Some bounds on the computational power of Piecewise Constant Derivative systems.

Olivier Bournez*
LIP, ENS-Lyon,
46, Alle d'Italie
69364 Lyon Cedex 07
FRANCE
e-mail: Olivier Bournez@ens-lyon.fr

Abstract

We study the computational power of Piecewise Constant Derivative (PCD) systems. PCD systems are dynamical systems defined by a piecewise constant differential equation and can be considered as computational machines working on a continuous space with a continuous time. We show that the computation time of these machines can be measured either as a discrete value, called discrete time, or as a continuous value, called continuous time. We relate the two notions of time for general PCD systems. We prove that general PCD systems are equivalent to Turing machines and linear machines in finite discrete time. We prove that the languages recognized by purely rational PCD systems in dimension d in finite continuous time are precisely the languages of the $d-2^{th}$ level of the arithmetical hierarchy. Hence the reachability problem of purely rational PCD systems of dimension d in finite continuous time is Σ_{d-2}-complete.

1 Introduction

There has been recently an increasing interest in the community of control and verification theory about hybrid systems. A hybrid system is a system that combines discrete and continuous dynamics. Several models have been proposed in literature. Hybrid systems can also be considered as computational machines [1, 2, 3, 8, 9]: they can be seen either as machines working on a continuous space with a discrete time or as machines working on a continuous space with a continuous time.

Several theoretical models of machines working on a continuous space with a discrete time are known: in [5], Blum, Shub and Smale introduced the real Turing machine. Many papers have been devoted to the study of computability and complexity in this model. See [13] for an up-to-date survey. In [14], Meer introduced a restricted class of real Turing machines called the linear machines. Meer proved that $P \neq NP$ in this class of systems. In [11, 12], Koiran characterized the boolean part of the languages recognized by linear machines as $P/poly$ in polynomial time and as unbounded in exponential time. In [1, 2, 3] the attention is focused on a very simple type of hybrid systems: Piecewise Constant Derivative Systems (PCD systems) are dynamical systems defined by a piecewise constant differential equation. It is shown that the reachability problem for PCD systems is decidable in dimension $d = 2$ and undecidable for dimensions $d \geq 3$ [1, 3]. In [8], the computational power of Piecewise Constant Derivative systems is characterized as $P/poly$ in polynomial discrete time, and as unbounded in exponential discrete time.

However, hybrid systems are very interesting models since they provide natural computational machines working on a continuous space and with a continuous time. Studies of this type of machines are only beginning. In [15], Moore proposed a recursion theory for computations on the reals in continuous time . Recently, Asarin and Maler [2] showed, using Zeno's paradox, that every set of the arithmetical hierarchy can be recognized in finite continuous time by a PCD system of finite dimension: every set of

*Support by Esprit Project 8556. NeuroColt is acknowledged.

the arithmetical hierarchy in $\Sigma_k \cup \Pi_k$ can be recognized by a rational PCD system of dimension $5k+1$. Unfortunately, no precise characterization of the sets recognizable by PCD systems is given in [2]: this paper improves the results of Asarin and Maler and shows that the sets recognized by purely rational PCD systems in dimension d in finite continuous time are precisely the sets of the $d-2^{th}$ level of the arithmetical hierarchy. For discrete time, we prove that PCD systems of dimension d are equivalent to linear machines of dimension $d-1$.

Section 2 is devoted to general definitions: PCD systems, computations on PCD systems, discrete and continuous time and linear machines. In section 3, we study the links that exist between the discrete and the continuous time: in dimension d, a finite continuous time can correspond to any discrete time whose ordinal is bounded by ω^{d-1}. Section 4 proves that PCD systems of dimension d are equivalent to linear machines of dimension $d-1$. In section 5 we focus on computations in finite continuous time: we improve the bounds given by Asarin and Maler: any arithmetical set in Σ_k can be recognized in dimension $2+k$ by a purely rational PCD system. Then, we prove that this bound is optimal. No other set can be recognized in that dimension by a purely rational PCD system. Hence, we get a full characterization of the computational power of purely rational PCD systems: $\Sigma_k = \textit{dimension } k+2$.

2 Definitions

2.1 PCD systems

A convex polyhedron of $\mathbb{R}^d$ is any finite intersection of open or closed half spaces of $\mathbb{R}^d$. A polyhedron of $\mathbb{R}^d$ is a finite union of convex polyhedral of $\mathbb{R}^d$. In particular, a polyhedron may be unbounded or flat. For $V \subset \mathbb{R}^d$, we denote by $\overline{V}$ the topological closure of V. We denote by d some distance of $\mathbb{R}^d$.

Definition 2.1 (PCD System)
- *A* dynamical system *is a pair $\mathcal{H} = (X, f)$ where $X = \mathbb{R}^d$ and f is a function from X to X. X is called the* space *and d is called the* dimension *of $\mathcal{H}$. A* trajectory *of $\mathcal{H}$* starting from x_0 *is a continuous solution to the differential equation $\dot{x}_d = f(x)$, with initial condition x_0, where $\dot{x}_d$ denotes the right derivative: that is $\Phi : D \subset \mathbb{R}^+ \to X$ where D is an interval of $\mathbb{R}^+$ containing 0, $\Phi(0) = x_0$, and $\forall t \in D, \dot{\Phi}_d(t) = f(\Phi(t))$. Trajectory Φ is said to* continue for ever *if $D = \mathbb{R}^+$.*
- *A* piecewise constant derivative (PCD) system *[2, 3] is a dynamical system $\mathcal{H} = (X = \mathbb{R}^d, f)$ where the range of f is a finite set $C \subset X$, such that for any $c \in C$ (c is called a* slope*) $f^{-1}(c)$ is a finite union of convex polyhedral sets (called* regions*).*

In other words a PCD system consists in partitioning the space into convex polyhedral sets, called *regions*, and assigning a constant derivative c, called *slope* to all the points sharing the same region. The trajectories of such systems are broken lines with the breakpoints occurring on the boundaries of the regions [2]. See figure 1. The *signature* of a trajectory is the sequence of the regions that are crossed by the trajectory.

Definition 2.2 (Rational and Purely rational PCD systems)
- *A PCD system is said to be* rational *if all the slopes as well as all the polyhedral regions can be described using only rational coefficients. A* real *PCD system is any PCD system: that is, any rational or non-rational PCD system.*
- *A PCD system is called* purely rational*, if in addition, for all trajectories Φ starting from a rational point, each time Φ enters a region in a point x, necessarily x has rational coordinates.*

Some comments are in order: one must understand that a trajectory Φ can enter a region either by a discrete transition or by converging to a point of the region: see figure 3. Thus, in other words, in a purely rational PCD system any converging process converges to a point with rational coordinates. Note that one can construct a rational PCD system of dimension 5 that is not purely rational [6].

We can say some words on the existence of trajectories in a PCD system: let $x_0 \in X$. We say that x_0 is *trajectory well-defined* if there exists a $\epsilon > 0$ such that $f(x) = f(x_0)$ for all $x \in [x_0, x_0 + \epsilon * f(x_0)]$. It is clear that, for any $x_0 \in X$, there exists a trajectory starting from x_0 iff x_0 is trajectory well-defined. Given a rational PCD system $\mathcal{H}$, one can effectively compute the set $NoEvolution(\mathcal{H})$ of the points of X that are not trajectory well-defined. Observe that a trajectory can continue for ever iff it does not reach $NoEvolution(\mathcal{H})$.

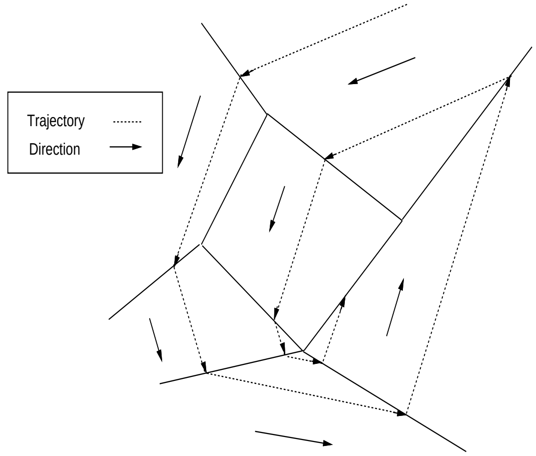

Figure 1: A PCD system in dimension 2.

Let Φ be a trajectory. Assume that $(t_i)i \in I$ is a sequence of reals where $\Phi(t_i) = x_i$ is defined for all i. Denote $t^* = sup_{i \in I} t_i$. Observe that if Φ is defined on $D = [0, t^*)$, Φ can be extended to a continuous function defined on all of $[0, t^*]$ (by a well known result of real analysis, a continuous function on an open interval with a bounded right derivative on this interval can be extended to a continuous function on the cloture of the interval). Hence, for any sequence $(t_i)i \in I$ of reals, we can assume that $\Phi(sup_i t_i) = x^*$ for some x^*. We will use implicitly this fact several times in this paper to prove our assertions. Note that from the continuity of Φ we have also $x^* = lim_i x_i$.

2.2 Computations on PCD systems

Definition 2.3 (Computation [2])

- *Let $\mathcal{H} = (X, f)$ be a PCD system of dimension d. Let $I = [0, 1]$ and let $r : \mathbb{N} \to I$ be an injective coding function*[1] *, let x^1, x^0 be two distinct points of $\mathbb{R}^d$. A computation of system $\hat{H} = (\mathbb{R}^d, f, r, I, x^1, x^0)$ on input $n \in \mathbb{N}$ is a trajectory of $\mathcal{H} = (X, f)$ starting from $(r(n), 0, \ldots, 0)$ that can continue forever. The computation is accepting if the trajectory eventually reaches x^1, and rejecting if it reaches x^0. It is assumed that the derivatives at x^1 and x^0 are zero.*
- *Language $L \subset \mathbb{N}$ is semi-recognized by $\hat{H}$ if, for every $n \in \mathbb{N}$, there is a computation on input n and the computation is accepting iff $n \in L$. L is said to be (fully-)recognized by $\hat{H}$ when, in addition, this trajectory reaches x^0 iff $n \notin L$.*

2.3 Time and PCD systems

Two different notions of time are distinguished in this paper:

Definition 2.4 (T_Φ) *Let $\Phi(.)$ be a trajectory of PCD system $\mathcal{H} = (X, f)$. We denote by T_Φ the subset of the domain of Φ defined by $T_\Phi = \{t/\Phi(t)$ enters a region at time $t\}$.*

Lemma 2.1 *For any trajectory Φ, T_Φ is a well-ordered set: T_Φ does not contain any infinite decreasing subsequence.*

Proof: Assume that $(t_n)_n$ is an infinite decreasing subsequence. Denote $t_\infty = \inf_n(t_n)$. Since $\Phi(t_\infty)$ must be a point where the trajectory is well-defined, there must exist ϵ such that Φ does not change of region at a time in $(t_\infty, t_\infty + \epsilon)$. This is a contradiction, since t_∞ must be the limit of the $(t_n)_n$.

□

We are now ready to define:

[1]We assume r to be a "very easily computable" encoding function: we do no want to give a formal definition of what it means. But using a function r that would encode all the solving power in the coding function has no interest for our discussion.

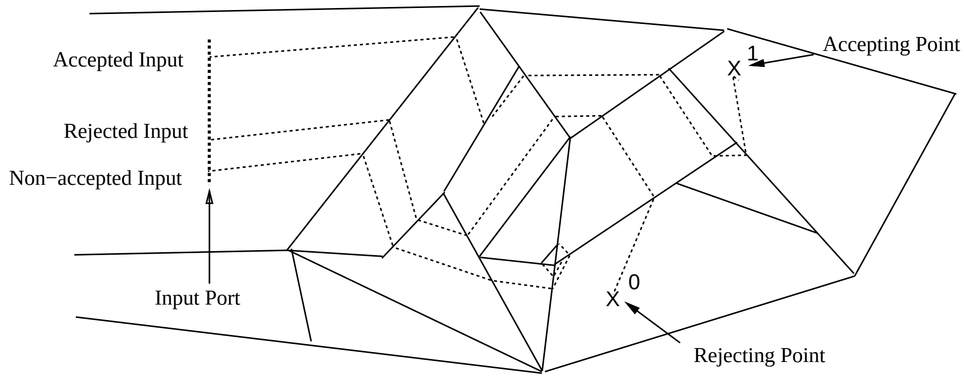

Figure 2: Some examples of computations by a PCD system.

Definition 2.5 (Continuous and discrete time) *Let $\Phi_n : \mathbb{R}^+ \to X$ be an accepting computation on input $n \in E$.*

- *The* continuous time $T_{cont}(n)$ *of the computation is* $T_{cont} = \min\{t \in \mathbb{R}^+ / \Phi_n(t) = x^1\}$
- *The* discrete time $T_{discr}(n)$ *of the computation is defined as the order type of well ordered set T_{Φ_n} (= the ordinal corresponding to T_{Φ_n}).*

Language $L \subset \mathbb{N}$ is said to be recognized in polynomial (discrete) time, if L is recognized in discrete time $T_{discr}(n)$ such that $T_{discr}(n)$ is polynomial in the bit size needed to represent integer $n \in \mathbb{N}$.

Note that Zeno's paradox appears: to a continuous finite time can correspond a transfinite discrete time: see figure 3.

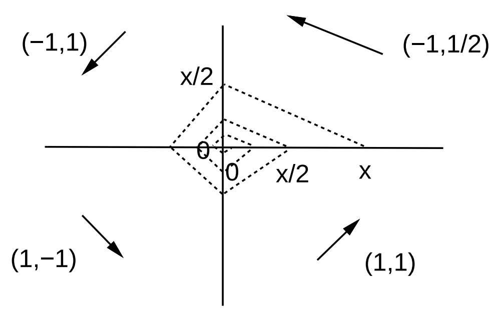

Figure 3: Zeno's paradox: at finite continuous time $5x = 2.5(x + x/2 + x/4 + \ldots)$ the trajectory is in $(0, 0)$, but it takes a transfinite discrete time ω to reach this point.

2.4 Linear machines

We use in this paper a restriction of the real Turing machine defined in [5]: multiplications between variables are forbidden. These machines are called *linear machines* and were first introduced in [14]. We will only consider in this paper finite dimensional linear machines with registers that stay in $[-1, 1]$. Note that the notion of polynomial time of [5] has no meaning in our context since we do not deal with inputs on $\mathbb{R}^\infty$ but with inputs on $\mathbb{R}^k$ for a given fixed k.

Definition 2.6 (Linear machine [14]) *A (bounded finite dimensional)* linear machine of dimension *d is a finite control part machine M with a finite number d of real registers $x_1, x_2, \ldots, x_d$ that can contain any real in $[-1, 1]$ in unbounded precision. M has also a finite number of constants $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. The operations of type $x_i := x_j + x_k, x_i := -x_j, x_i = \lambda_j * x_k, x_i := \lambda_j$ and the tests of type $x_i = x_j, x_i < x_j$ are considered as admissible and are executed in time 1, independently of the bit size of the arguments.*

A program of M is made of all the usual instructions of a RAM-machine (if,while,goto,...) with the addition of the above-mentioned admissible operations. M starts a computation with the input n in its first real register x_1, evolves according to its program, and stops when it reaches a final state. The computation time is defined as the number of operations done in the computation, independently of the bit-size of the operands on which the operations were applied.

