamma] \xrightarrow{B} [C]$, etc.

The terminology associated with focused proof systems varies a lot as well. For example, Andreoli referred to the two phases of focus-proof construction using the adjectives *asynchronous* and *synchronous*. Describing these phases as the *invertible* and *non-invertible* phases seems natural, although sometimes an inference rule in the synchronous phase can be invertible (see the $\wedge_r^+$ rule in the LJF proof system below). So these days I favor calling these phases simply the *negative* and *positive* phases.

As another example, there is the *religious* term "stoup" (i.e., "blessed") used by Girard and others to denote the formula under focus. More *dynamic* terminology was used by Andreoli (e.g., reaction rules) and the Carnegie Mellon group (e.g., active, neutral, passive, focus, blur). I have settled on terminology that is more motivated by an *administrative* perspective of focusing as a process for building synthetic inference rules. This perspective suggests the following terminology. (I assume we are building proofs in a conclusion-to-premise fashion.)

1. The first step in building a synthetic inference rule is to *decide* on which formula in *storage* should be selected to provide the material from which the synthetic rule is built. That formula is placed into a temporary part of the sequent (the *staging area*).

2. Applications of non-invertible (positive) introduction rules in the staging area form the first phase of this process. When this phase is exhausted, the focus phase ends with either an initial or *release* rule.

3. Applications of invertible (negative) inference rules in the staging area forms the second phase. Any formula an invertible rule cannot introduce is *stored*, meaning it is moved from the staging area to a storage zone.

For these reasons, I prefer using the terms *decide*, *store*, and *release* to describe certain administrative inference rules in a focused proof system.

2 Selecting the notation

The notation I settled on is simply the following. I write one-sided sequents as

$$\vdash \Delta \Uparrow \Delta' \quad \text{and} \quad \vdash \Delta \Downarrow \Delta'$$

and two-sided sequents as

$$\Gamma' \Uparrow \Gamma \vdash \Delta \Uparrow \Delta' \quad \text{and} \quad \Gamma' \Downarrow \Gamma \vdash \Delta \Downarrow \Delta'.$$

The outer zones Γ' and Δ' are the storage areas, while the inner zones, namely Γ and Δ, are the staging areas. Also, Γ' and Δ' might be further decomposed into multiple zones with different structural rules available (as Andreoli did in his original focused proof system). When these sequents are used with intuitionistic logic, the right side of sequents are restricted so that the multiset union of Δ and Δ' contains exactly one formula. A sequent never contains an occurrence of both $\Uparrow$ and $\Downarrow$.

There are several positive points of this notation. It only employs the special symbols $\Uparrow$ and $\Downarrow$, and these symbols have seldom been used in proof theory outside focusing. This notation also supports multifocusing [15]: that is, a $\Downarrow$-sequent can have more than one formula in the staging area ($\Gamma \cup \Delta$). The only downside I see with this notation is that (with two-sided sequents) always writing two occurrences of an up or down arrow seems heavy. For this reason, I also adopt the following conventions (assuming that an empty zone is written using the dot $\cdot$). We can drop writing $\cdot \Downarrow$ and $\cdot \Uparrow$ when they appear on the right, and we drop writing $\Downarrow \cdot$ and $\Uparrow \cdot$ when they appear on the left. Thus, a *border* sequent $\Gamma' \Uparrow \cdot \vdash \cdot \Uparrow \Delta'$ (i.e., a sequent with an empty staging area) can be written as $\Gamma' \vdash \Delta'$. In this way, a border sequent can be confused with a sequent without focusing markings, and such confusion is a happy accident since border sequents appear as both premises and conclusions of *synthetic inference rules*. This way, focusing notation only appears in sequents used to build the internals of a synthetic inference rule. In the proof systems I have considered, sequents of the form $\Gamma' \Downarrow \Gamma \vdash \Delta \Downarrow \Delta'$ are such that the multiset union $\Gamma \cup \Delta$ is never empty.

