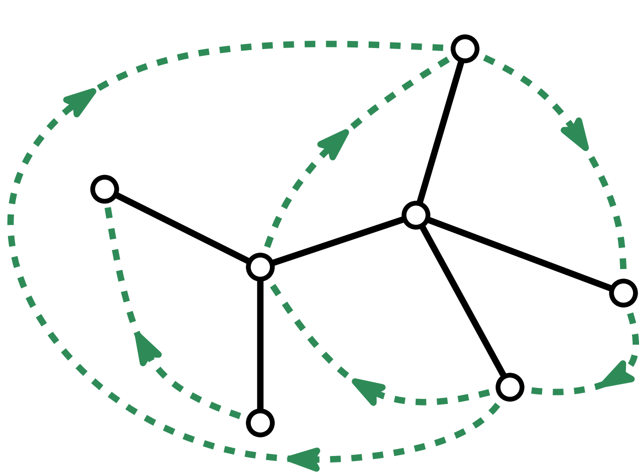
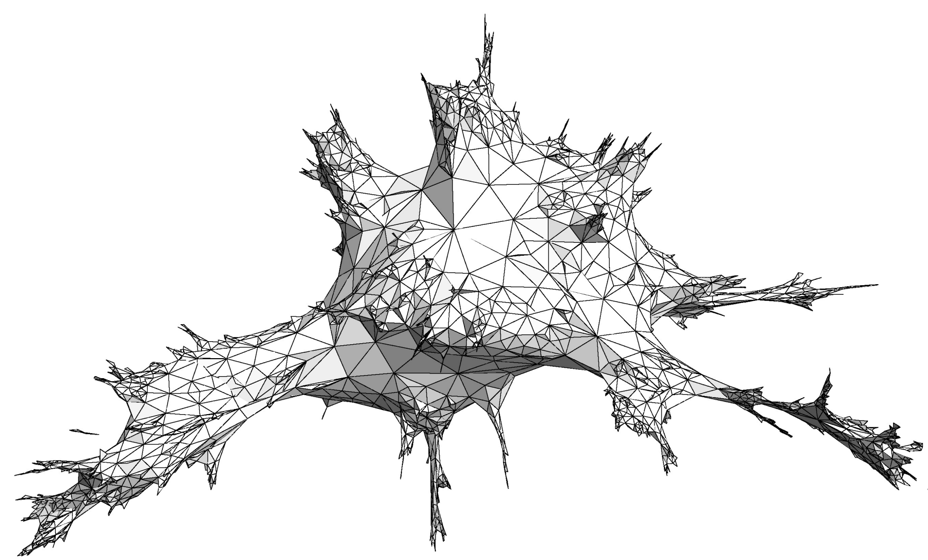
Maps: at the interface between combinatorics and probability

Marie Albenque (CNRS, LIX, École Polytechnique)