The computational power of linear machines has been investigated in [11, 12]. It has been proved that the discrete languages recognized in polynomial time are precisely the languages belonging to the complexity class *P/poly*, and that any discrete language $L \subset \mathbb{N}$ can be recognized in exponential time. For a definition of *P/poly* see [4]. Note that linear machines with rational constants are equivalent to Turing machines.

2.5 Arithmetical hierarchy

We recall:

Definition 2.7 (Arithmetical hierarchy [18]) *The classes $\Sigma_k, \Pi_k, \Delta_k$, for $k \in \mathbb{N}$, are defined inductively by:*

- *Σ_0 is the class of the languages that are recursive.*
- *For $k \geq 1$, Σ_k is the class of the languages that are recursively enumerable in a set in Σ_{k-1} (that is semi-recognized by a Turing machine with an oracle in Σ_{k-1})*
- *For $k \in \mathbb{N}$, Π_k is defined as the class of languages whose complement are in Σ_k, and Δ_k is defined as $\Delta_k = \Pi_k \cap \Sigma_k$.*

The arithmetical hierarchy is well understood: the hierarchy is strict and complete problems are known for each of the Σ and Π levels [17, 18].

Several different characterizations of the arithmetical sets are known: see [18, 17]. In particular we will assume the reader familiar with Tarski–Kuratowski computations: assume a first order formula F, over some recursive predicates, characterizing the elements of a set $S \subset \mathbb{N}$, is given. Then S is in the arithmetical hierarchy and the Tarski–Kuratowski algorithm on formula F returns a level of the arithmetical hierarchy containing S: see [17, 18] for the full details.

Following standard notations, we fix a bijective recursive encoding of $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$: we denote by $< n, m >$ the integer encoding integers n, m [17, 18].

3 On the links between continuous and discrete time

We show in this section that, to a finite continuous time T_{cont} can correspond a transfinite discrete time T_{discr}, but for any dimension d, the ordinal of T_{discr} is less than ω^{d-1}.

3.1 Case $d = 2$

In [3], the authors mentioned that their results can be generalized to a more general class of dynamical systems: we will use later this fact.

Definition 3.1 *A* planar PCD like system *is a dynamical system $\mathcal{H} = (X = \mathbb{R}^2, f)$ where f is bounded on X and the following property holds: any straight line of $\mathbb{R}^2$ can be divided into finitely many segments, each of which can be traversed by any trajectory in at most one direction [3].*

Of course, a planar PCD system is a PCD like system. From [3] we get:

Theorem 3.1 ([3])

- *Let $\mathcal{H} = (X, f)$ be a planar PCD like system. Then, every trajectory of $\mathcal{H}$ is ultimately either a contracting or an expanding spiral, or cyclic.*
- *Let $\mathcal{H} = (X, f)$ be any planar PCD system. Then every trajectory of $\mathcal{H}$ has an ultimately periodic signature, i.e, a signature of the form $F_1, F_2, \ldots, F_i, (F_{i+1}, \ldots, F_{i+j})^\omega$ for some finite i, j, where j is at most the number of regions.*

We are ready now to prove the result for $d = 2$.

Corollary 3.1 *Let $\mathcal{H} = (X, f)$ be a real PCD system in dimension $d = 2$. Let Φ be a trajectory of $\mathcal{H}$ of finite continuous time T_{cont}.*

- *the discrete time T_{discr} of Φ is necessarily such that $T_{discr} \leq \omega$.*
- *If $T_{discr} = \omega$, then Φ is ultimately a spiral contracting to a point at the intersection of at least three edges.*

Proof: First point is nothing but theorem 3.1. For second point, observe that any trajectory has an ultimately periodic signature $(F_{i+1}, \dots, F_{i+j})^\omega$. Since T_{cont} is finite, the time to go from F_i to F_{i+1} must converge to 0. As a consequence Φ must be ultimately a spiral contracting to a point x^* and all the $(F_i)_{i \in \{i+1, \dots, i+j\}}$ must intersect and must contain x^*.

□

The bound is sharp: figure 3 gives an example of a purely rational planar PCD system with a finite continuous time trajectory of ordinal ω.

3.2 Case $d \geq 3$

A PCD system $\mathcal{H}$ of dimension d can be made of a PCD system $\mathcal{H}'$ of dimension $d' < d$ embedded into dimension d. The following proposition shows that, if it is the case, we can restrict our attention to $\mathcal{H}'$: see figure 4. This proposition will be restated in different terms later on.

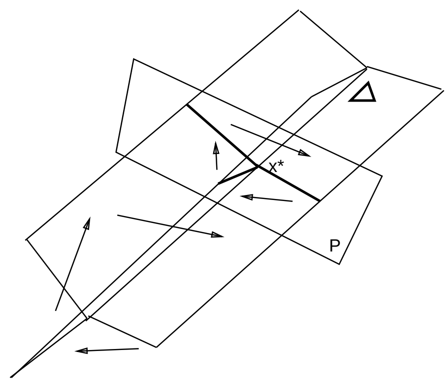

Figure 4: Construction of the PCD projection when surfaces are organized as a pencil. See also figure 17

Proposition 3.1 *Let $\mathcal{H} = (X, f)$ be a real PCD system in dimension d. Let x^* be a point of $X = \mathbb{R}^d$. Assume that there exist polyhedral subset $\Delta \subset X$ of dimension $d - d'$ $(1 \leq d' \leq d)$ and open convex polyhedron $V \subset X$, with $x^* \in \Delta$ and $x^* \in V$, such that, for any region F^j of $\mathcal{H}$, $F^j \cap V \neq \emptyset$ implies $\Delta \subset \overline{F^j}$.*

Then, if $d' < d$ and if we call P the affine variety of dimension d' which is the orthogonal of Δ in x^*, it is possible construct a PCD system $\mathcal{H}' = (X' = \mathbb{R}^{d'}, f')$ in dimension d' such that the trajectories of $\mathcal{H}'$ in a neighborhood of $(0, \dots, 0)$ are precisely the projection on P of the trajectories of $\mathcal{H}$ in V.

We say in that case that *$\mathcal{H}$ is made of the local embedding of $\mathcal{H}'$ in V*. If F^j is a region of $\mathcal{H}$, the projection of F^j on P is called *its trace on $\mathcal{H}'$*: see figure 4.

Proof: Choose an affine basis of $\mathbb{R}^d$ of the form $(x^*, e_1, e_2, \dots, e_{d'}, \dots, e_d)$ with $(x^*, e_1, e_2, \dots, e_{d'})$ taken as a basis of P and $(x^*, e_{d'+1}, \dots, e_d)$ taken as a basis of Δ. Let $p : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ be the projection that sends $(x_1, x_2, \dots, x_d)$ to $(x_1, \dots, x_{d'})$. By hypothesis, in V the regions are organized as a 'pencil of regions': therefore speed in point $(x_1, x_2, \dots, x_{d'}, \dots, x_d) \in V$ does not depend on the coordinates $x_{d'+1}, x_{d'+2}, \dots, x_d$. The reader can check that $\mathcal{H}' = (X' = \mathbb{R}^{d'}, f')$ where $f'(x_1, x_2, \dots, x_{d'}) = p(f(x_1, x_2, \dots, x_{d'}, 0, \dots, 0))$ is a solution. See figure 4.

□

We are now ready to prove the main assertion of the section:

Theorem 3.2 *Let $\mathcal{H} = (X, f)$ be a real PCD system in dimension d. Let Φ be a trajectory of $\mathcal{H}$ of finite continuous time T_{cont}. Denote by T_{discr} the discrete time of Φ.*

- *Assume $d = 2$, then necessarily $T_{discr} \leq \omega$.*
- *Assume $d \geq 3$ then necessarily $T_{discr} < \omega^{d-1}$.*

Proof: Case $d = 2$ is corollary 3.1. Assume $d \geq 3$ and $T_{discr} = \omega^{d-1}$. Denote $t_1, t_2, \ldots, t_\omega, t_{\omega+1}, \ldots, t_{T_{discr}=\omega^{d-1}}, \ldots$ the elements of the well ordered set T_Φ (see definition 2.4 for the definition of T_Φ). Sequence $(t_{\omega^{d-2}i})_{i\in\mathbb{N}}$ is an increasing sequence bounded by $t_{\omega^{d-1}}$. Apply lemma 3.1: there exists $i_0 \in \mathbb{N}$, V_0 such that for any region F of $\mathcal{H}$, $F \cap V_0 \neq \emptyset \Rightarrow \overline{F} \ni x^{*\prime} = \Phi(t_{\omega^{d-1}})$ and for any $t_{\omega^{d-2}i_0} \leq t \leq t_{\omega^{d-1}}$, $\Phi(t) \in V_0$. Consider now sequence $(t_{\omega^{d-2}i_0+\omega^{d-3}j})_{j\in\mathbb{N}}$. Using again lemma 3.1, there exists $j_0 \in \mathbb{N}$, V_1 such that for any region F of $\mathcal{H}$, $F \cap V_1 \neq \emptyset \Rightarrow \overline{F} \ni x^* = \Phi(t_{\omega^{d-2}(i_0+1)})$ and for all $t_{\omega^{d-2}i_0+\omega^{d-3}j_0} \leq t \leq t_{\omega^{d-2}(i_0+1)}$ $\Phi(t) \in V_1$.

Denote $\Delta = [x^*, x^{*\prime}]$ and $V = V_0 \cap V_1$. We get by convexity of the regions that $F \cap V \neq \emptyset \Rightarrow \Delta \subset \overline{F}$. We can apply proposition 3.1: there exits a PCD system $\mathcal{H}'$ of dimension $d-1$ such that $\mathcal{H}$ is made of the local embedding of $\mathcal{H}'$ in V. $\mathcal{H}'$ have the projection Φ' of Φ on the orthogonal P of Δ in x^* as one of its trajectory, so $\mathcal{H}'$ has a finite continuous time trajectory Φ' of discrete time ω^{d-2} in dimension $d-1$.

By induction, the only case we have to pay attention is the case $d = 3$. Assume $d = 3$. $\mathcal{H}$ must still be made of the local embedding of a PCD system $\mathcal{H}'$ in V . But $\mathcal{H}'$ must be a planar PCD system that has finite continuous time trajectory Φ' of discrete time ω. By corollary 3.1 Φ' must be a spiral converging to x^*. Once on Δ, Φ must necessarily evolve with a slope collinear to $\overrightarrow{x^*x^{*\prime}}$. That is, T_{discr} would be equal to $\omega + 1$ instead of ω^2, which is a contradiction.

□

Lemma 3.1 *Let $(t_i)_{i\in\mathbb{N}}$ be any bounded increasing subsequence of points of T_ϕ converging to t_∞. Necessarily there exists $i_0 \in \mathbb{N}$, a convex open polyhedron V such*

- *For any $t \in [t_{i_0}, t_\infty]$, $\Phi(t) \in V$.*
- *For any region F of $\mathcal{H}$, $F \cap V \neq \emptyset \Rightarrow \overline{F} \ni x^* = \Phi(t_\infty)$.*

Proof: Let V be any convex polyhedral open subset containing x^*, of dimension d, included in the ball centered in x^* of radius

$$r = 1/2 * inf_{F \ region \ of \ \mathcal{H}}(\mathbb{R}^+ \cap \{d(x^*, F)\})$$

By definition of V, we have immediately $F \cap V \neq \emptyset \Rightarrow x^* \in \overline{F^j}$ for all region F. Now, observe that by continuity of ϕ and convergence of (t_i) there exists a rank i_0 such that $t \geq t_{i_0} \Rightarrow \Phi(t) \in V$.

□

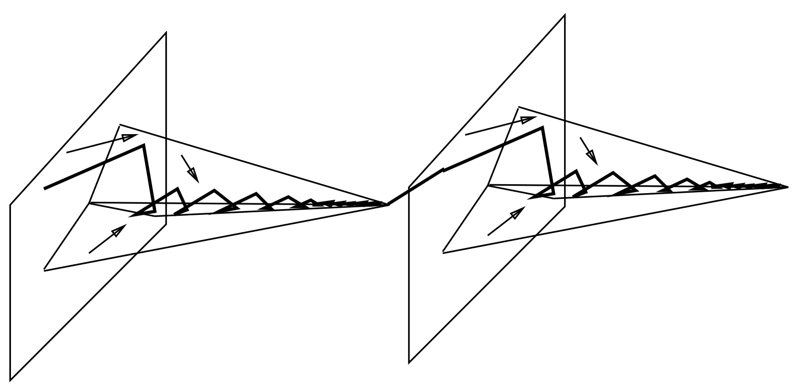

Figure 5: A PCD system with a finite continuous time trajectory of discrete time $\omega 2$. It is possible to construct a PCD system of dimension 3 with a finite continuous time trajectory of discrete time $\omega p + q$, for all $p, q \in \mathbb{N}$, but it is not possible to construct a PCD system of dimension 3 with a finite continuous time trajectory of discrete time ω^2.

The bound given by theorem 3.2 is sharp: for any $d \geq 3$, for any ordinal $o < \omega^{d-1}$ one can construct a purely rational d-dimensional PCD system that has a trajectory of ordinal o. See figure 5.

4 Computability in finite discrete time

We prove in this section that rational PCD systems are equivalent to Turing machines and that real PCD systems are equivalent to linear machines.

4.1 Affine maps realized by some PCD systems.

Let $\mathcal{H}$ be a PCD system of dimension d. Assume that F is a polyhedral subset of dimension $d-1$: that is, F is a polyhedral subset of an affine hyper-plane A of $\mathbb{R}^d$. Let $k \neq 0$ be a vector of $\mathbb{R}^d$ orthogonal to A. Let Φ be a trajectory that enters F or that leaves F with slope k_0. If the dot product of k and k_0 is positive (respectively: negative), we say that k (resp. $-k$) gives the *moving orientation*.

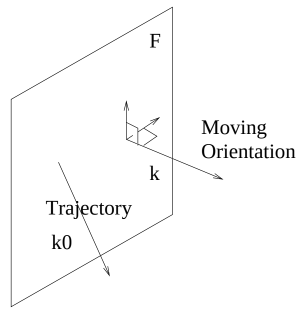

Figure 6: A trajectory leaving F of dimension $d-1$ in a PCD system of dimension $d=3$ and a vector giving the moving orientation

First, we show that any affine map realized by some regions of a PCD systems have a non-negative determinant: see figure 7.

