A range of object-level proof systems from focusing a linear logical framework

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Outline

1. Basics: linear logic and focused proofs
2. A word about object-logic / meta-logic
3. Assigning polarities to atoms
4. Specification of various object-level proof systems

Joint work with Vivek Nigam. Based on a paper in IJCAR 2008.

Review of Linear Logic

Literals are either atomic formulas or their negations.

We write $\neg B$ to denote the *negation normal form* of the formula B: use de Morgan dualities to push negation to have only atomic scope.

The connectives $\otimes$ and $\invamp$ and their units 1 and $\perp$ are *multiplicative*; the connectives $\oplus$ and $\&$ and their units 0 and $\top$ are *additive* connectives; $\forall$ and $\exists$ are (first-order) quantifiers; and $!$ and $?$ are the exponentials.

$B \equiv C$ to denotes the formula $(\neg B \invamp C) \& (\neg C \invamp B)$.

The formula B is *derived using theory* $\mathcal{X}$ if $\vdash B, ?\mathcal{X}$ is provable in linear logic.

LLF: the negative connectives

Negative connectives: $\bot$, $\parr$, $\top$, $\&$, $\forall$, $?$.

Their right introduction rules are *invertible*: *i.e.*, if the conclusion is provable the premises are provable.

All sequents use the up-arrow $\Uparrow$.

$$\frac{\vdash \Theta : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, \bot}\ [\bot] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G}\ [\parr] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F[c/x]}{\vdash \Theta : \Gamma \Uparrow L, \forall x\, F}\ [\forall]$$

$$\frac{}{\vdash \Theta : \Gamma \Uparrow L, \top}\ [\top] \quad \frac{\vdash \Theta : \Gamma \Uparrow L, F \quad \vdash \Theta : \Gamma \Uparrow L, G}{\vdash \Theta : \Gamma \Uparrow L, F \,\&\, G}\ [\&] \quad \frac{\vdash \Theta, F : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, ?F}\ [?]$$

Here, L is a list; Θ and Γ are multisets.

LLF: the positive connectives

Positive connectives: 1, $\otimes$, 0, $\oplus$, $\exists$, $!$.

Their right introduction rules are *not necessarily invertible*.

All sequents use the down-arrow $\Downarrow$.

$$\frac{}{\vdash \Theta :\Downarrow 1}\ [1] \qquad \frac{\vdash \Theta : \Gamma \Downarrow F \quad \vdash \Theta : \Gamma' \Downarrow G}{\vdash \Theta : \Gamma, \Gamma' \Downarrow F \otimes G}\ [\otimes] \qquad \frac{\vdash \Theta, F : \Gamma \Downarrow F[t/x]}{\vdash \Theta : \Gamma \Downarrow \exists x\, F}\ [\exists]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow F}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_l] \qquad \frac{\vdash \Theta : \Gamma \Downarrow G}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_r] \qquad \frac{\vdash \Theta :\Uparrow F}{\vdash \Theta :\Downarrow\, !\, F}\ [!]$$

LLF: structural rules

Each inference rule mentions both $\Uparrow$ and $\Downarrow$ but no logical connectives.

$$\frac{\vdash \Theta : \Gamma \Uparrow N}{\vdash \Theta : \Gamma \Downarrow N}\ [R\Downarrow] \quad \frac{\vdash \Theta : \Gamma, S \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, S}\ [R\Uparrow] \qquad \text{Release rules}$$

$$\frac{\vdash \Theta : \Gamma \Downarrow P}{\vdash \Theta : \Gamma, P \Uparrow}\ [D_1] \quad \frac{\vdash \Theta, P : \Gamma \Downarrow P}{\vdash \Theta, P : \Gamma \Uparrow}\ [D_2] \qquad \text{Decide rules}$$

Here, N is a negative formula, P is a positive formula, and S is a positive formula or a negative atom.

There are some choices to make in the design of these kind of rules. In particular, the *decide* rules D_1 and D_2 here provides a *single* focus: they could be extended to allow for *multiple* foci.

LLF: initial rules

We need literals to fit into one phase or the other.