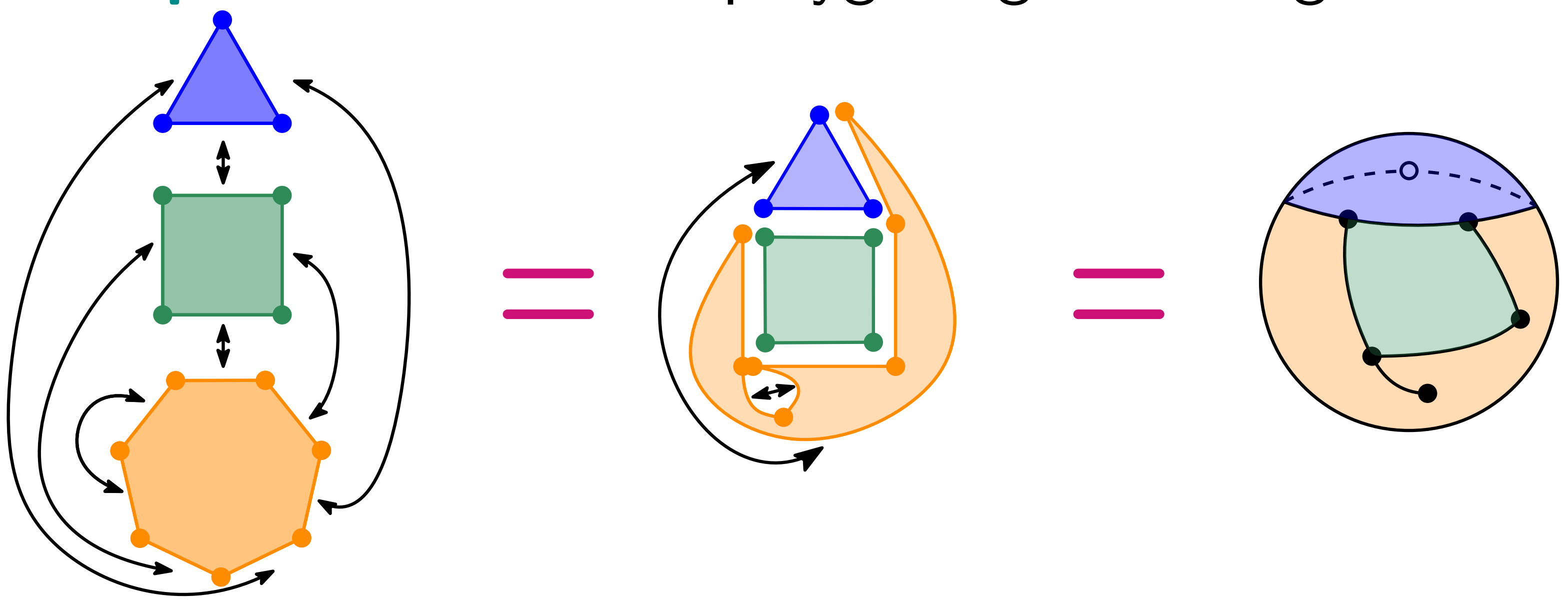




Soutenance d'habilitation, 16 décembre 2020

Maps – Definition(s)

A **map** is a collection of polygons glued along their sides (with some technical conditions).



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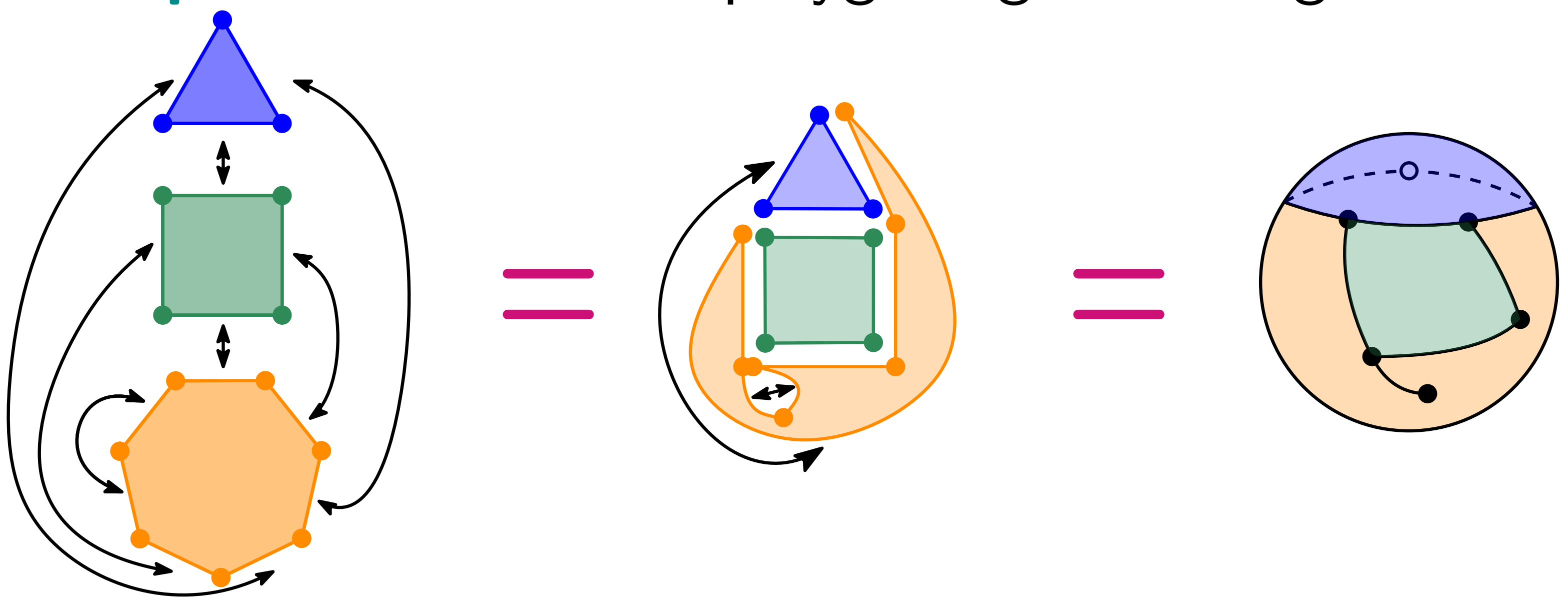