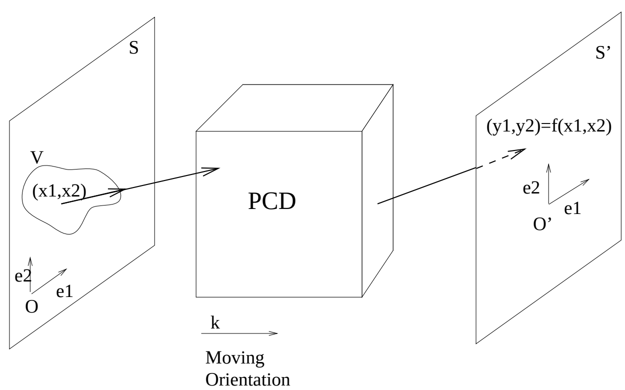

Figure 7: Some regions of a PCD system realizing a partial mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ in dimension 3: $f(x_1, x_2) = (y_1, y_2)$ iff the trajectory starting from the point of coordinates (x_1, x_2) on S reaches S' in the point of coordinates (y_1, y_2) in finite discrete time. We assume that there exists k that gives the moving orientation of any trajectories leaving S or reaching S'. Lemma 4.1 proves that f restricted to some open subset V of $\mathbb{R}^2$ is an affine map with non-negative determinant. Lemma 4.2 shows that any affine map f with non-negative determinant can be realized by some regions of a PCD system.

Lemma 4.1 *Let $\mathcal{H} = (X, f)$ be a PCD system of dimension d. Let P and P' be two parallel affine hyper-planes of $\mathbb{R}^d$. Let S be a polyhedral convex subset of P and S' a polyhedral convex subset of P'. Choose $O, O', e_1, \ldots, e_{d-1}$ such that $B = (O, e_1, \ldots, e_{d-1})$ is an affine basis of P and $B' = (O', e_1, \ldots, e_{d-1})$ is an affine basis of P'. Denote by $f : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ the partial mapping such that $f(x_1, \ldots, x_{d-1}) = (y_1, \ldots, y_{d-1})$ iff point x of coordinates $(x_1, \ldots, x_{d-1})$ in basis B is in S, and the trajectory Φ_x starting* *from x reaches S' at some finite discrete time in point y of coordinates $(y_1, \dots, y_{d-1})$ in basis B'. Denote by Q the domain of f. Assume that $\overline{Q}$ have an non empty interior. Assume that there exists $k \in \mathbb{R}^d$ such that for all x, Φ_x leaves S with moving orientation k and reaches S' with moving orientation k.*

Then there exists a neighborhood $V \subset Q$ such that f restricted to V is an affine map with non negative determinant.

In particular, if f is an affine map or the restriction of an affine map, then f has a non negative determinant.

Proof: Subdivide the regions of $\mathcal{H}$ if necessary such that all the regions of $\mathcal{H}$ are convex. Let γ be a finite sequence of (convex) regions. Denote by $Q_\gamma \subset Q$ the subset of the points $x \in S$ such that Φ_x has signature γ between S and S'. Each $Q_\gamma \neq \emptyset$ is a convex polyhedral subset of Q. We have $Q = \cup_{\gamma \text{ finite sequence of regions}} Q_\gamma$. Since $\overline{Q}$ has an non empty interior, from Baire theorem [19], there exists a γ_0 such that $\overline{Q_{\gamma_0}}$ has an non empty interior. Since $Q_{\gamma_0} \neq \emptyset$ is a convex polyhedral subset, Q_{γ_0} has an non empty interior and contains a ball V. It is easy to observe that f restricted to V must be an affine map.

Denote by $F_0, F_1, \dots, F_l$ the successive convex boundaries of the regions intersected by the trajectories starting from V. Without loss of generality, assume $O \in V$. The successive images $B_0, \dots, B_l$ of basis $B = (e_1, \dots, e_{d-1})$ on $F_0, \dots, F_l$ are such that B_{i+1} is the projection of B_i on F_{i+1} with projection direction given by the slope k_i between F_i and F_{i+1}. If one of the B_i has a dimension $< d-1$ the determinant of f is 0 and the result is true. Assume that all the B_i are of rank $d-1$. That implies that each F_i is a convex polyhedral subset of some hyper-plane H_i of dimension $d-1$ and that $k_i \notin H_i$. Observe by an easy induction on i that the sign of the determinant of (B_i, k_i) is independent of i since once on F_{i+1} one can only leave by the side of hyper-plane H_{i+1} opposite to the side the trajectory is coming from. As a consequence, since k_0 and k_{l-1} must make the trajectories to reach and leave P and P' with same moving orientation, the determinant of f restricted to V is positive.

□

Observe now that any affine map of non-negative determinant can be realized by some regions of a PCD system.

Lemma 4.2 *Assume that $f : [0,1]^{d-1} \to [0,1]^{d-1}$ is (the restriction to $[0,1]^{d-1}$ of) an affine map with non-negative determinant. Let S_0 be a given polyhedral subset of dimension $d-1$ and let $B = (0, e_1, \dots, e_{d-1})$ be an affine basis of S_0. Then it is possible to construct some regions $R_0, \dots, R_d$ in dimension d, a polyhedral subset S_d of dimension $d-1$ parallel to S_0 and to find a point $O' \in S_d$ such that any trajectory starting from point of coordinates $(x_1, \dots, x_{d-1})$ on B reaches the point of S_d of coordinates $f(x_1, \dots, x_{d-1})$ in basis $B' = (O', e_1, \dots, e_{d-1})$. Moreover the trajectory leaves S_0 and reaches S_d with same moving orientation.*

Proof: Let k be the dimension of the image of f. Choose a basis of S_0 of type $(0, b_1, b_2, \dots, b_{d-1})$ such that $(b_{k+1}, \dots, b_{d-1})$ is a basis of the kernel of f. Let $k^0, k^1, \dots, k^{d-1}$ be d vectors, still to be determined, satisfying a set of $d+1$ inequalities $(D_l)_{l=0,1,\dots,d}$. We want to find some polyhedral subsets $S_1, S_2, \dots, S_d$ of dimension $d-1$. We will take R_l, for all l, as the region between S_l and S_{l+1} and we want each k^l to be the slope in region R_l. Inequality D_d is $det(k^0, k^1, \dots, k^{d-1}) \neq 0$. So we take the $(k^l)_l$ linearly independents. For all $i \in \{1, 2, \dots, k\}$ denote $b'_i = f(b_i)$. Then for each $j \in \{0, 1, \dots, d\}$ denote $b_i^j = b_i + \lambda_i^0 k^0 + \lambda_i^1 k^1 + \dots + \lambda_i^{j-1} k^{j-1}$. We want b_j^i to give the image of b_i on S_j. We want $b_i^d = b'_i$ for all i. Since $(k^0, k^1, \dots, k^{d-1})$ is taken as a basis of $\mathbb{R}^d$ this implies that all the λ_i^j are given by the Cramer rule as $det(k^0, k^1, \dots, k^{j-1}, b'_i - b_i, k^{j+1}, \dots, k^{d-1}) / det(k^0, k^1, \dots, k^{d-1})$. Thus the $(\lambda_i^j)_{i,j}$ are rational functions in the coordinates of the $(k^l)_l$s. Hence the coordinates of the $(b_i^j)_{i,j}$ are also rational functions in the coordinates of the $(k^l)_l$s. We write, for $j = 0, 1, \dots, d-1$, inequality (D_j) as $k^j \notin Vect(b_1^j, b_2^j, \dots, b_k^j)$. For $j \in \{1, 2, \dots, d-1\}$, it is easy to show that if all the previous $(D_l)_{l<j}$ are verified then necessarily $rank(b_1^j, b_2^j, \dots, b_k^j) = k$. Now, observe that (D_j), assuming the $(D_l)_{l<j}$, can be written as the non-nullity of the sum of the squares of all the square sub-determinants of matrix $(k^j, b_1^j, b_2^j, \dots, b_k^j)$, so as a polynomial inequality on the coordinates of the $(k^l)_l$s. And since $b_1^j, b_2^j, \dots, b_k^j$ are independent, this polynomial is non-null.

There must exist some $(k^l)_l$ verifying all the inequalities $D_0, D_1, \dots, D_d$ since the converse would imply that the product P of all the polynomial inequalities $(D_j)_j$, for $j = 0, 1, \dots, d$ would be a non null polynomial which is always null. Now, observe that if $k^0, k^1, \dots, k^{j-1}, k^j, k^{j+1}, \dots, k^{d-1}$ is a solution, for all j, $k^0, k^1, \dots, k^{j-1}, -k^j, k^{j+1}, \dots, k^{d-1}$ is also a solution of the inequalities. Construct inductively

$S_1, S_2, \ldots, S_d$: assume $S_l, l \geq 0$ is constructed. Let s^- (respectively s^+) be the side of S_l the trajectory is coming from (resp. is going to). If k_l is not going from s^- to s^+: replace k^l by $-k^l$. In any case, take S_{l+1} as any bounded polyhedral subset of any affine variety V whose induced vector space is the vector space spanned by vectors $b_1^{l+1}, b_2^{l+1}, \ldots, b_k^{l+1}$, such that S_{l+1} does not intersect the projection of $[0,1]^{d-1}$ on S_l and such that S_{l+1} is big enough to contain the projection of $[0,1]^{d-1}$ on V (we call *projection of* $[0,1]^{d-1}$ *on* S_l (respectively V) the subset of the points of S_l (resp. V) that are reached by a trajectory starting from a point of S_0 of coordinates in $[0,1]^d$ on basis B.).

□

4.2 PCD systems and linear machines

Let M be a linear machine in dimension d. As usual for computational machines, at every time of the evolution of M, we can write the global state of M as a tuple, called *ID*, $ID = (q, x_1, x_2, \ldots, x_d)$, where q is the internal state of the control part of M and the $(x_j)_{j=1\ldots d} \in \mathbb{R}$ denote the values of the real registers of M. The binary relation $\vdash$ between IDs is defined by $ID^1 \vdash ID^2$ if when machine M is in the state given by ID^1 then M makes an immediate transition to the state given by ID^2. Linear machine M is said to have *the connectivity property* if for all IDs ID^1, ID^2, ID^3 such that $ID^1 = (q^1, x_1^1, x_2^1, \ldots, x_d^1) \vdash ID^3$ and $ID^2 = (q^2, x_1^2, x_2^2, \ldots, x_d^2) \vdash ID^3$ with $q^1 \neq q^2$ we have necessarily $x_d^1 = x_d^2 = 0$.

Theorem 4.1
- *If language L is recognized by a PCD system in dimension d in discrete time T, then L is recognized by a linear machine in dimension $d-1$ in time $O(T)$.*
- *If language L is recognized by a linear machine in dimension d in time T, then L is recognized by a PCD system in dimension $d+2$ in discrete time $O(T)$.*
- *If language L is recognized by a linear machine M with the connectivity property in dimension d in time T, then L is recognized by a PCD system in dimension $d+1$ in discrete time $O(T)$.*

Proof: First point is straightforward since it is easy to simulate a PCD system by a linear machine. Second point is immediate from third point, since one can always add a real register that stays equal to 0 to any linear machine.

Now for the third point, observe that we can suppose without loss of generality that linear machine M does only linear operations of non-negative determinant: if it is not so, simulate M by a linear machine M' that replaces any linear operation l of negative determinant by the composition of l with the operation that changes x_d in its opposite. M' stores at any time in its control part the information whether the value of its last variable is equal or opposite to the corresponding variable in M at same time. M' is able to simulate M and does only linear operations of non-negative determinant.

A *k-dimensional box* is a pair $I = (P, B)$ where P is any polyhedral subset of an affine variety V of $\mathbb{R}^d$ of dimension k, and B is a affine basis $(O, e_1, e_2, \ldots, e_d)$ of $\mathbb{R}^d$ such that $(O, e_1, \ldots, e_k)$ is an affine basis of V. The *point of coordinates* $(x_1, \ldots, x_d)$ *on* I denotes the point of P (if it exists) of coordinates $(x_1, \ldots, x_d, 0, \ldots, 0)$ in basis B.

Let l be the number of internal states of M. By renaming, we can assume that the set of the internal states of M is $\{1, 2, \ldots, l\}$. We work in $\mathbb{R}^{d+1}$ with its canonical basis $(e_1, e_2, \ldots, e_{d+1})$. Denote by Q the polyhedral subset $[-1,1] \times [-1,1] \times \ldots [-1,1] \times \{0\}$. For $i \in \{1, 2, \ldots, l\}$, denote O_i the point of coordinates $O_i = (4*(i-1), 0, \ldots, 0)$. Denote $P_i = O_i + Q$ and $B_i = (O_i, e_1, e_2, \ldots, e_d)$. So, for all i, $\hat{P}_i = (P_i, O_i)$ is a d-dimensional box. Denote $\tilde{P} = \cup_{i=1\ldots l} P_i$. Let P any polyhedral bounded subset of the hyper-plane containing $\tilde{P}$. We construct a PCD system $\mathcal{H}$ that has M as a 'Poincare Map'[10]: see figure 8. We want the trajectories of $\mathcal{H}$ to intersect again and again P in $\tilde{P}$ with direction $x_{d+1} > 0$. We want the t^{th} intersection of the trajectory with P to correspond to the ID of M at time t: if the intersection happens in the point of coordinates $(x_1, x_2, \ldots, x_d)$ on d-dimensional box $\hat{P}_i$ then M has ID $(i, x_1, x_2, \ldots, x_d)$ at time t.

Let $i \in \{1, 2, \ldots, l\}$. We can assume that in internal state i, M does first a linear operation f_i of non-negative determinant on its variables (set f_i as the identity if needed), makes a linear test T_i (assume T_i to be a trivial test if needed), and then makes a transition to some internal state $\delta(i)^+$ or $\delta(i)^-$ according to the result of test T. For any i, by lemma 4.2, it is possible to construct a sequence of regions such that f_i is computed with input port d-dimensional box $\hat{P}_i$ and with output port a parallel d-dimensional box $\hat{P}'_i = (P'_i, B'_i)$. Denote by $P_i'^{+}$ and $P_i'^{-}$ the polyhedral partition of P'_i in the subset of values $(x'_1, x'_2, \ldots, x'_d)$ that respectively satisfies and does not satisfy test T. It is easy to build for every

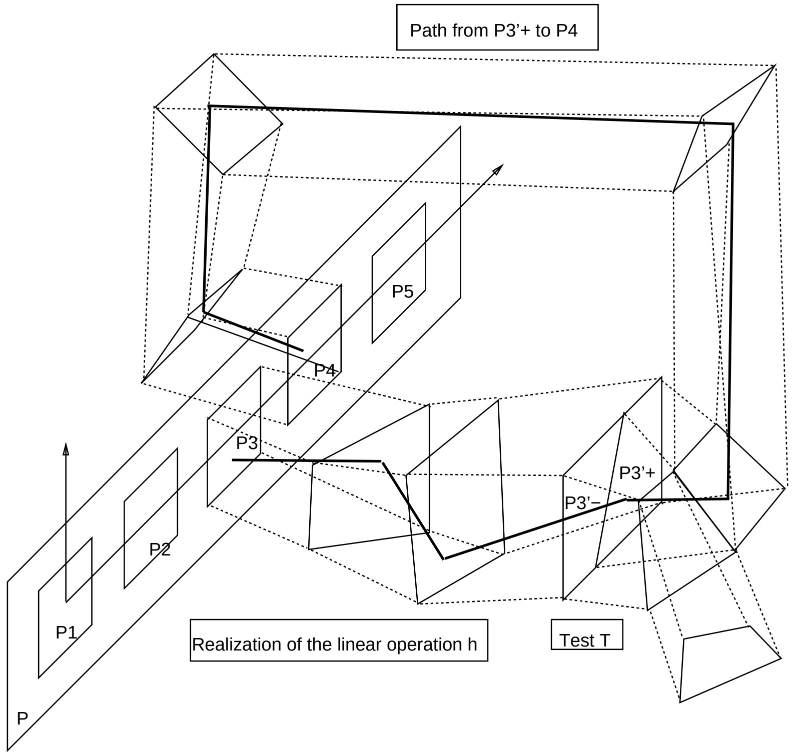

Figure 8: A PCD system simulating a linear machine. Only one path is represented.