Atoms can be classified as positive or negative *arbitrarily*. Negation flips the assigned polarity: *e.g.*, A pos and $A^\perp$ neg.

$$\frac{}{\vdash \Theta : K^\perp \Downarrow K}\ [I_1] \qquad\qquad \frac{}{\vdash \Theta, K^\perp :\Downarrow K}\ [I_2]$$

provided K is a positive literal.

Only half of the initial rules are present: if we encounter a negative atom, we don't check the context for its dual.

Completeness of Focusing [Andreoli 92]: Let formula B be provable in linear logic and assume some (arbitrary) polarization of the atoms. Then there is a focused proof of $\vdash :\Uparrow B$.

The roles of linear logic

Old story: Linear logic can be used to model resources: *e.g.*, multiset rewriting, Petri nets, side-effects, etc.

New story: Linear logic is used for structuring proofs.

Focusing in MALL without atoms is clear and fixed. When one adds atoms, exponentials, or fixed points, or moves to intuitionistic or classical logics, focusing is not fixed.

Polarity assignment to atomic formulas does not affect *provability* but it can have a huge impact on the *structure* of proofs.

> Something that proof theory cannot make canonical can be exploited by a computer scientist.

Think to first-order quantification, modal operators, and the $!, ?$ -exponentials themselves.

A (partial) defense of meta-logic

> The word 'meta-logic' should not be used in front of small children. - J.-Y. Girard

We do not mean any logical obscenity but a common reflex in mathematics and computer science:

> design (and implement) an *abstraction* that is used to obtain numerous *concrete* instances.

Our results in a nutshell

The abstraction: focused proofs in linear logic

Various concretion: proofs systems for classical & intuitionistic logics: sequent calculus, natural deduction, tableaux, free deduction, etc.

The gap: polarities assignments and trivial equivalences.

Intuitionistic meta-logics for natural deduction

Natural deduction inference rules such as

$$\frac{A \quad B}{A \wedge B} \qquad\qquad \frac{A \vee B \quad \begin{array}{c} A \\ \vdots \\ C \end{array} \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C}$$

Can be encoded as formulas in intuitionistic logic as:

$$\forall A \forall B\ [(\mathtt{pv}(A) \wedge \mathtt{pv}(B)) \supset \mathtt{pv}(A \wedge B)]$$

$$\forall A \forall B \forall C\ [(\mathtt{pv}(A \vee B) \wedge (\mathtt{pv}(A) \supset \mathtt{pv}(C)) \wedge (\mathtt{pv}(B) \supset \mathtt{pv}(C)) \supset \mathtt{pv}(C)]$$

Object-level connectives are black; meta-level connectives are red.
We have one meta-level predicate $\mathtt{pv}$.

Such encodes are standard in: Isabelle, λProlog, Twelf, Coq, etc.

Encoding the object-logic

We shall only consider two object-logics here: (first-order) intuitionistic and classical logics.

Most (object-level) proof systems mention (object-level) formulas in *two senses*. For example,

- Sequent calculus: left-hand-side, right-hand-side
- Natural deduction: hypothesis, conclusion
- Tableaux: positive or negative signed formulas

These two senses are represented as the two meta-level predicates $\lfloor \cdot \rfloor$ (left) and $\lceil \cdot \rceil$ (right), both of type $bool \to o$.

The two-sided, object-level sequent $B_1, \ldots, B_n \vdash C_1, \ldots, C_m$ as the one-sided, meta-level sequent $\vdash \lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil$.

Convention: $\lfloor \Gamma \rfloor$ denotes $\{\lfloor F \rfloor \mid F \in \Gamma\}$, etc.

The theory $\mathcal{L}$: introduction rules

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (\lfloor A \rfloor \parr \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil)$
$(\vee_L)$	$\lfloor A \vee B \rfloor^\perp \otimes (\lfloor A \rfloor \,\&\, \lfloor B \rfloor)$	$(\vee_R)$	$\lceil A \vee B \rceil^\perp \otimes (\lceil A \rceil \oplus \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^\perp \otimes \lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil$
$(\exists_L)$	$\lfloor \exists B \rfloor^\perp \otimes \forall x \lfloor Bx \rfloor$	$(\exists_R)$	$\lceil \exists B \rceil^\perp \otimes \lceil Bx \rceil$
$(\perp_L)$	$\lfloor \perp \rfloor^\perp$	(t_R)	$\lceil t \rceil^\perp \otimes \top$

[really the ? and existential closure of these formulas]

The meanings of the two senses for object-level connectives are supplied by these formulas.