3 LJF: A focused version of LJ

To illustrate this proposed notation, I present below my preferred way to write the LJF proof system [12]. Here, P is positive, N is negative, C is a negative formula or positive atom, D a positive formula or negative atom, N_a is a negative atom, P_a is a positive atom, and B is an arbitrary formula. In the rules $\vee_r^+$ and $\wedge_l^-$, i is either 1 or 2. In the rules $\forall_r$ and $\exists_l$, the eigenvariable y does not occur free in any formula of the conclusion. The display of these rules can be simplified somewhat using the conventions mentioned before that allow some up and down arrows next to empty zones to be elided.

Negative Introduction Rules

$$\frac{\Gamma \Uparrow B_1, \Theta \vdash B_2 \Uparrow \cdot}{\Gamma \Uparrow \Theta \vdash B_1 \supset B_2 \Uparrow \cdot}\ \supset_r \qquad \frac{\Gamma \Uparrow B_1, \Theta \vdash \Delta_1 \Uparrow \Delta_2 \quad \Gamma \Uparrow B_2, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow B_1 \vee^+ B_2, \Theta \vdash \Delta_1 \Uparrow \Delta_2}\ \vee_l^+$$

$$\frac{\Gamma \Uparrow B_1, B_2, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow B_1 \wedge^+ B_2, \Theta \vdash \Delta_1 \Uparrow \Delta_2}\ \wedge_l^+ \qquad \frac{\Gamma \Uparrow \Theta \vdash B_1 \Uparrow \cdot \quad \Gamma \Uparrow \Theta \vdash B_2 \Uparrow \cdot}{\Gamma \Uparrow \Theta \vdash B_1 \wedge^- B_2 \Uparrow \cdot}\ \wedge_r^-$$

$$\frac{}{\Gamma \Uparrow f^+, \Theta \vdash \Delta_1 \Uparrow \Delta_2}\ f_l^+ \qquad \frac{}{\Gamma \Uparrow \Theta \vdash t^- \Uparrow \cdot}\ t_r^- \qquad \frac{\Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow t^+, \Theta \vdash \Delta_1 \Uparrow \Delta_2}\ t_l^+$$

$$\frac{\Gamma \Uparrow \Theta \vdash [y/x]B \Uparrow \cdot}{\Gamma \Uparrow \Theta \vdash \forall x.B \Uparrow \cdot}\ \forall_r \qquad \frac{\Gamma \Uparrow [y/x]B, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow \exists x.B, \Theta \vdash \Delta_1 \Uparrow \Delta_2}\ \exists_l$$

Positive Introduction Rules

$$\frac{\Gamma \Downarrow \cdot \vdash B_1 \Downarrow \cdot \quad \Gamma \Downarrow B_2 \vdash \cdot \Downarrow D}{\Gamma \Downarrow B_1 \supset B_2 \vdash \cdot \Downarrow D}\ \supset_l \qquad \frac{\Gamma \Downarrow \cdot \vdash B_i \Downarrow \cdot}{\Gamma \Downarrow \cdot \vdash B_1 \vee^+ B_2 \Downarrow \cdot}\ \vee_r^+$$

$$\frac{\Gamma \Downarrow B_i \vdash \cdot \Downarrow D}{\Gamma \Downarrow B_1 \wedge^- B_2 \vdash \cdot \Downarrow D}\ \wedge_l^- \qquad \frac{\Gamma \Downarrow \cdot \vdash B_1 \Downarrow \cdot \quad \Gamma \Downarrow \cdot \vdash B_2 \Downarrow \cdot}{\Gamma \Downarrow \cdot \vdash B_1 \wedge^+ B_2 \Downarrow \cdot}\ \wedge_r^+$$

$$\frac{}{\Gamma \Downarrow \cdot \vdash t^+ \Downarrow \cdot}\ t_r^+ \qquad \frac{\Gamma \Downarrow [t/x]B \vdash \cdot \Downarrow D}{\Gamma \Downarrow \forall x.B \vdash \cdot \Downarrow D}\ \forall_l \qquad \frac{\Gamma \Downarrow \cdot \vdash [t/x]B \Downarrow \cdot}{\Gamma \Downarrow \cdot \vdash \exists x.B \Downarrow \cdot}\ \exists_r$$

Identity rules: Initial

$$\frac{}{\Gamma \Downarrow N_a \vdash \cdot \Downarrow N_a}\ \mathsf{I}_l \qquad \frac{}{\Gamma, P_a \Downarrow \cdot \vdash P_a \Downarrow \cdot}\ \mathsf{I}_r$$