$P_i'^k$, $k \in \{+,-\}$, a "path" that brings any trajectory crossing $P_i'^k$ in coordinates x on d -dimensional box $(P_i'^k, B_i')$ to cross $P_{\delta(i)^k}$ on the point of same coordinates x on d-dimensional box $\hat{P_{\delta(i)^k}}$. See figure 8.

We want to do the construction for all i. Unfortunately, it might happen that there exists i_1 and i_2, $k_1, k_2 \in \{+,-\}$, $i_1 \neq i_2$ or $k_1 \neq k_2$ such that $\delta(i_1)^{k_1} = \delta(i_2)^{k_2}$. In this case, we need to realize the fusion between the paths coming from $P_{i_1}'^{k_1}$ and $P_{i_2}'^{k_2}$. If M is a linear machine with the connectivity property this is always possible since this property asserts precisely that when a fusion is needed, the fusion is between paths of dimension strictly less than d. See [3] or figure 9.

□

The above-mentioned simulations preserve constants: a PCD system that can be described using rational and irrational constants $\gamma_1, \dots, \gamma_k \in \mathbb{R}$ is simulated by a linear machine that has the constants $\gamma_1, \dots, \gamma_k$, and conversely.

We get:

Corollary 4.1

- *The languages recognized in polynomial discrete time by real PCD systems in dimension $d \geq 3$ are precisely the languages of $P/poly$. Every language $L \subset \mathbb{N}$ can be recognized by a real PCD system in dimension $d \geq 3$ in finite discrete exponential time.*
- *The languages recognized in polynomial (respectively: exponential) discrete time by rational PCD systems in dimension $d \geq 3$ are precisely the languages that are recognized by a Turing machine in polynomial (resp. exponential) time.*

Proof: These results have been proved for linear machine of dimension 2 in [11, 12]. They are already proved using bounded finite dimensional linear machines. Now observe, that we can also assume that the linear machines used have the connectivity property: replace the Turing machines in the simulation used in [11, 12] by either some reversible machines [8] or by machines that empty one of their stacks before making a non-reversible transition [3]. The results follow from theorem 4.1.

□

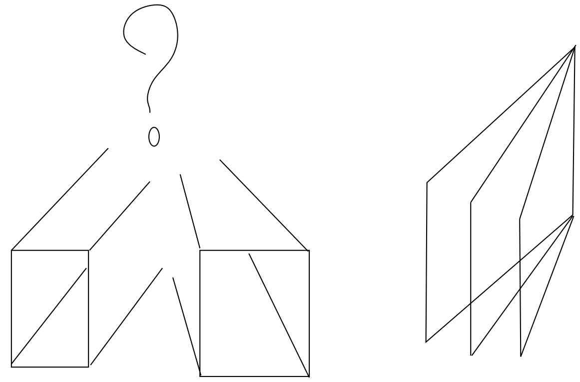

Figure 9: Realizing a fusion between several paths in dimension $d+1$. If the paths have dimension d they can not be merged. Nevertheless, if they have dimension $d-1$ they can be merged.

5 Computability in finite continuous time

5.1 Lower bounds

It was shown in [2] that every set of the arithmetical hierarchy can be recognized in finite continuous time: more precisely, it is shown that $L \in \Sigma_k \cup \Pi_k$ can be recognized by a PCD system of dimension $5k+1$: see figure 10 for the general idea of the "Zeno construction". Therefore, five dimensions are used

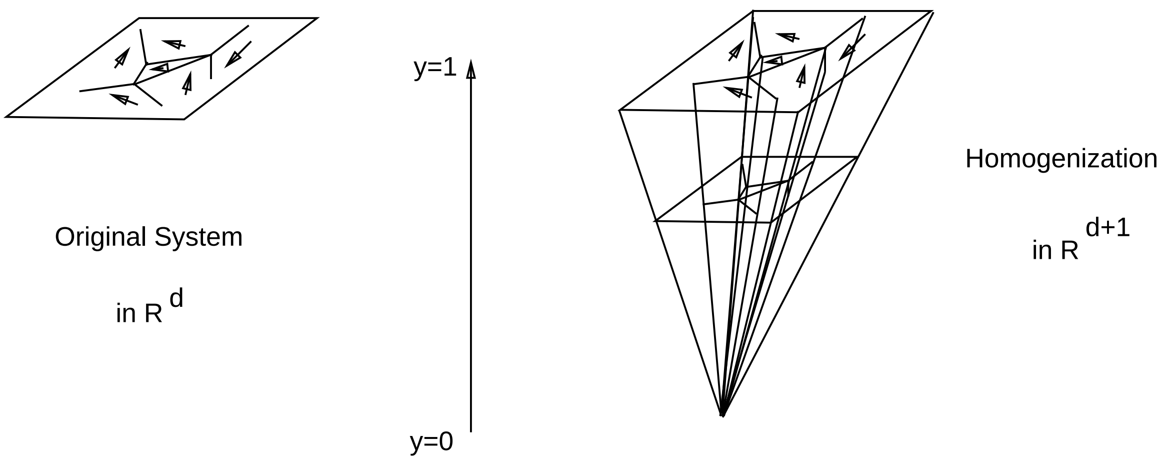

Figure 10: The general principle of the Zeno construction: assume that PCD system $\mathcal{H} = (\mathbb{R}^d, f)$ semi-recognizes L in finite continuous time. A pyramid $\mathcal{H}'$ is constructed in dimension $d+1$ from the original system $\mathcal{H}$ in dimension d: in $\mathcal{H}'$, when the variables are divided by two, the simulation of $\mathcal{H}$ by $\mathcal{H}'$ goes two times faster. The original system $\mathcal{H}$ is simulated by $\mathcal{H}'$ during time 1 at speed 1, time 1/2 at speed 2, etc... If $\mathcal{H}$ accepts, $\mathcal{H}'$ is made such that the process stops and such that $\mathcal{H}'$ accepts. If $\mathcal{H}$ never accepts, the trajectory converges in finite continuous time to $(0, \dots, 0)$ which is taken as the point of rejection of $\mathcal{H}'$. Thus, L is now fully recognized by $\mathcal{H}'$. See lemma 5.1 for the full details.

in [2] to climb each level of the arithmetical hierarchy: one for a timer, one used for the divisions by 2, one used to do the homogenization, and two dimensions used to go from quantifier elimination to semi-recognition. We show here that only one dimension is needed (the one used to do the homogenization), and that the construction only requires purely rational PCD systems. The proof is rather technical but is detailed up to the end of this subsection.

We shall use PCD systems with a special structure: see figure 11.

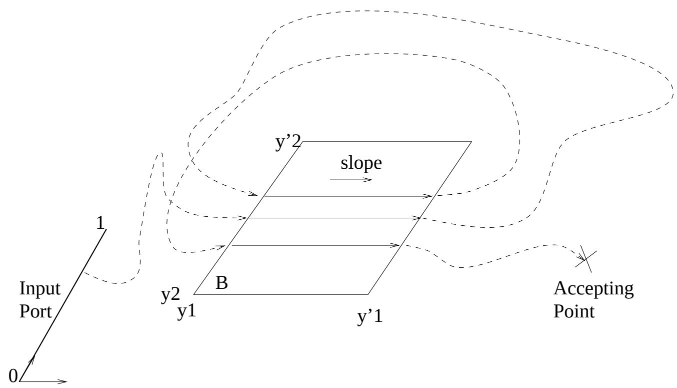

Figure 11: A PCD system with the box property: there exists a "box" $\mathcal{B}$ of equation $\Pi_{i=1}^{d}[y_i, y'_i]$ that is crossed at regular time interval by any trajectory (The dot lines represent symbolically the trajectory evolving somewhere in the space of the PCD).

Definition 5.1 *We say that a PCD system $\mathcal{H} = (X, f)$ of dimension d has the box property if, in some basis of $\mathbb{R}^d$,*

- *there exists a region $\mathcal{B}$ of equation $\Pi_{i=1}^{d}[y_i, y'_i]$ for some $y_1, \ldots, y_d,\ y'_1, \ldots, y'_d \in \mathbb{R}$, with for all $i \in \{1, \ldots, d\}$, $y_i < y'_i$.*
- *f has a constant value on $\mathcal{B}$ of type $(v, 0, \ldots, 0)$ for some $v \in \mathbb{R}^+$.*
- *Any trajectory of $\mathcal{H}$ crosses $\mathcal{B}$ at regular time intervals: there exists $t_0 \in \mathbb{R}^+$ such for any $t \in \mathbb{R}^+$, for any trajectory Φ, if Φ is defined up to time $t + t_0$, then Φ crosses $n_t \in \mathbb{N}, n_t \geq 1$ times $\mathcal{B}$ on time interval $[t, t + t_0]$.*
- *The input port of $\mathcal{H}$ has equation $\{0\} \times [0, 1] \times \{0\}^{d-2}$.*

The main construction used in this subsection is contained in the following lemma: see figure 13 and figure 12.

Lemma 5.1 *Let $\mathcal{H} = (X, f, r, I, x^1, x^0)$ be a bounded PCD system with the box property in dimension $d \geq 2$ that semi-recognizes language L. Then there exists an bounded PCD system $\mathcal{H}' = (X' = \mathbb{R}^{d+1}, f')$ in dimension $d + 1$, a point $x'^0 \in X'$, a segment $J' = \Pi_{i=1\ldots d}\{x'^1_i\} \times (1/2, 1]$ of X', for some given $(x'^1_i)_{i=1\ldots d} \in \mathbb{R}$, such that:*

- *the trajectory Φ' of $\mathcal{H}'$ starting from $x = (0, x_2, 0, \ldots, 0, x_{d+1}) \in X'$ with $x_{d+1} \in (1/2, 1]$ and $y = x_2/x_{d+1} \in [0, 1]$ reaches x'^0 (respectively: J') if and only if the trajectory Φ of $\mathcal{H}$ starting from $\{0\} \times \{y\} \times \{0\}^{d-2} \in I$ does not reach x^1 (resp: reaches x^1).*
- *When Φ' reaches J', the intersection happens in $(x'^1_1, x'^1_2, \ldots, x'^1_d, x_{d+1})$.*

In particular, if I' denotes $\{0\} \times [0, 1] \times \{0\}^{d-2} \times \{1\}$, if $x'^1 = (x'^1_1, x'^1_2, \ldots, x'^1_d, 1)$, then $\mathcal{H}' = (X', f', r, I', x'^1, x'^0)$ is a bounded PCD system in dimension $d + 1$ that fully recognizes L.

Proof: We work in $\mathbb{R}^{d+1}$ with its canonical basis and use the notations of definition 5.1. First we start by modifying $\mathcal{H}$: divide region $\mathcal{B}$ into d regions: for all $i \in \{2, \ldots, d + 1\}$, set $\mathcal{B}_i = [y_1 + (y'_1 - y_1) * (i - 1)/d, y_1 + (y'_1 - y_1) * i/d] \times \Pi_{i=2}^{d}[y_i, y'_i]$. Build in each $\mathcal{B}_i$, for $i \in \{2, \ldots, d\}$ two regions $\mathcal{B}_i^1, \mathcal{B}_i^2$ with respective slopes v_i^1, v_i^2 that makes x_i to be divided by 2: take for example $\mathcal{B}_i^1 = \{(x_1, \ldots, x_d) \in \mathcal{B}_i | y_1 + (y'_1 - y_1) * (i - 1)/d \leq x_1 < y_1 + (y'_1 - y_1) * (i - 1)/d + (x_i - y_i)/(y'_i - y_i) * (y'_1 - y_1)/d\}$, $\mathcal{B}_i^2 = \mathcal{B}_i - \mathcal{B}_i^1$, $v_i^1 = ((y'_1 - y_1)/(2d)), 0, \ldots, 0, x_i = -(y'_i - y_i)/2, 0, \ldots, 0)$ and $v_i^2 = (1, 0, \ldots, 0)$.

We construct PCD $\mathcal{H}' = (X' = \mathbb{R}^{d+1}, f')$ as follows. Denote $Z = X - \mathcal{B}_{d+1} \subset \mathbb{R}^d$. First $\mathcal{H}'$ is constructed as a pyramid of $\mathcal{H}$ on Z: formally $f'(x_1, \ldots, x_{d+1})$ is defined, for all $x_1, \ldots, x_d \in \mathbb{R}, 0 < x_{d+1} \leq 1$ with $(x_1/x_{d+1}, x_2/x_{d+1}, \ldots, x_d/x_{d+1}) \in Z$, by $f'(x_1, \ldots, x_{d+1}) = f(x_1/x_{d+1}, \ldots, x_d/x_{d+1})$. The main property of this construction is that for any $0 < \mu \leq 1$, $y_1, y_2 \in Z$, if there is a trajectory of

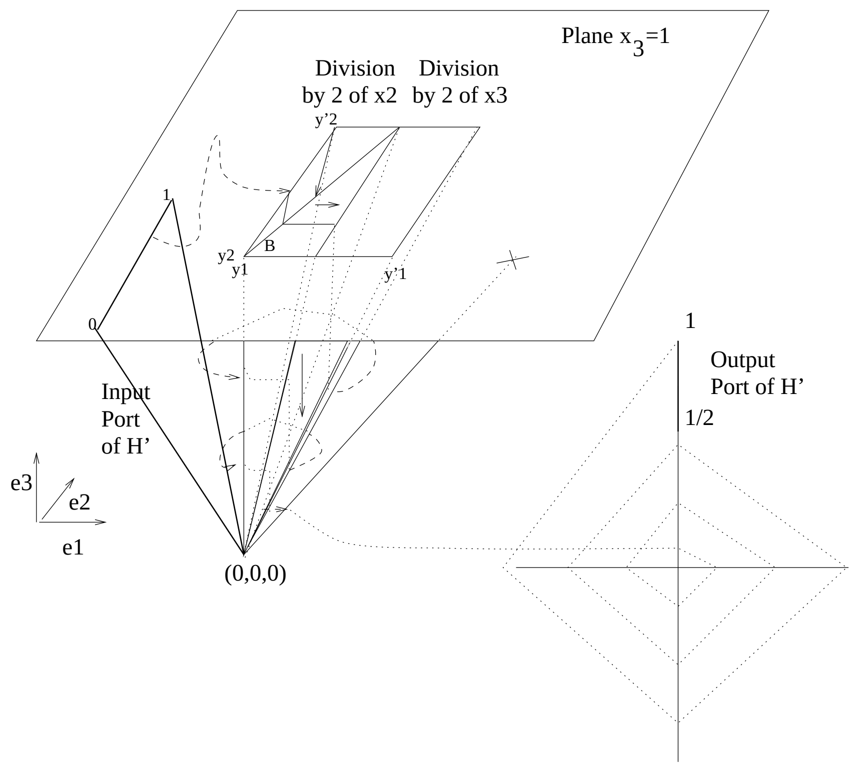

Figure 12: PCD system $\mathcal{H}'$ obtained by applying lemma 5.1 on PCD system $\mathcal{H}$ of figure 11. $\mathcal{H}'$ simulates $\mathcal{H}$, but now each time a trajectory crosses box B all the variables are divided by 2. Then the simulation goes on but two times faster.