The theory $\mathcal{L}$: structural and identity rules

$$
\begin{array}{llll}
(Id_1) & \lfloor B \rfloor^{\perp} \otimes \lceil B \rceil^{\perp} & (Id_2) & \lfloor B \rfloor \otimes \lceil B \rceil \\
(Str_L) & \lfloor B \rfloor^{\perp} \otimes ?\lfloor B \rfloor & (Str_R) & \lceil B \rceil^{\perp} \otimes ?\lceil B \rceil \\
(W_R) & \lceil C \rceil^{\perp} \otimes \perp & &
\end{array}
$$

Specification of the identity rules (*e.g.*, cut and initial), the structural rules (weakening and contraction), and just weakening (on the right).

Note: Mix would correspond to the formula $\perp \otimes \perp$, *i.e.*, the smallest positive formula B of MALL (without atoms) such that neither $\vdash B$ nor $\vdash B^{\perp}$.

Proving dualities

The Id_1 and Id_2 formulas can prove the duality of the $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ predicates: in particular, one can prove in linear logic that

$$\vdash \forall B(\lceil B \rceil \equiv \lfloor B \rfloor^{\perp}) \,\&\, \forall B(\lfloor B \rfloor \equiv \lceil B \rceil^{\perp}), Id_1, Id_2$$

Similarly, the formulas Str_L and Str_R allow us to prove the equivalences $\lfloor B \rfloor \equiv ?\lfloor B \rfloor$ and $\lceil B \rceil \equiv ?\lceil B \rceil$.

Adequacy level of encodings

Following Girard, we identify three "levels of adequacy" in comparing two proof systems.

Level 0 / Relative completeness: the two systems have the same theorems.

Level -1 / Full completeness of proofs: The proofs of a formula in one proof system are in one-to-one correspondence with proofs in the other proof system.

Level -2 / Full completeness of derivations: The derivations (*i.e.*, open proofs) in one system are in one-to-one correspondence with the other proof system.

We aim for Level -2 encodings; we sometimes settle for Level -1. Here, all Level 0 proofs are trivial using linear logic equivalences.

Annotations of atoms

Before we can build focused proofs, we need to polarize the atoms: which literal belongs to the positive phase and which belong to the negative phase?

We shall provide global and fixed polarity assignments.

Surely, more flexible assignment is possible:

- the polarity of atoms can even change within a single proof,
- the polarity can depend on an atoms occurrences in a proof.

The impact of different bias assignments

Consider the specification Γ of the Fibonacci series:

$$\{\mathit{fib}(0,0)^\perp, \mathit{fib}(1,1)^\perp, \exists n, f, f'[\mathit{fib}(n,f) \otimes \mathit{fib}(n+1,f') \otimes \mathit{fib}(n+2,f+f')^\perp]\}$$

Consider focused proofs of

$$\frac{\vdash \Gamma; \mathit{fib}(n, f_n) \Downarrow \exists n, f, f'[\mathit{fib}(n,f) \otimes \mathit{fib}(n+1,f') \otimes \mathit{fib}(n+2,f+f')^\perp]}{\vdash \Gamma; \mathit{fib}(n, f_n) \Uparrow \cdot}\ D_2$$

If all atoms are negative, there there is a unique focused proof of size *exponential* in n (goal-directed, backchaining).

If all atoms are positive, then there are infinitely many proofs and the smallest one is of *linear* size in n (program-directed, forward-chaining).

Changes in the polarization of atoms can have important consequences on proof structure.