Structural rules: Decide, Release, Store

$$\frac{\Gamma, N \Downarrow N \vdash \cdot \Downarrow D}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow D}\ \mathsf{D}_l \quad \frac{\Gamma \Downarrow \cdot \vdash P \Downarrow \cdot}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P}\ \mathsf{D}_r \quad \frac{\Gamma \Uparrow P \vdash \cdot \Uparrow D}{\Gamma \Downarrow P \vdash \cdot \Downarrow D}\ \mathsf{R}_l \quad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \cdot}{\Gamma \Downarrow \cdot \vdash N \Downarrow \cdot}\ \mathsf{R}_r$$

$$\frac{C, \Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow C, \Theta \vdash \Delta_1 \Uparrow \Delta_2}\ \mathsf{S}_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow D}{\Gamma \Uparrow \cdot \vdash D \Uparrow \cdot}\ \mathsf{S}_r$$

Since the order in which the negative introduction rules are applied does not matter (don't care nondeterminism), it is sometimes convenient to modify this proof system so that a negative, right-introduction rule is applied only when the left staging area is empty.

If one is only interested in using some negative connectives (i.e., t^-, $\wedge^-$, $\supset$, and $\forall$), then LJF can be simplified to the point where every two-sided sequent only needs to display at most one arrow symbol. In such a setting, it makes sense to call these two phases the *right* phase and the *left* phase.

4 Additional observations

I would argue that the adjective "focused" should be used only with proof systems: in particular, a formula and a logic are not focused. However, it make sense to say that an *occurrence* of a formula in the staging area of a $\Downarrow$ sequent is called a *focus* of that sequent.

Focusing is a way to take the micro-rules given by Gentzen and arrange them into macro or synthetic rules. In particular, a synthetic rule is the result of taking a (partial) derivation built using micro-rules with border sequents as its premises and conclusion, with no border sequents elsewhere, and then hiding all of the micro-rules. Cut-elimination can be an automatic consequence for synthetic inference rules built using focusing [13].

The style of focusing described here is sometimes called *strong* focusing: that is, the decide rule is not selected until the staging area of the $\Uparrow$-phase is empty. A weak focusing proof system would allow the decide rule to select a focus even if not all invertible rules have been applied. The notation given here does not immediately accommodate weak focusing proof rules.

A synthetic *inference rule* is *bipolar*: its internal structure is defined by both positive and negative phases. If we are to introduce a notion of synthetic *connectives*, we would probably insist that they are *monopolar*, i.e., composed of either only negative or only positive connectives.

Focusing has also been applied in natural deduction [2, 16, 17] and deep inference [3]. I have not considered how the notational conventions mentioned above might be applied in those settings.

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5 Additional observations

Some observations that appear as comments on the blog or occurred to me later.

27 October 2022: In my original post, I should have pointed out that Girard used this "inner/outer rather than left/right" distinction in his paper on *On the unity of logic* (APAL, 1993). In that paper, he had two zones on both the left and the right: the inner zones are the classical-maintenance zones, and the outer zones are the linear-maintenance zones. I switch these zones around in my writing: inner zones are linear, and outer zones are classical. I do this since linear resources (switches, registers, tokens, etc.) seem to be the center of the action, while the classical zone (the logic program) does not change much during proof search. The "center of action" means near the center of the sequent, which is the turnstile. Putting the linear zone next to the turnstile in (unfocus) sequents is similar to putting the $\Uparrow$ and $\Downarrow$ staging areas next to the turnstile in focused systems: the staging area is a linear maintenance zone as well.

Regarding other kinds of sequent arrows, I should mention that both Girard and Jean Gallier tried to introduce (in the early 1990s) a new turnstile symbol, where the foot was shorter and thicker, as a replacement for the sequent arrow (see Gallier's "On the Correspondence Between Proofs and Lambda-Terms," p. 10, (UPenn MS-CSE report MS-CIS-93-01). But, unfortunately, it doesn't seem that that symbol has caught on.

1 November 2022: Also, note that when working only with negative connectives, it is possible to avoid the $\Uparrow$ entirely since it is easy to tell when a formula on the right-hand side is stored or not: a right-hand side formula occurrence is stored if and only if it is atomic.