$\mathcal{H}$ from y_1 to y_2 of time t, then there is a trajectory of $\mathcal{H}'$ from $(\mu y_1, \mu)$ to $(\mu y_2, \mu)$ of time μt. Hence, for any $k \in \mathbb{R}^+$, $\mathcal{H}'$ simulates $\mathcal{H}$ but goes k times faster when all the variables are divided by k.

Denote $\mathcal{B}'_{d+1} = \{(x_1, \ldots, x_{d+1}) | 0 < x_{d+1} \leq 1 \wedge (x_1/x_{d+1}, \ldots, x_d/x_{d+1}) \in \mathcal{B}_{d+1}\}$. Now set f' to value $((y'_1 - y_1)/(2d), 0, \ldots, 0, -1/2)$ on $\mathcal{B}'_{d+1}$: see figure 12.

Denote by x''^1 the region $\{(x_1, \ldots, x_{d+1}) | (x_1/x_{d+1}, \ldots, x_d/x_{d+1}) = x^1 \wedge 0 < x_{d+1} \leq 1\}$. Add to $\mathcal{H}'$ some regions that bring the points of x''^1 of coordinates $(x_1, \ldots, x_{d+1})$ to points of coordinate $(x'_1, 0, \ldots, 0, x_{d+1})$ for some big enough x'_1. Add to $\mathcal{H}'$ four regions as in figure 3: set f' to $(1/2, 0, \ldots, 0, -1)$ on $R_1 = \{(x_1, \ldots, x_{d+1}) | 0 < x_{d+1} \leq 1/2 \wedge x'_1 \leq x_1 \leq x'_1 + 1\}$, to $(-1, 0, \ldots, 0, -1)$ on $R_2 = \{(x_1, \ldots, x_{d+1}) | -1 \leq x_{d+1} \leq 0 \wedge x'_1 < x_1 \leq x'_1 + 1\}$, to $(-1, 0, \ldots, 0, 1)$ on $R_3 = \{(x_1, \ldots, x_{d+1}) | -1 \leq x_{d+1} < 0 \wedge x'_1 - 1 \leq x_1 \leq x'_1\}$ and to $(1, 0, \ldots, 0, 1)$ on $R_4 = \{(x_1, \ldots, x_{d+1}) | 0 \leq x_{d+1} \leq 1 \wedge x'_1 - 1 \leq x_1 < x'_1\}$.

By hypothesis, $\mathcal{H}$ is such that any non accepting trajectory Φ of $\mathcal{H}$ starting from $(0, x_2, \ldots, 0)$ crosses infinitely often $\mathcal{B}$. Hence, for any $0 < \mu \leq 1$, the trajectory Φ' starting from $(0, \mu x_2, \ldots, 0, \mu)$ crosses $\mathcal{B}' = \{(x_1, \ldots, x_{d+1}) | 0 < x_{d+1} \leq 1 \wedge (x_1/x_{d+1}, \ldots, x_d/x_{d+1}) \in \mathcal{B}\}$ infinitely often. Each time Φ' crosses $\mathcal{B}'$ all the variables are divided by 2. Hence the simulation of $\mathcal{H}$ by $\mathcal{H}'$ goes two times faster: see figure 12. As the time between two intersections of Φ with $\mathcal{B}$ is bounded by some t_0, if Φ is non accepting, Φ will cross an infinite number of time $\mathcal{B}$ and Φ' will reach $(0, \ldots, 0)$ at a finite continuous time ($\Sigma_k t_0/2^k$ is a convergent series). Take $x'_0 = (0, \ldots, 0)$. Assume now that trajectory Φ is accepting. Then Φ' reaches x''^1, hence is brought to regions R_1, R_2, R_3, R_4. These regions multiply x_{d+1} by 2 until the result falls in $(1/2, 1]$. Take $x'^1_2 = x'^1_3 = \ldots, = x'^1_d = 0$. We get a PCD that fulfills all the assertions.

□

Denote $I'' = \{(0, x_2, 0, \ldots, 0, x_{d+1}) | x_2/x_{d+1} \in [0, 1] \wedge x_{d+1} \in (1/2, 1]\}$. We say that I'' and J' are respectively the *input and output ports* of $\mathcal{H}'$: see figure 13.

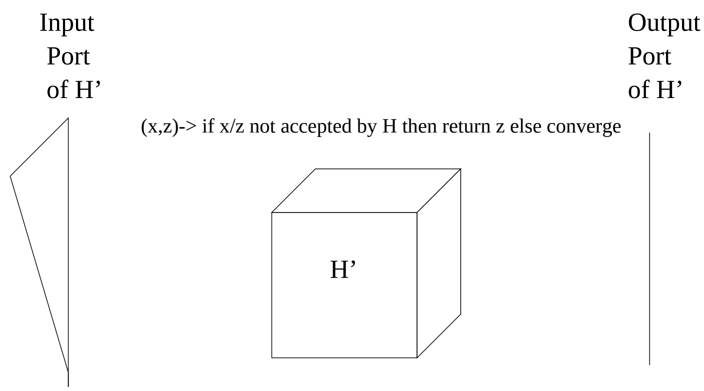

Figure 13: Symbolic representation of PCD system $\mathcal{H}'$ obtained by lemma 5.1 from PCD system $\mathcal{H}$: $\mathcal{H}'$ simulates $\mathcal{H}$: if $\mathcal{H}'$ is started with $x_2 = x, x_{d+1} = z$ then $\mathcal{H}'$ simulates $\mathcal{H}$ on input x/z. If $\mathcal{H}$ does not accept then $\mathcal{H}'$ converges to some limit point. If $\mathcal{H}$ accepts, then $\mathcal{H}'$ returns z.

We now prove:

Lemma 5.2 *There exists $\mathcal{E} : \mathbb{N} \to (1/2, 1]$ such that:*

- *$\mathcal{E}$ is injective and computable by a linear machine, and $\mathcal{E}^{-1}$ is computable by a linear machine on all points of the range of $\mathcal{E}$.*
- *There exists a linear machine M of dimension 2 that computes the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps (x, n), $x \in \mathbb{R}$, $n \in \mathbb{N}$, to $(x * \mathcal{E}(n), \mathcal{E}(n))$.*

Proof: For all $n \in \mathbb{N}$, define $\mathcal{E}(n)$ as the unique point which is simultaneously in interval $(1/2, 1]$ and in the set $\{2^k 3^n | k \in \mathbb{Z}\}$. Clearly, $\mathcal{E}$ is injective and computable by a linear machine: compute 3^n and divide or multiply by 2 until it falls in $(1/2, 1]$. Now, observe that $\mathcal{E}^{-1}(y)$, for y in the range of $\mathcal{E}$, is also computable by the following algorithm: for $k = 1, 2, \ldots, \infty$, test if $\mathcal{E}(k) = y$. if it so, return k and stop. Since y is assumed to be on the range of $\mathcal{E}$ the algorithm stops.

For the second assertion, consider linear machine M that on input $x \in \mathbb{R}$, $n \in \mathbb{N}$ performs the following algorithm:

Algorithm 1

```
y:=1
for k = 1, 2, ..., n do
    x:=x*3
    y:=y*3
end for
while y ∉ (1/2, 1] do
    if y > 1 then
        y:=y/2
        x:=x/2
    else
        y:=2*y
        x:=2*x
    end if
end while
```

□

We are now ready to prove the main result of this section:

Theorem 5.1 • *Any language L of Σ_k is semi-recognized by a purely rational PCD system in dimension $2+k$.*

• *Any language L of Δ_k is fully-recognized by a purely rational PCD system in dimension $2+k$.*

Proof: We prove the first assertion by induction on $k \geq 1$: we prove that any language L of Σ_k can be semi-recognized by a PCD system of dimension $2+k$ with the box property.

The assertion for $k = 1$ is immediate from theorem 4.1 since a linear machine with the connectivity property in dimension 2 can simulate any arbitrary Turing machine [11]. Observe in the proof of theorem 4.1, that we can easily choose all the paths such that the PCD system has the box property: take $\mathcal{B}$ as the region between $P - (0, \ldots, 0, \epsilon)$ and P.

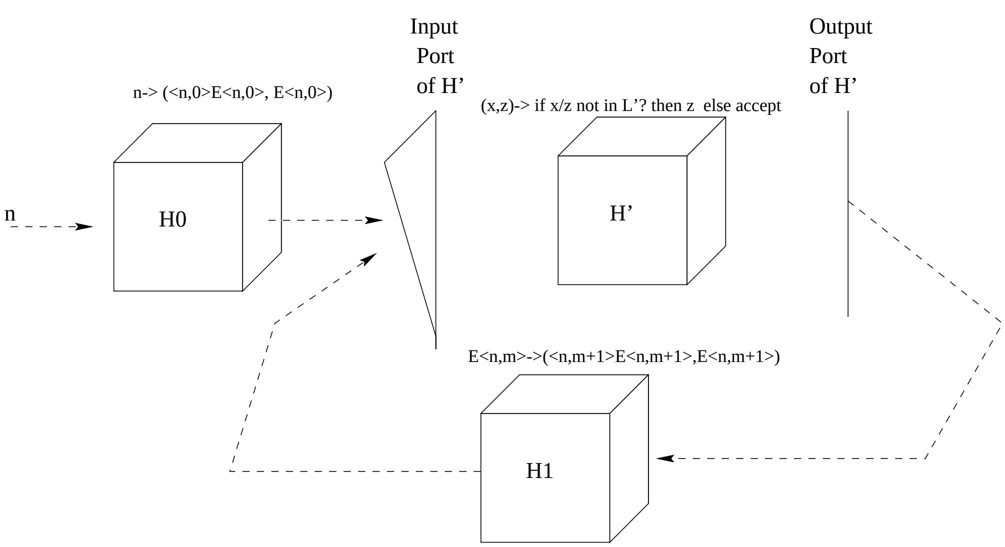

Figure 14: The PCD system constructed in the proof of the first assertion of theorem 5.1.

Assume that the induction hypothesis is true at rank k. Let L be a language of Σ_{k+1}. There exists $L' \in \Pi_k$ such that $n \in L$ if and only if there exist $m \in \mathbb{N}$, such that $< n, m > \in L'$ [18]. By induction hypothesis, the complement of L' is semi-recognized by a PCD system $\mathcal{H}$ with the box property in dimension $2+k$. By lemma 5.1, the complement of L' is also fully-recognized by a PCD system $\mathcal{H}'$ in dimension $3+k$. Consider function $f:\mathbb{R}\to\mathbb{R}^2$ that maps, for all $n,m\in\mathbb{N}$, $\mathcal{E}(<n,m>)$ to $(<n,m+1> *\mathcal{E}(<n,m+1>),\mathcal{E}(<n,m+1>))$. Function f is computable by a linear machine in dimension 2: using lemma 5.2, first compute $<n,m>$ from $\mathcal{E}(<n,m>)$, then compute $<n,m+1>$, and then compute $(<n,m+1> *\mathcal{E}(<n,m+1>),\mathcal{E}(<n,m+1>))$. As a consequence, by theorem 4.1, there exists a PCD system $\mathcal{H}_1$ in dimension 4 that computes f. In a similar way, using theorem 4.1 there exists a PCD system $\mathcal{H}_0$ in dimension 4 that computes the function that maps $n\in\mathbb{N}$ to $(<n,0> *\mathcal{E}(<n,0>),\mathcal{E}(<n,0>))$. Now connect the two dimensional output ports of $\mathcal{H}_0$ and $\mathcal{H}_1$ to the input port of $\mathcal{H}'$. Connect the output port of $\mathcal{H}'$ to the input port of $\mathcal{H}_1$. We get a bounded purely rational PCD system that semi-recognizes L. Its input port is the input port of $\mathcal{H}_0$ and its accepting point is the rejecting point of $\mathcal{H}'$: see figure 14. Now, since $\mathcal{H}_1$ has the box property, and since any evolution in $\mathcal{H}'$ takes a finite bounded continuous time, this PCD system also have the box property. Therefore, the first assertion is proved.

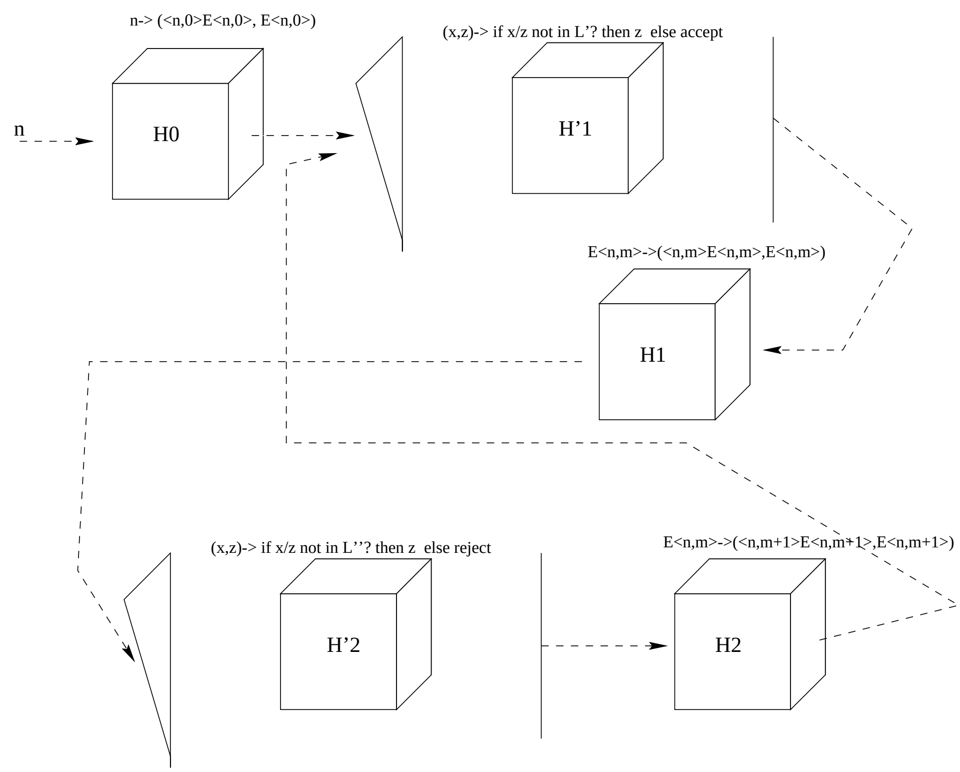

Figure 15: The PCD system constructed in the proof of the second assertion of theorem 5.1.