Sequent Calculus

Proposition. If we fix the polarity of all (meta-level) atoms to be negative, then

1) $\Gamma \vdash_{lm} C$ iff $\vdash \mathcal{L}_{lm}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$, *2)* $\Gamma \vdash_{lj} C$ iff $\vdash \mathcal{L}_{lj}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$, and
3) $\Gamma \vdash_{lk} \Delta$ iff $\vdash \mathcal{L}_{lk}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil : \Uparrow$ where

$$\mathcal{L}_{lk} = \mathcal{L} \cup \{Id_1, Id_2, Str_L, Str_R\},$$

$$\mathcal{L}_{lm} = \mathcal{L} \cup \{Id_1, Id_2, Str_L, \Rightarrow'_L\} \setminus \{\bot_L, \Rightarrow_L\},$$

$$\mathcal{L}_{lj} = \mathcal{L} \cup \{Id_1, Id_2, Str_L, \Rightarrow'_L, W_R\} \setminus \{\Rightarrow_L\}, \text{ and}$$

$$\Rightarrow'_L \text{ is } ?\exists A \exists B [\lfloor A \Rightarrow B \rfloor^\perp \otimes (! \lceil A \rceil \otimes \lfloor B \rfloor)]$$

This Level 0 adequacy result actually can be strengthen to Level -2. The Level -2 result requires the ! in the encoding of the implication left-introduction rule for both minimal and intuitionistic logics. (Level -1 does not need the !.)

The simulation of an inference rule

Let $F \in \mathcal{L}_{lm}$ be $\exists A \exists B[\lfloor A \Rightarrow B \rfloor^{\perp} \otimes (!\lceil A \rceil \otimes \lfloor B \rfloor)]$ and let $\mathcal{K}$ denote the set $\mathcal{L}_{lm}, \lfloor \Gamma \rfloor$.

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\,[I_2] \qquad \dfrac{\dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\,[!, R\Uparrow] \quad \dfrac{\vdash \mathcal{K} : \lfloor B \rfloor, \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor B \rfloor}\,[R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \rceil \otimes \lfloor B \rfloor}\,[\otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}\,[2 \times \exists, \otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\,[D_2]$$

Thus, $A \Rightarrow B \in \Gamma$ and the following object-level inference rule is simulated.

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$$

Cut-free sequent calculus

Remove the formula Id_2 from the sets $\mathcal{L}_{lm}$, $\mathcal{L}_{lj}$, and $\mathcal{L}_{lk}$, obtaining the sets $\mathcal{L}^f_{lm}$, $\mathcal{L}^f_{lj}$, and $\mathcal{L}^f_{lk}$, respectively,

Let $\vdash^f_{lm}$, $\vdash^f_{lj}$, and $\vdash^f_{lk}$ be the cut-free provability judgments.

Proposition. Again, assume that all meta-level atomic formulas are given a negative polarity. Then

$$\begin{array}{l} \textit{1)}\ \Gamma \vdash^f_{lm} C \text{ iff } \vdash \mathcal{L}^f_{lm}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow \\ \textit{2)}\ \Gamma \vdash^f_{lj} C \text{ iff } \vdash \mathcal{L}^f_{lj}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow \\ \textit{3)}\ \Gamma \vdash^f_{lk} \Delta \text{ iff } \vdash \mathcal{L}^f_{lk}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow \end{array}$$

This Level 0 statement can be sharpened to Level -2.

Natural Deduction

$$\frac{}{\Gamma, A \vdash_{nd} A \downarrow}\ [\mathrm{Ax}] \qquad \frac{\Gamma \vdash_{nd} F \uparrow \quad \Gamma \vdash_{nd} G \uparrow}{\Gamma \vdash_{nd} F \wedge G \uparrow}\ [\wedge I] \qquad \frac{\Gamma \vdash_{nd} F \wedge G \downarrow}{\Gamma \vdash_{nd} F \downarrow}\ [\wedge E]$$

$$\frac{\Gamma \vdash_{nd} A_i \uparrow}{\Gamma \vdash_{nd} A_1 \vee A_2 \uparrow}\ [\vee I] \qquad \frac{\Gamma \vdash_{nd} A \vee B \downarrow \ \ \Gamma, A \vdash_{nd} C \uparrow (\downarrow)\ \Gamma, B \vdash_{nd} C \uparrow (\downarrow)}{\Gamma \vdash_{nd} C \uparrow (\downarrow)}\ [\vee E]$$

