Proof of the Millin series

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^k}} = \frac{7-\sqrt{5}}{2}$$

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First, show that

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1-x^{2^{k+1}}} = \frac{x}{1-x}$$

Remember that

$$\frac{x}{1-x} = x^1 + x^2 + x^3 + \cdots = \sum_{k=1}^{\infty} x^k$$

Expand $\dfrac{x^{2^k}}{1-x^{2^{k+1}}}$ for several values of k

$k=0$ $\quad \dfrac{x}{1-x^2} = x + x^3 + x^5 + \cdots$

$k=1$ $\quad \dfrac{x^2}{1-x^4} = x^2 + x^6 + x^{10} + \cdots$

$k=2$ $\quad \dfrac{x^4}{1-x^8} = x^4 + x^{12} + x^{20} + \cdots$

$k=3$ $\quad \dfrac{x^8}{1-x^{16}} = x^8 + x^{24} + x^{40} + \cdots$

For general k, the series contains x^j where j is 2^k times an odd number.

Every $n \in \mathbb{N}$ $(n>0)$ can be written uniquely as $2^k m$ where $k \in \mathbb{N}$, m is odd.

Hence, for every $n \in \mathbb{N}$, x^n occurs exactly once in

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1-x^{2^{k+1}}}$$ which means it

is equal to $$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n$$

The following variation is needed next.

$$\sum_{k=1}^{\infty} \frac{x^{2^k}}{1-x^{2^{k+1}}} = \frac{x}{1-x} - \frac{x}{1-x^2} = \frac{x(1+x)}{1-x^2} - \frac{x}{1-x^2}$$

$$= \frac{x^2}{1-x^2}$$

Recall: Binet formula $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ where

α & β are roots of $y^2 = y+1$ $(n \geq 0)$

Recall: $\alpha\beta = -1$

$\alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$

$\alpha = \dfrac{1+\sqrt{5}}{2}$ $\beta = \dfrac{1-\sqrt{5}}{2}$

Now we just calculate.

$$\sum_{k=0}^{\infty} \frac{1}{F_{2^k}} = \sum_{k=0}^{\infty} \frac{\alpha - \beta}{\alpha^{2^k} - \beta^{2^k}} = \sqrt{5} \sum_{k=0}^{\infty} \frac{1}{\alpha^{2^k} - \beta^{2^k}}$$

Since $\alpha = \frac{-1}{\beta}$,

$$\sum_{k=0}^{\infty} \frac{1}{\left(\frac{-1}{\beta}\right)^{2^k} - \beta^{2^k}} = \frac{1}{-\frac{1}{\beta} - \beta} + \sum_{k=1}^{\infty} \frac{1}{\left(-\frac{1}{\beta}\right)^{2^k} - \beta^{2^k}}$$

$$= \frac{\sqrt{5}}{5} + \sum_{k=1}^{\infty} \frac{\beta^{2^k}}{1 - \beta^{2^{k+1}}} = \frac{\sqrt{5}}{5} + \frac{\beta^2}{1 - \beta^2}$$

$$= \frac{\sqrt{5}}{5} + \frac{\beta + 1}{-\beta} = \frac{\sqrt{5}}{5} + \alpha(\beta + 1)$$

Thus $$\sum_{k=0} \frac{1}{F_{2^k}} = \sqrt{5}\left(\frac{\sqrt{5}}{5} + \alpha\beta^2\right) =$$

$$= 1 + \sqrt{5}\,(-\beta) = \frac{2}{2} + \frac{\sqrt{5}\,(-1 + \sqrt{5})}{2}$$

$$= \frac{7 - \sqrt{5}}{2}$$

QED