For the second assertion, consider now $L\in\Delta_k$. We have to show that L is fully recognized in dimension $2+k$. Case $k=1$ has already been established in theorem 4.1. Assume $k>1$. There exists $L'\in\Pi_{k-1}$ and L'' also in Π_{k-1} such that $n\in L$ if and only if there exists $m\in\mathbb{N}$ such that $<n,m>\in L'$, and $n\notin L$ if and only if there exists $m\in\mathbb{N}$ such that $<n,m>\in L''$ [18]. We have already observed that it is possible to construct a PCD system $\mathcal{H}_0$ in dimension 4 that computes $n\mapsto(<n,0> *\mathcal{E}(<n,0>),\mathcal{E}(<n,0>))$. Using theorem 4.1 and lemma 5.2, construct also a PCD system $\mathcal{H}_1$ that computes the map that sends $\mathcal{E}(<n,m>)$ to $(<n,m> *\mathcal{E}(<n,m>),\mathcal{E}(<n,m>))$. Construct now a PCD system $\mathcal{H}_2$ of dimension 4 that computes the map that sends $\mathcal{E}(<n,m>)$ to $(<n,m+1> *\mathcal{E}(<n,m+1>),\mathcal{E}(<n,m+1>))$. Using the first assertion, we also know that the complement of L' can be semi-recognized in dimension $1+k$ by a PCD system with the box property. By lemma 5.1, there exists a PCD system $\mathcal{H}'_1$ that fully-recognizes the complement of L' in dimension $2+k$. Similarly, using lemma 5.1, there exists a PCD system $\mathcal{H}'_2$ that fully-recognizes the complement

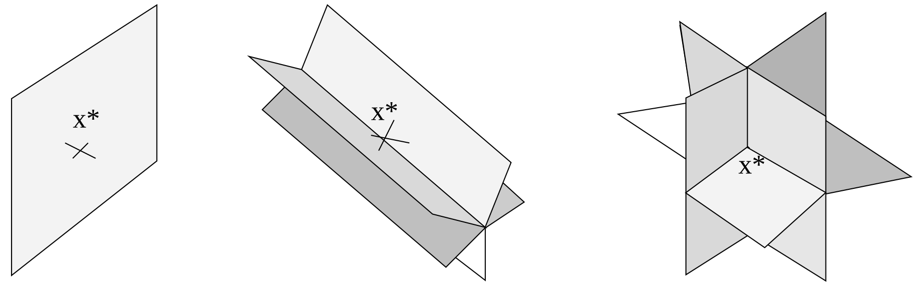

Figure 16: From left to right: x^* is of local dimension $1^+, 2^+, 3$ in a PCD system of dimension 3.

of L'' in dimension $2+k$. Now connect the output port of $\mathcal{H}_0$ and $\mathcal{H}_2$ to the input port of $\mathcal{H}'_1$. Connect the output port of $\mathcal{H}'_1$ to the input port of $\mathcal{H}_1$, the output port of $\mathcal{H}_1$ to the input port of $\mathcal{H}'_2$ and the output port of $\mathcal{H}'_2$ to the input port of $\mathcal{H}_2$: we get a PCD system in dimension $2+k$ that fully recognizes L. Its accepting point is the rejecting point of $\mathcal{H}'_1$ and its rejecting point is the rejecting point of $\mathcal{H}'_2$: see figure 15.

□

5.2 Upper bounds

5.2.1 Local dimension

We define:

Definition 5.2 (Local dimension) *Let $\mathcal{H} = (X, f)$ be a PCD system in dimension d. Let x^* be a point of X. Let Δ be a polyhedral subset $\Delta \subset X$ of maximal dimension $d - d'$ $(1 \leq d' \leq d)$ such that there exists an open convex polyhedron $V \subset X$, with $x^* \in \Delta \cap V$, and such that, for any region F of $\mathcal{H}$, $F \cap V \neq \emptyset$ implies $\Delta \subset \overline{F}$ ($\overline{F}$ is the topological closure of F). In other words Δ is the polyhedral subset of maximal dimension $d - d'$ such that there exists V that makes the hypothesis of proposition 3.1 to hold.*

If $d' < d$ then x^* is said to be of local dimension *d'^+. If $d' = d$ then x^* is said to be of* local dimension *d' and we can always choose V small enough such that x^* is the only point of local dimension d' in $\overline{V}$: see figure 16.*

Note that given a rational PCD system $\mathcal{H} = (X, f)$ and $k = d'$ or $k = d'^+$ one can effectively compute $LocDim(\mathcal{H}, k)$ defined as the set of the points $x \in X$ that have a local dimension equals to k.

The idea is that if a point x^* is of local dimension $(d')^+$ in a PCD of dimension d, to study the trajectories in a neighborhood of x^*, one can restrict the attention to a PCD system of dimension d'. Proposition 3.1 can be restated as:

Proposition 5.1 *Let $\mathcal{H} = (X, f)$ be a PCD system in dimension d. Let x^* be a point of local dimension $(d')^+$ with $d' < d$. Let P be the affine variety of dimension d' which is the orthogonal of Δ in x^*. It is possible to construct a PCD system $\mathcal{H}' = (X' = \mathbb{R}^{d'}, f')$ in dimension d' such that the trajectories of $\mathcal{H}'$ are the orthogonal projections on P of the trajectories of $\mathcal{H}$ in V.*

For any point x^*, the corresponding V is denoted by V_{x^*}. $\mathcal{H}'$, Δ are respectively denoted by $\mathcal{H}_{x^*}$ and Δ_{x^*}. If $d' < d$ we denote by p_{x^*} and q_{x^*} the functions that map all point $x \in X$ onto its orthogonal projection on P and onto its orthogonal projection on Δ respectively. If $d' = d$, we define p_{x^*} and q_{x^*} as respectively the identity function and the null function. We assume the natural order $1 < 1^+ < 2 < 2^+ < \ldots$.

Lemma 5.3 *Let $\mathcal{H} = (X, f)$ be a PCD system of dimension d. Let Φ be a trajectory of $\mathcal{H}$ that reaches x^* at finite continuous time T_{cont}. Assume that x^* is of local dimension $k = d'$ or $k = (d')^+$. For any l, denote by S_l the set of the points $x \in X$ that are reached by Φ at some time $0 \leq t < T_{cont}$ and that have local dimension l. Assume $S_l = \emptyset$, for all $l > k$.*

- *S_k is a finite set.*

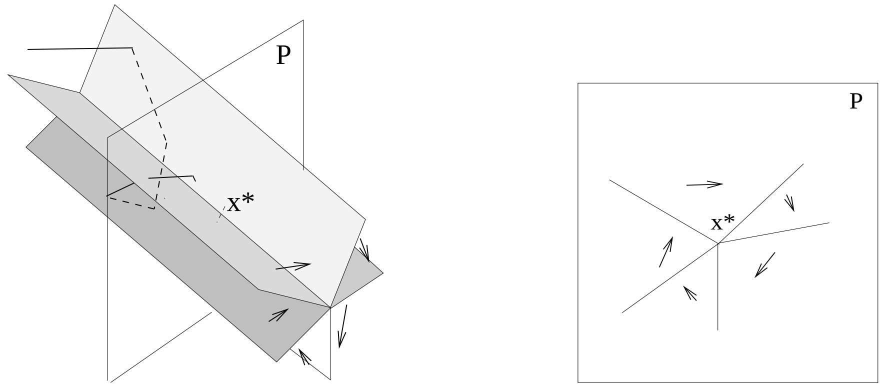

Figure 17: Proposition 3.1: if x^* is of local dimension 2+ in a PCD of dimension 3, the projections of the trajectories in neighborhood V of x^* on P are precisely the trajectories of a PCD system of dimension 2.

- *Assume $S_k = \emptyset$. Fix the origin in x^*. Then either $S_{(d'-1)^+}$ is a finite set or there exist $y_1, y_2 \in X$ that are reached by Φ, there exists $0 < \lambda < 1$ such that $p_{x^*}(y_2) = \lambda p_{x^*}(y_1)$ and such that, for all $n \geq 1$, Φ reaches at a time $t_n \leq T_{cont}$ the point y_n defined by $p_{x^*}(y_n) = \lambda^n p_{x^*}(y_1)$ and $q_{x^*}(y_n) = q_{x^*}(y_1) + \sum_{i=1}^{n} \lambda^i (q_{x^*}(y_2) - q_{x^*}(y_1))$.*

Proof: Let $m \leq k$. We prove first that if S_m is not a finite set, then Φ reaches a point of local dimension $> m$ at some time $\leq T_{cont}$: assume that S_m is not a finite set. $T_m = \{t|\Phi(t) \in S_m\}$ is a well ordered set. Denote its elements by $t_1^m, t_2^m, \ldots, t_\omega^m, \ldots$. Take $t_\infty^m = sup_{i \in \mathbb{N}} t_i^m$. We have $t_\infty^m \leq T_{cont}$. Consider $x_\infty^m = \Phi(t_\infty^m)$. By continuity of Φ, there exists $t^m < t_\infty^m$ such that $t \in [t^m, t_\infty^m] \Rightarrow \Phi(t) \in V_{x_\infty^m}$. Take $t \in [t^m, t_\infty^m] \cap S_m$. From considerations of dimensions about point $\Phi(t)$ of local dimension m in $V_{x_\infty^m}$, we get that the local dimension d'' of x_∞^m is $\geq m$. From the definition of t_∞^m, we get $d'' \neq m$. Hence $d'' > m$ and our claim is proved: if S_m is not a finite set then Φ reaches some x_∞^m of local dimension $> m$.

The first assertion of the lemma is an easy consequence of this claim with $m = k$.

For the second assertion, take $m = (d'-1)^+$, and assume that $S_{(d'-1)^+}$ is not a finite set. From $S_k = \emptyset$, we must have $x_\infty^m = x^*$ and $t_\infty^m = T_{cont}$. If $k < d$ denote $\mathcal{H}' = \mathcal{H}_{x^*}$ else take $\mathcal{H}' = \mathcal{H}$. Define Φ' as $p_{x^*}(\Phi)$. From time t^m up to time T_{cont}, Φ' is a trajectory of $\mathcal{H}' = (X', f')$ (apply proposition 3.1 for $k < d$), reaching $p_{x^*}(x^*)$ at time T_{cont}. Let $\mathcal{L}$ be the set of the one-dimensional regions of $\mathcal{H}'$ that intersect $V'_{x^*} = p_{x^*}(V_{x^*})$. We claim that each time Φ' reaches a point of $S_{(d'-1)^+}$, Φ' reaches an element of $\mathcal{L}$: if Φ' reaches some point $x^{*\prime} \in X'$ of local dimension $(d-1)^+$ at some time $t \in [t^m, T_{cont}]$, then $p_{x^*}(\Delta_{x^{*\prime}})$ is an element of $\mathcal{L}$ and contains $x^{*\prime}$. See figure 18.

Since Φ' converges to $p_{x^*}(x^*)$, since $\mathcal{L}$ is a finite set, since $S_{(d'-1)^+}$ is infinite, $p_{x^*}(\Phi)$ reaches two times the same element of $\mathcal{L}$ in $p_{x^*}(y_1)$ and $p_{x^*}(y_2)$ with $p_{x^*}(y_2) = \lambda p_{x^*}(y_1)$ for some $0 < \lambda < 1$, at some times t_{y_1}, t_{y_2} with $t^m \leq t_{y_1} < t_{y_2} < T_{cont}$. Now observe that by definition of V'_{x^*} all the regions of $\mathcal{H}'$ intersecting V'_{x^*} contain $p_{x^*}(x^*)$ in their topological closure. Hence we have $f'(x) = f'(\mu x)$, for all $x \in V'_{x^*}, \mu \in (0,1]$. If $\Phi'(t)$ is solution to differential equation $\dot{x}_d = f'(x)$, $\Psi'(t) = \lambda \Phi'(t/\lambda)$ is also solution. As a consequence trajectory Φ' must reach $\lambda^n p_{x^*}(y_1)$ for all n. From the definition of $\mathcal{H}'$ this implies that Φ reaches the y_n of the lemma for all n : see figure 18.

□

5.2.2 Problems *Reach* and *Conv*

We will deal with the following problems up to the end of this section:

Definition 5.3 (Problems $Reach_{d'}, Reach_{d'^+}$) *Let k be either of type $k = d'$ or of type $k = d'^+$, where d' is an integer.*

- Instance*: A purely rational PCD system $\mathcal{H} = (X, f)$ of dimension d, a polyhedral convex subset $V \subset X$, a rational polygon $x^1 \subset X$, a rational number $t_{sup} \in \mathbb{Q}$, a rational number $t_{inf} \in \mathbb{Q}$, a rational point $x_0 \in X$.*

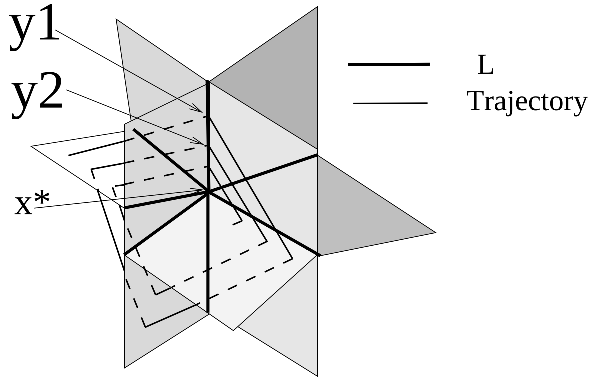

Figure 18: Proof of lemma 5.3: here $d = d' = 3$. $\mathcal{L}$ is defined as the set of the one dimensional regions that intersect $p_{x^*}(V_{x^*})$. $\mathcal{L}$ is made of a finite number of segments. Each time the trajectory reaches a point of local dimension 2^+, it reaches $\mathcal{L}$. If the trajectory reaches two times $\mathcal{L}$ in a same segment then the trajectory is ultimately cycling.

Question *"$Reach_k(\mathcal{H}, V, x_0, x^1, t_{inf}, t_{sup})$": "Do all the following conditions hold simultaneously:*

- *trajectory Φ starting from x^0 reaches x^1 at some finite continuous time T_{cont}*
- $t_{inf} < T_{cont} \leq t_{sup}$
- *for any $0 \leq t \leq T_{cont}$, $x = \Phi(t)$ is in V and is of local dimension $\leq k$."*

- Instance*: A purely rational PCD system $\mathcal{H} = (X, f)$ of dimension d, a polyhedral convex subset $V \subset X$, a rational point $x^* \in X$, a rational number $t_{sup} \in \mathbb{Q}$, a rational number $t_{inf} \in \mathbb{Q}$, a rational point $x_0 \in X$.*

  Question *"$Conv_k(\mathcal{H}, V, x_0, x^*, t_{inf}, t_{sup})$": "Do all the following conditions hold simultaneously:*

  - *the trajectory Φ starting from x_0 reaches point x^* at some finite continuous time T_{cont}*
  - *x^* is of local dimension k and is in V*
  - $t_{inf} < T_{cont} \leq t_{sup}$
  - *for any $0 \leq t < T_{cont}$, $x = \Phi(t)$ is in V and is of local dimension $< k$."*

5.2.3 Case $d = 3$

We start by some topological considerations:

Lemma 5.4 *Let S be a sphere of space $\mathbb{R}^d$ of radius r centered in x^*, with $d \geq 2$. Let t_0, t_1 be two reals. Then any injective continuous mapping $\Psi : [t_0, t_1) \rightarrow S$ is not surjective.*

Proof: For all $n \in \mathbb{N}$, denote $I_n = [t_0, t_1 - 1/n]$. Ψ realizes an homeomorphism between I_n and $J_n = \Psi(I_n)$ for all n, since Ψ is continuous and injective on compact subset I_n [16]. With two exceptions (the endpoints of I_n) the suppression of one point of I_n disconnect I_n, so the same must hold for J_n. As a consequence J_n must have an empty interior in the induced topology on S [16]. Now observe that the range J of Ψ is the enumerable union of the J_n, for $n \in \mathbb{N}$. By Baire theorem [16, 19], J has also an empty interior. Therefore, J can not be the whole sphere S.