$$\frac{\Gamma, A \vdash_{nd} B \uparrow}{\Gamma \vdash_{nd} A \Rightarrow B \uparrow}\ [\Rightarrow I] \qquad \frac{\Gamma \vdash_{nd} A \Rightarrow B \downarrow \quad \Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} B \downarrow}\ [\Rightarrow E] \qquad \frac{}{\Gamma \vdash_{nd} t \uparrow}\ [tI]$$

$$\frac{\Gamma \vdash_{nd} A\{c/x\} \uparrow}{\Gamma \vdash_{nd} \forall x\, A \uparrow}\ [\forall I] \qquad \frac{\Gamma \vdash_{nd} \forall x\, A \downarrow}{\Gamma \vdash_{nd} A\{t/x\} \downarrow}\ [\forall E] \qquad \frac{\Gamma \vdash_{nd} A \downarrow}{\Gamma \vdash_{nd} A \uparrow}\ [\mathrm{M}] \qquad \frac{\Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} A \downarrow}\ [\mathrm{S}]$$

$$\frac{\Gamma \vdash_{nd} \exists x\, A \downarrow \quad \Gamma, A\{a/x\} \vdash_{nd} C \uparrow (\downarrow)}{\Gamma \vdash_{nd} C \uparrow (\downarrow)}\ [\exists E] \qquad \frac{\Gamma \vdash_{nd} A\{t/x\} \uparrow}{\Gamma \vdash_{nd} \exists x\, A \uparrow}\ [\exists I]$$

Rules for minimal natural deduction - NM. In $[\vee L]$, $i \in \{1, 2\}$.

Natural deduction and normal proofs

The sequent $\Gamma \vdash_{nd} C \uparrow$ is encoded as $\vdash \Sigma, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$. These are obtained from the conclusion by a derivation (from bottom-up) where C is not the major premise of an elimination rule.

The sequent $\Gamma \vdash_{nd} C \downarrow$ is encoded as $\vdash \Sigma, \lfloor\Gamma\rfloor : \lfloor C\rfloor^{\perp} \Uparrow$. These are obtained from the set of hypotheses by a derivation (from top-down) where C is extracted from the major premise of an elimination rule.

These two types of derivations meet either with a *match* rule M or with a *switch* rule S.

A natural deduction proof is *normal* if it is without the switch rule.

The judgment $\vdash_{nm}$ denotes the existence of a natural deduction proof and $\vdash_{nm}^{n}$ denotes the existence of a normal proof.

Correctness of encoding

Proposition. Assign all $\lceil \cdot \rceil$ atomic formulas negative polarity and all $\lfloor \cdot \rfloor$ atomic formulas positive polarity. Then

1) $\Gamma \vdash_{nm} C\uparrow$ iff $\vdash \mathcal{L}_{lm}, \lfloor\Gamma\rfloor : \lceil C \rceil \Uparrow$ *2)* $\Gamma \vdash^{n}_{nm} C\uparrow$ iff $\vdash \mathcal{L}^{f}_{lm}, \lfloor\Gamma\rfloor : \lceil C \rceil \Uparrow$

3) $\Gamma \vdash^{n}_{nm} C\downarrow$ iff $\vdash \mathcal{L}^{f}_{lm}, \lfloor\Gamma\rfloor : \lfloor C \rfloor^{\perp} \Uparrow$

A corresponding Level -1 statement can also be proved.

Since the polarity assignment in a focused system does not affect provability, we obtain for free the following (Level 0) equivalences between LM and NM.

Corollary.

$$\Gamma \vdash_{lm} C \text{ iff } \Gamma \vdash_{nm} C \quad \text{and} \quad \Gamma \vdash^{f}_{lm} C \text{ iff } \Gamma \vdash^{n}_{nm} C.$$

ID_2 changes from "cut" to "switch"

$$
\dfrac{\dfrac{}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor}\ [I_1] \quad \dfrac{\vdash \Sigma, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow}{\vdash \Sigma, \lfloor \Gamma \rfloor :\Downarrow \lceil C \rceil}\ [R \Downarrow, R \Uparrow]}{\dfrac{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor \otimes \lceil C \rceil}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Uparrow}\ [D_2, \exists]}\ [\otimes]
$$

We skip the natural deduction treatment of negation in intuitionistic and classical logics: it is not so natural.