□

Lemma 5.5 *Let $\mathcal{H} = (X, f)$ be a PCD system of dimension d. Let Φ be a trajectory of $\mathcal{H}$ of finite continuous time T_{cont} and discrete time $T_{discr} \geq \omega$ converging toward $x^* = \Phi(T_{cont})$. Assume that x^* is of local dimension $d' \leq 3^+$. Then necessarily the signature of Φ is ultimately cyclic.*

Proof: Assume first $d \leq 3$. Denote $t_1, t_2, \ldots, t_\omega, \ldots$ the elements of well ordered set $T_\Phi = \{t | \Phi$ crosses a boundary of a region at time $t\}$. Denote by d_0 the infimum of the distance of x^* to all the regions of $\mathcal{H}$ that are at a strictly positive distance of x^*. Take V as any convex open polyhedral included in the ball centered in x^* or radius $d_0/2$. Since Φ is converging to x^*, there exists t_{i_0} such that for all

$t_{i_0} \leq t \leq T_{cont}$, $\Phi(t) \in V$. Let $0 < \epsilon < d_0/4$. Denote by S the sphere centered in x^* of radius ϵ. Denote by θ the function that maps $x \in \mathbb{R}^3 - \{0\}$ to $\theta(x) = x^* + x^*x/\|x^*x\| * \epsilon$. θ is the central projection of $\mathbb{R}^3$ onto S.

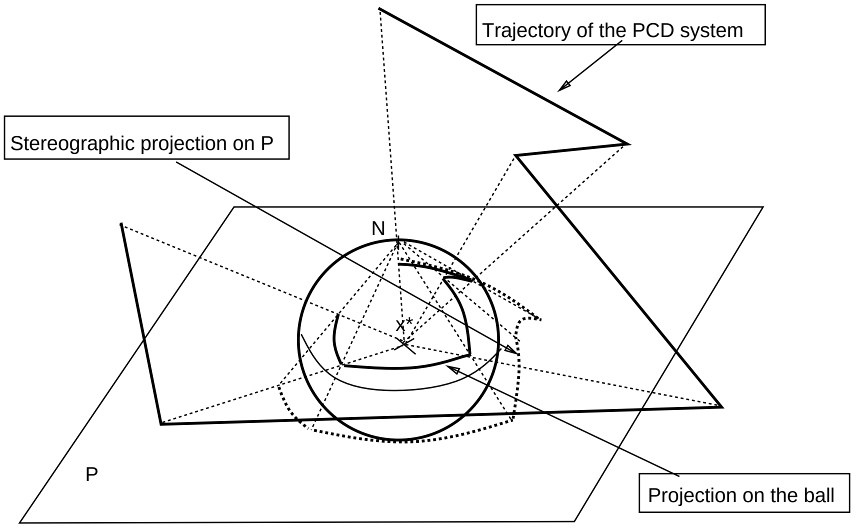

Figure 19: We first project the trajectory on sphere S by central projection θ. Then, we project the image by stereographic projection ψ onto plane P. We get a trajectory of a PCD like system $\mathcal{H}'$.

The key point is the following: from the definition of V, every region intersecting V contains x^* in its topological closure: two points $x, x' \in V$ with same image by θ belong necessarily to the same region of $\mathcal{H}$, and thus have same slope. As a consequence, the signature of trajectory Φ on $[t_{i_0}, T_{cont})$ is identical to the signature of trajectory $\Psi = \theta(\Phi)$.

If we assume that Ψ is not injective on $[t_{i_0}, T_{cont})$, Ψ is ultimately cyclic and thus have an ultimately cyclic signature. Assume now that Ψ is injective on $[t_{i_0}, t_\infty)$. By lemma 5.4, there exists $N \in S$ not in the range of Ψ. Let P the affine plane orthogonal to x^*N in x^*. It is known that sphere S minus point N is diffeomorphic to plane P. Denote by ψ the stereographic projection that sends $S - \{N\}$ to P in a diffeomorphic way: $\psi(x) = N - (Nx^*.N)/(Nx^*.Nx)Nx$. Now observe that $\psi(\Psi)$ is a trajectory of planar PCD like system $\mathcal{H}' = (P = \mathbb{R}^2, f')$ defined by $f'(x) = f(\psi^{-1}(x))$. As a consequence, by lemma 3.1, $\psi(\Psi)$ is ultimately either a contracting or an expanding spiral or cyclic. In any case, the signature of Ψ, hence the signature of Φ is ultimately cyclic: see figure 19.

Assume now that $d > 3$. By proposition 5.1, there must exist a PCD system $\mathcal{H}'$ of dimension 2 or 3 that has a projection Φ' of Φ as a finite continuous time trajectory of discrete time $T_{discr} \geq \omega$. The signature of Φ is given by the signature of Φ' which is ultimately cyclic by the case $d \leq 3$ applied on PCD $\mathcal{H}'$ and trajectory Φ'.

□

Following lemma is rather technical. Its proof is not very interesting and relies mostly on some computations about the convergence of the iterates of a certain linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ that describes the evolution of the trajectory after each cycle.

Lemma 5.6 *The following problem is decidable:*

Instance: a rational PCD system $\mathcal{H} = (X, f)$ of dimension d, a finite sequence of distinct regions $(F_0, F_1, \ldots, F_j)$ of $\mathcal{H}$, a rational point $x_0 \in X$.

Question: "Does the trajectory Φ starting from x_0 have a periodic signature of type $(F_0, F_1, \ldots, F_j)^\omega$ and then reach a point $x^* \in X$ of local dimension $d'' \leq 3^+$ at a finite continuous time t^*"

Moreover, given a positive instance, one can effectively computes t^* and x^* as a function of the coordinates of x_0.

Proof: Without loss of generality, by renaming of the regions, we can assume that F_0 has a dimension $d_0 < d$. Use the following algorithm:

Algorithm 2

1 compute $\Delta = \cap_{0,\ldots,j}\overline{F_j}$ and denote by d_Δ its dimension. Denote $d' = d - d_\Delta$ and $d'_0 = d_0 - d_\Delta$. **if** $\Delta = \emptyset$ **then** reply "false"

*/*the subsequence of the intersections of Φ with each F_k must converge toward x^*. Hence x^* must belong to Δ*/.*

2 **if** $d' > 3$ **then** reply "false"

/x^* must be of local dimension $d' \leq 3$.*/*

*/*We have at this point $d'_0 = 1$ or $d'_0 = 2$*/.*

3 Choose $x^*_0 \in \Delta$. Choose an affine basis of $\mathbb{R}^d$ of type $B = (x^*_0, e_1, \ldots, e_d)$ such that $(x^*_0, e_{d'+1}, \ldots, e_d)$ is an affine basis of Δ and $B' = (x^*_0, e_1, \ldots, e_{d'_0}, e_{d'+1}, \ldots, e_d)$ is an affine basis of F_0. For $x = (x_1, x_{d'_0}) \in \mathbb{R}^{d'_0}$, denote by Φ_x the trajectory of $\mathcal{H}$ that starts from point $x \in F_0$ of coordinates $(x_1, x_{d'_0}, 0, \ldots, 0)$ in basis B'. **if** Φ_x has a signature starting by $F_0, \ldots, F_j$ **then** denote by $(y_1, y_{d'_0}, y_{d'+1}, \ldots, y_d)$ the coordinates in basis B' of the next intersection of Φ_x with F_0 and by T_x the time of this intersection. Compute the coefficients of the linear maps $A : \mathbb{R}^{d'_0} \to \mathbb{R}^{d'_0}$, $T : \mathbb{R}^{d'_0} \to \mathbb{R}$, $Off_k : \mathbb{R}^{d'_0} \to \mathbb{R}$ for $k \in \{d'+1, \ldots, d\}$ that respectively map $x = (x_1, x_{d'_0}) \in \mathbb{R}^{d'_0}$ to $(y_1, y_{d'_0})$, T_x and to y_k.
Denote by $x'_0 = (x^0_1, x^0_{d'_0}, x^0_{d'+1}, \ldots, x^0_d)$ the coordinates of x_0 in basis B.

4 Check that the sequence of the iterates of A given by $(x^n_1, x^n_{d'_0}) = A^n(x^0_1, x^0_{d'_0})$, is converging to 0 when n tends to infinity as follows: if A has a real or complex eigen value λ with $|\lambda| > 1$ and if x'_0 is not a eigen vector of A with eigen value $|\lambda'| < 1$ return "false"

*/*for 2×2 matrices all these tests are easily computable. The determinant of A is always non-negative from lemma 4.1*/.*

5 Compute $x^* = (x^*_1, \ldots, x^*_d) \in \mathbb{R}^d$ by $x^*_i = 0$, for $i \leq d'$, $x^*_i = x^0_i + \Sigma^\infty_{j=0} Off_i(x^j_1, x^j_{d'_0})$. Compute $t^* \in \mathbb{R}$ by $t^* = \Sigma^\infty_{j=0} T(x^j_1, x^j_{d'_0})$

*/*If the signature of Φ is really $(F_0, F_1, \ldots, F_j)^\omega$ then Φ must converge at time t^* to the point x^* with these coordinates in basis B . These series are always computable by working in a basis where A is tridiagonal and by using the identities $\Sigma\lambda^n = 1/(1-\lambda)$ and $\Sigma n\lambda^{n-1} = 1/(1-\lambda)^2$). Check that all the x^*_i, $1 \leq i \leq d$, are rational numbers, and that t^* is a rational number. */*

6 Compute the subset $Out \subset F_0$ of points $z \in \mathbb{R}^d$ such that the trajectory starting from z has not a signature starting by $F_0, \ldots, F_j, F_0, \ldots, F_j$

*/*Out is made of an union of polyhedral $O_1, \ldots, O_m$.*/*

7 Compute d'' the local dimension of x^*. **if** $d'' > 3^+$ **then** return "false"

8 Choose a local dimension neighborhood V for x^* that is also included in the ball centered in x^* of radius $\inf\{d(x^*, O_j) | j \in \{1, \ldots, m\} \wedge d(x^*, O_j) > 0\}$. Compute n_0 such that $n \geq n_0 \Rightarrow (x^n_1, x^n_{d'_0}) \in V$. If the signature of Φ does not start by $(F_0, \ldots, F_j)^{n_0} F_0$ **then** return "false".

9 Construct $\mathcal{H}' = (X', f')$ as a local dimension PCD system for x^*: that is, $\mathcal{H}' = \mathcal{H}_{x^*}$. For $k \in \{0, \ldots, j\}$, let $F'_k = F_k \cap X'$ and d''_k the dimension of F'_k. **if** there is a $i \in \{0, \ldots, j\}$ with $d''_i = 1$, and the signature of Φ starts by $F_0, \ldots, F_j$ **then** return "true"

*/*in this case, Φ will always intersect F_0 on segment $[x_0, x^*]$ and we will never reach Out by definition of V.*/*

*/*We are now sure that $d'' = 3$, and $d_i = 2$ for $i \in \{0, \ldots, j\}$*/.*

10 Choose an affine basis B'' of F'_0 of type $B'' = (x^*, e_1, e_2)$.

11 **if** there exists $(x_1, x_2) \in F'_0 - \{x^*\}$ with $(-x_1, -x_2) \in F'_0$ **then** return "false"

*/*by convexity of F_0 that would imply that x^* is in the topological interior of F_0 and that at some rank big enough, F_0 would delimit the space in two, and that it would be impossible to get from one half of space to the other.*/*

12 Compute $Out' \subset \mathbb{R}$ by $Out' = \{z = x_1/x_2 | (x_1, x_2) \in F'_0 \cap Out\}$

*/*since all the regions F_k that intersect V have x^* in their topological cloture, testing if (x_1, x_2) is in Out can be done by only considering the ratio x_1/x_2*/.*

Computes the limit z^* of the ratio $z^n = {x'_1}^{2n}/{x'_2}^{2n}$ where $({x'_1}^n, {x'_2}^n)$ is the sequence of the coordinates in basis B'' of the intersections of Φ with F'_0

*/*As before, $({x'_1}^{n+1}, {x'_2}^{n+1}) = A'({x'_1}^n, {x'_2}^n)$ for some linear map A'. Since the determinant of A' is non-negative this limit always exists and z^n is either decreasing or increasing*/*

13 **if** z^* is in the topological interior of Out' **then** return "false".

14 **if** z^n is an increasing (respectively: decreasing) subsequence **then** choose $\epsilon > 0$ such that interval $I = (z^* - \epsilon, z^*) \subset Out'$ (resp: $I = (z^*, z^* + \epsilon) \subset Out'$). Compute n_1 such that $n \geq n_1 \Rightarrow z^n \in I$. Check that Φ has a signature starting by $(F_0, \ldots, F_j)^{n_1}$. **if** it so **then** return "true" **else** return "false".

□

With these lemmas, we prove:

Theorem 5.2 *The problems $Reach_3$ and $Reach_{3^+}$ are in Σ_1.*

Proof: $Reach_{3^+}(\mathcal{H}, V, x_0, x^1, t_{inf}, t_{sup})$ is answered by the following algorithm:

Algorithm 3

---

1 Compute V' as the union of the complement of V, of $NoEvolution(\mathcal{H})$ and of the union of all the $LocDim(\mathcal{H}, k)$ for all $k > 3^+$.

2 Compute $x^{1'}$ (respectively: $x^{1''}$) made of the union of all the points $x \in X$ that are mapped to x^1 (resp. to V') in discrete time less than 1.

/$x^{1'}$ and $x^{1''}$ are made of an union of polygons.*/*

3 If $t_{inf} < 0$ or $t_{sup} < t_{inf}$ **then** return "false"

4 $t_0 := 0; i := 0$.

5 **while** $x_i \notin x^{1''} \wedge (x_i \notin x^{1'} \vee t_i \leq t_{inf})$ **do**

6 $i = i + 1$.

7 Compute x_{i+1} and t_{i+1} corresponding respectively to the first intersection of Φ_{i+1} with a region of $\mathcal{H}$ and to the time of this intersection, where Φ_{i+1} is the trajectory starting from x_i at time t_i. Denote the region intersected by F_i.

8 **if** $t_i > t_{sup}$ **then** stop and return "false".

9 If $i' = \sup\{j < i | F_j = F_i\}$ exists **then** using lemma 5.6, test if the trajectory Φ_{i+1} has a cyclic signature of type $(F_{i'+1} \cap (x^{1''} \cup x^{1'})^c, F_{i'+2} \cap (x^{1''} \cup x^{1'})^c, \ldots, F_i \cap (x^{1''} \cup x^{1'})^c)^\omega$ and then reach a point x^* at a finite continuous time t^*. If it is so and $t^* \leq t_{sup} - t_i$, set $x_i = x^*$ and $t_i = t^*$.

10 **end while**

11 **if** $x_i \in x^{1'} \wedge \forall k > 3^+ x_i \notin LocDim(\mathcal{H}, k)$ **then** return "true" **else** return "false".

By lemma 5.5 each point x^* reached by Φ of local dimension less than 3^+ is reached by an ultimately cyclic signature. Hence each such point x^* is detected and dropped by the algorithm. As a consequence, if the answer should be positive, the algorithm always halt with a correct answer. Assume that this algorithm does not halt: take $t^* = sup_i t_i$. We have $t^* \leq t_{sup}$. By continuity of Φ, Φ reaches at time t^* the point $x^* = \Phi(t^*)$. x^* is necessary of local dimension $d' > 3^+$ since if it would not be so, x^* would be dropped by the algorithm. Hence the answer should be negative and the algorithm is correct.

□

5.2.4 Case $d \geq 4$

We generalize theorem 5.2 to higher dimensions. We prove first:

Lemma 5.7 *Let $d' \geq 4$. Assume that $Reach_{(d'-1)^+} \in \Sigma_p$ and that $Reach_{(d'-2)^+} \in \Sigma_q$ for some integers p, q. Then*

- *$Conv_{d'} \in \Sigma_{max(p,q+2)}$.*
- *$Conv_{d'^+} \in \Sigma_{max(p,q+2)}$.*

Proof:

Denote by $B(x^*, 1/n_1)$ the ball of radius $1/n_1$ centered in x^* for the norm of $\mathbb{R}^d$ defined by $d(x,y) = max_i|x_i - y_i|$. For a subset $U \subset X$, denote its complement by U^c. Let $k = d'$ or $k = {d'}^+$. We claim:

$$
\begin{array}{l}
Conv_k(\mathcal{H}, V, x_0, x^*, t_{inf}, t_{sup}) \\
\Leftrightarrow \quad x^* \in LocDim(\mathcal{H}, k) \ \wedge \ x^* \in V \ \wedge \ t_{inf} < t_{sup} \\
\quad \wedge \quad \exists y_1 \in \mathbb{Q}^d \ \exists t_1, t_2 \in \mathbb{Q} \ y_1 \in V_{x^*} \ \wedge \ Reach_{(d'-1)^+}(\mathcal{H}, V, x_0, y_1, t_1, t_2) \\
\quad \wedge \left\{ \begin{array}{l}
\quad \exists y_2 \in \mathbb{Q}^d \ \exists t_3, t_4 \in Q \ \exists \lambda \in \mathbb{R}^+ \\
\quad \left\{ \begin{array}{l}
Reach_{(d'-1)^+}(\mathcal{H}, V \cap V_{x^*}, y_1, y_2, t_3, t_4) \\
p_{x^*}(y_2) = \lambda p_{x^*}(y_1) \\
\lambda < 1 \\
t_1 + \sum_{i=1}^{\infty} \lambda^i t_3 > t_{inf} \\
t_2 + \sum_{i=1}^{\infty} \lambda^i t_4 \leq t_{sup} \\
q_{x^*}(y_1) + \sum_{i=1}^{\infty} \lambda^i (q_{x^*}(y_2) - q_{x^*}(y_1)) = q_{x^*}(x^*)
\end{array} \right. \\
\vee \quad \forall n_1 \in \mathbb{N} \\
\quad Reach_{(d'-2)^+}(\mathcal{H}, V, y_1, B(x^*, 1/n_1), t_{inf} - t_1, t_{sup} - t_2)
\end{array} \right.
\end{array}
$$

Assume that we have a positive instance to formula $Conv_k$: use the notations of definition 5.3. Denote by S the set of the points that are reached by Φ before time T_{cont} and that have a local dimension $(d'-1)^+$. Since Φ converges to x^*, there must exist an $y_1 = \Phi(t_{y_1}) \in V_{x^*}$, $t_{y_1} < T_{cont}$ that is reached by Φ, and such that Φ stays in V_{x^*} between time t_{y_1} and time T_{cont}. y_1 is reached using only points of local dimension $\leq (d'-1)^+$. If S is not a finite set, by lemma 5.3 the first clause of the disjunction is true. Assume now that S is a finite set: we can assume that t_{y_1} is chosen big enough such that Φ does not reach any point of S between time t_{y_1} and time T_{cont}. For all $n_1 \in \mathbb{N}$ we get that the trajectory starting from y_1 reaches $B(x_*, 1/n_1)$ using only points of local dimension $\leq (d'-2)^+$. Hence the second clause of the disjunction is true.

Conversely, assume that the right hand side of the formula is true. If the first clause of the disjunction is true, it is clear than the formula $Conv_k$ should be true. Assume now that the second clause is true. For all n_1, we get that there exists t_{n_1} such that $\Phi(t_{n_1}) \in B(x^*, 1/n_1)$. Denote $T_{cont} = sup_{n_1 \in \mathbb{N}} t_{n_1}$. From the continuity of Φ we get that $\Phi(T_{cont}) = x^*$. Hence Φ reaches x^* of local dimension k and formula $Conv_k$ must be true.

The result is now immediate by applying the Tarski-Kuratowski algorithm on the formula [18].

□

We also prove:

Lemma 5.8 *Let $d' \geq 4$. Assume $Reach_{(d'-1)^+} \in \Sigma_p$ for some integer p. Then $Conv_{d'} \in \Sigma_{p+1}$.*

Proof: For a point $x^* \in X$ of local dimension d, define Out_{x^*} as the set of the points $x \in X$ such that the trajectory starting from x intersects the complement of V_{x^*} at a discrete time less or equal to one. We claim:

$$
\begin{array}{l}
Conv_{d'}(\mathcal{H}, V, x_0, x^*, t_{inf}, t_{sup}) \\
\Leftrightarrow \quad x^* \in LocDim(\mathcal{H}, k) \\
\qquad \wedge \ x^* \in V \ \wedge \ t_{inf} < t_{sup} \ \wedge \ dimension(\mathcal{H}) = d' \\
\quad \wedge \quad \exists y_1 \in \mathbb{Q}^d \ \exists t_1, t_2 \in \mathbb{Q} \ y_1 \in V_{x^*} \ \wedge \ Reach_{(d'-1)^+}(\mathcal{H}, V, x_0, y_1, t_1, t_2) \\
\quad \left\{ \begin{array}{ll}
 & Reach_{(d'-1)^+}(\mathcal{H}, X, y_1, X, t_{inf} - t_1, t_{inf} - t_1 + 1) \\
\wedge & \neg Reach_{(d'-1)^+}(\mathcal{H}, X, y_1, V^c, 0, t_{sup} - t_2) \\
\wedge & \neg Reach_{(d'-1)^+}(\mathcal{H}, X, y_1, NoEvolution(\mathcal{H}), 0, t_{sup} - t_2) \\
\wedge & \neg Reach_{(d'-1)^+}(\mathcal{H}, X, y_1, Out_{x^*}, 0, t_{sup} - t_2) \\
\wedge & \neg Reach_{(d'-1)^+}(\mathcal{H}, X, y_1, X, t_{sup} - t_2, t_{sup} - t_2 + 1)
\end{array} \right.
\end{array}
$$

Assume that we have a positive instance to formula $Conv_k$: use the notations of definition 5.3. Φ reaches V_{x^*} and does not reach $V^c \cup$ $NoEvolution(\mathcal{H}) \cup Out_{x^*}$. Φ must reach some points at a time $t > t_{inf}$ but does not reach any point at a time $t > t_{sup}$ using only points of local dimension $\leq (d'-1)^+$. Hence the right hand side of the formula is true.

Conversely, assume that the right hand side of the formula is true. That means that the dimension of $\mathcal{H}$ is equals to d'. Define T_{cont} as the greatest t such that $Reach_{(d'-1)^+}(\mathcal{H}, X, x_0, X, t, t+1)$ is true. Since Φ is a continuous function, Φ reaches $x^{*\prime} = \Phi(T_{cont})$. Now observe that, by definition, $x^{*\prime}$ is necessarily a point of local dimension $> (d'-1)^+$. Now, since Φ does not reach Out_{x^*}, Φ must stay in $\overline{V_{x^*}}$. By definition x^* is the only point of local dimension $> (d'-1)^+$ in $\overline{V_{x^*}}$. Hence $x^{*\prime} = x^*$. We have clearly $t_{inf} < T_{cont} \leq t_{sup}$. As a consequence, $Conv_{d'}$ should be true.

The result is immediate by applying the Tarski-Kuratowski algorithm on the formula [18].

□

We get:

Theorem 5.3 *Let $d' \geq 3$.*

- $Reach_{d'}$ *is in* $\Sigma_{d'-2}$.
- $Reach_{d'+}$ *is in* $\Sigma_{d'-1}$ *if* d' *is even.*
- $Reach_{d'+}$ *is in* $\Sigma_{d'-2}$ *if* d' *is odd.*

Proof: We prove the assertion by recurrence over d': cases $d' = 2, 3$ are theorem 5.2. Assume $d' > 4$ and assume the hypothesis at rank d'' for all $d'' < d'$. From lemma 5.3 we have, for $k = d'$ or $k = d'^+$:

$$\begin{array}{l} Reach_k(\mathcal{H}, V, x_0, x^1, t_{inf}, t_{sup}) \\ \Leftrightarrow \quad Reach_{(d'-1)^+}(\mathcal{H}, V, x_0, x^1, t_{inf}, t_{sup}) \\ \qquad \left\{ \begin{array}{l} x_0^* = x_0 \\ \forall 0 \leq i < n \; Conv_k(\mathcal{H}, V, x_i^*, x_{i+1}^*, t_i, t_i') \\ Reach_{(d'-1)^+}(\mathcal{H}, V, x_n^*, x^1, t_n, t_n') \\ t_0 + t_1 + \ldots + t_n > t_{inf} \\ t_0' + t_2' + \ldots + t_n' \leq t_{sup} \end{array} \right. \end{array}$$

The result is immediate by applying the Tarski-Kuratowski algorithm on this formula since

- If d' is even, $Reach_{d'} \in \Sigma_{d'-2}$ from lemma 5.8 and the hypothesis at rank $d' - 1$, and $Reach_{d'+} \in \Sigma_{d'-1}$ from lemma 5.7 and the hypothesis at ranks $d' - 1$ and $d' - 2$.
- If d' is odd, $Reach_{d'} \in \Sigma_{d'-2}$ and $Reach_{d'+} \in \Sigma_{d'-2}$ from lemma 5.7 and the hypothesis at ranks $d' - 1$ and $d' - 2$.

□

We get the main result of this section:

Corollary 5.1

- *If L is semi-recognized by a purely rational PCD system of dimension d, then $L \in \Sigma_{d-2}$.*
- *If L is recognized by a purely rational PCD system of dimension d, then $L \in \Delta_{d-2}$.*

Proof: With the notations of definition 2.3, we have

$$n \in L \Leftrightarrow \exists t_1 \in \mathbb{N} \; Reach_d(\mathcal{H}, X, r(n), x^1, 0, t_1)$$

The second assertion is immediate by considering the complement of L.

□

An by using theorem 5.1:

Corollary 5.2

- *The languages that are semi-recognized by purely rational PCD systems of dimension d in finite continuous time are precisely the languages of Σ_{d-2}*
- *The languages that are recognized by purely rational PCD systems of dimension d in finite continuous time are precisely the languages of Δ_{d-2}*

References

[1] Eugene Asarin and Oded Maler. On some Relations between Dynamical Systems and Transition Systems. In *Proceedings of ICALP*, pages 59–72, 1994. Lecture Notes in Computer Science, 820.

[2] Eugene Asarin and Oded Maler. Achilles and the Tortoise Climbing Up the Arithmetical Hierarchy. In *Proceedings of FSTTCS*, pages 471–483, 1995. Lecture Notes in Computer Science, 1026.

[3] Eugene Asarin, Oded Maler, and Amir Pnueli. Reachability analysis of dynamical systems having piecewise-constant derivatives. *Theoretical Computer Science*, 138:33–65, 1995.

[4] J. L. Balcázar, J. Diáz, and J. Gabarró. *Structural Complexity I & II.* EATCS Monographs on Theoretical Computer Science, 1988.

[5] Lenore Blum, Mike Shub, and Steve Smale. On a Theory of Computation and Complexity over the Real Numbers: NP-completeness, Recursive Functions and Universal Machines. *Bulletin of the American Mathematical Society*, 21(1):1–46, July 1989.

[6] Olivier Bournez. Achilles and the tortoise climbing up the hyperarithmetical hierachy. Technical report, LIP ENS-Lyon, 1997.

[7] Olivier Bournez. Some bounds on the computational power of piecewise constant derivative systems. In *Proceedings of ICALP'97*, pages 143–153, 1997. Lecture Notes in Computer Science, 1256.

[8] Olivier Bournez and Michel Cosnard. On the computational power of hybrid and dynamical systems. *Theoretical Computer Science*, 168(2):417–459, 1996.

[9] Michael S. Branicky. Universal computation and other capabilities of hybrid and continuous dynamical systems. *Theoretical Computer Science*, 138:67–100, 1995.

[10] Morris W. Hirsch and Stephen Smale. *Differential Equations, Dynamical Systems and Linear Algebra.* Accademic Press, San Diego, CA, 1974.

[11] P. Koiran. Computing over the reals with addition and order. *Theoretical Computer Science*, 133:35–47, 1994.

[12] Pascal Koiran. A Weak Version of the Blum Shub Smale Model. Technical Report 005, NeuroColt Technical Report Series, August 1994. A preliminary version can be found in *Proceedings of 34th IEEE Symposium on Foundations of Computer Science*, pages 486-495, 1993.

[13] K. Meer and C. Michaux. A Survey on real Structural Complexity Theory. To be published in *Bulletin of the Belgian Mathematical Society - Simon Stevin.*

[14] Klaus Meer. A note on a $P \neq NP$ Result for a Restricted Class of Real Machines. *Journal of Complexity*, 8:451–453, 1992.

[15] Cristopher Moore. Recursion theory on the reals and continuous-time computation. *Theoretical Computer Science*, 162:23–44, 1996.

[16] M. H. A. Newman. *Elements of the Topology of Plane Sets of points.* Greenwood Press, Publishers, 1985.

[17] P. Odifreddi. *Classical Recursion Theory*, volume 125 of *Studies in Logic and the foundations of mathematics.* Elsevier, 1992.

[18] H. Rogers. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, 1967.

[19] Walter Rudin. *Real and Complex Analysis, 3rd edition.* McGraw Hills, 1987.