ND with General Elimination Rules

$$\frac{\Gamma \vdash_{ge} A \vee B \quad \Gamma, A \vdash_{ge} C \quad \Gamma, B \vdash_{ge} C}{\Gamma \vdash_{ge} C} \qquad \frac{\Gamma \vdash_{ge} A \wedge B \quad \Gamma, A, B \vdash_{ge} C}{\Gamma \vdash_{ge} C}$$

$$\frac{\Gamma \vdash_{ge} A \Rightarrow B \quad \Gamma \vdash_{ge} A \quad \Gamma, B \vdash_{ge} C}{\Gamma \vdash_{ge} C} \qquad \frac{\Gamma \vdash_{ge} \forall x\, A \quad \Gamma, A\{t/x\} \vdash_{ge} C}{\Gamma \vdash_{ge} C}$$

Let $\mathcal{L}_{ge}$ result from changing $\mathcal{L}$ by

- replacing all occurrences of $\lfloor B \rfloor^{\perp}$ by $\lceil B \rceil$ (logically equivalent given the two identities rules), and
- replacing the additive version of $\wedge_L$ with a multiplicative version (equivalence in the given the Str_L rule).

Proposition. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash_{ge} C$ iff $\vdash \mathcal{L}_{ge}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$.

The corresponding Level -1 statement can also be proved.

Corollary. $\Gamma \vdash_{ge} C$ iff $\Gamma \vdash_{lm} C$.

Free Deduction

Use general elimination rules and the following style of general introduction rules.

$$\frac{\Gamma, A \vee B \vdash_{fd} \Delta \quad \Gamma \vdash_{fd} \Delta, A}{\Gamma \vdash_{fd} \Delta} [\vee GI] \qquad \frac{\Gamma, A \Rightarrow B \vdash_{fd} \Delta \quad \Gamma, A \vdash_{fd} \Delta, B}{\Gamma \vdash_{fd} \Delta} [\Rightarrow GI]$$

$$\frac{\Gamma, A \wedge B \vdash_{fd} \Delta \quad \Gamma \vdash_{fd} \Delta, A \quad \Gamma \vdash_{fd} \Delta, B}{\Gamma \vdash_{fd} \Delta} [\wedge GI]$$

$$\frac{\Gamma, \neg A \vdash_{fd} \Delta \quad \Gamma, A \vdash_{fd} \Delta}{\Gamma \vdash_{fd} \Delta} [\neg GI_1] \qquad \frac{\Gamma \vdash_{fd} \Delta, \neg A \quad \Gamma \vdash_{fd} \Delta, A}{\Gamma \vdash_{fd} \Delta} [\neg GI_2]$$

Encoding: roughly, take the classical logic system $\mathcal{L}_{lk} = \mathcal{L} \cup \{Id_1, Id_2, Str_L, Str_R\}$ and replace $\lfloor \cdot \rfloor^{\perp}$ with $\lceil \cdot \rceil$ and $\lceil \cdot \rceil^{\perp}$ with $\lfloor \cdot \rfloor$ and assign all meta-level atoms a negative polarity.

A Level -2 encoding can be proved for free deduction.

Polarity and focusing had appeared before

Using positive literals to terminate a "synthetic" connective is related to at least the following two system.

- Parigot's notion of "killing" different premises in free deduction as a way to recover sequent calculus and natural deduction corresponds to polarity assignment on (meta-level) atomic judgments.
- Negri & Plato defined normal forms in "generalized natural deduction" by requiring that major premises of elimination rules are immediately present as assumptions.

Other proof systems

The full version of the paper as treats

- the KE tableaux of D'Agostino and Mondadori, and
- a proof system of Smullyan's with many axioms and with cut as the only inference rule.

Conclusions

We have worked with essentially one "definition" of the two senses of a logical connective (here, $\mathcal{L}$).

We allowed either changes in polarity assignment to atoms or replacing specifications with logically equivalent formulas.

This simple meta-level tuning accounts faithfully for a number of (object-level) proof systems.

Classical systems can usually be encoded at Level -2.

Intuitionistic systems encodings were often only achieving Level -1.

There is a conflict between uses of exponentials to improve adequacy of encodings and the focusing discipline that is at the heart of getting adequacy results in the first place.

What about other logics? Hypersequents? Implementations.