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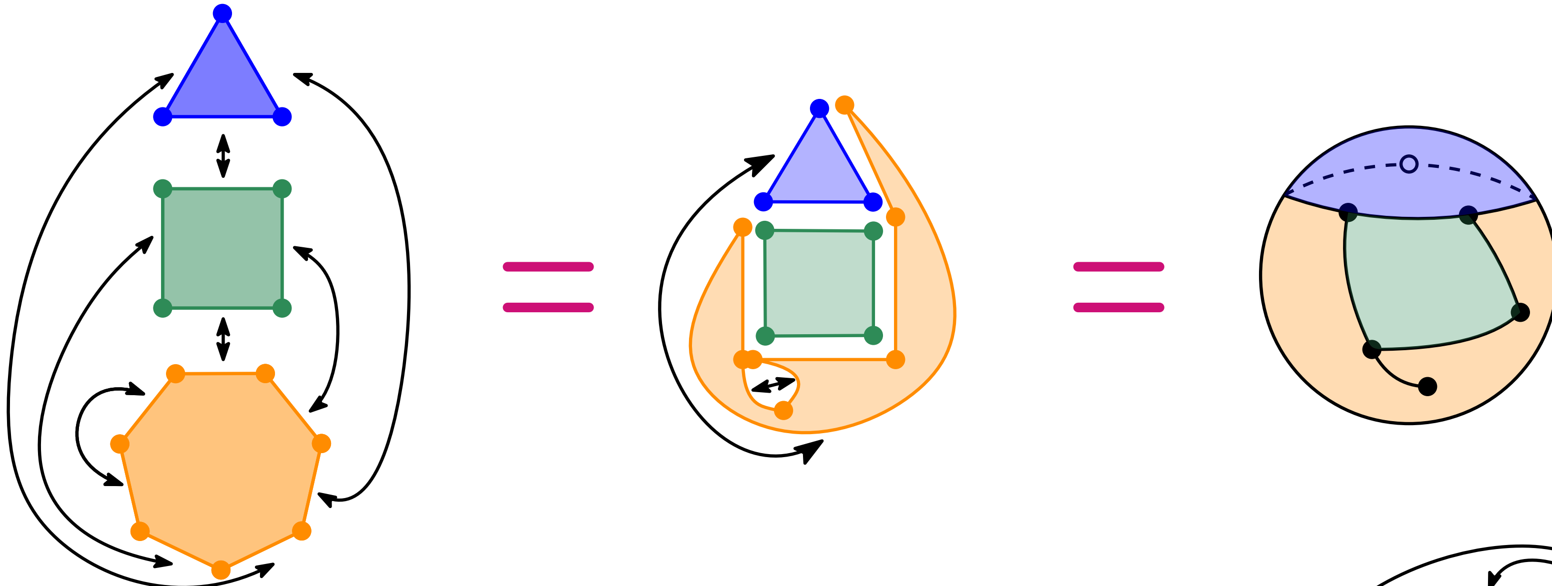

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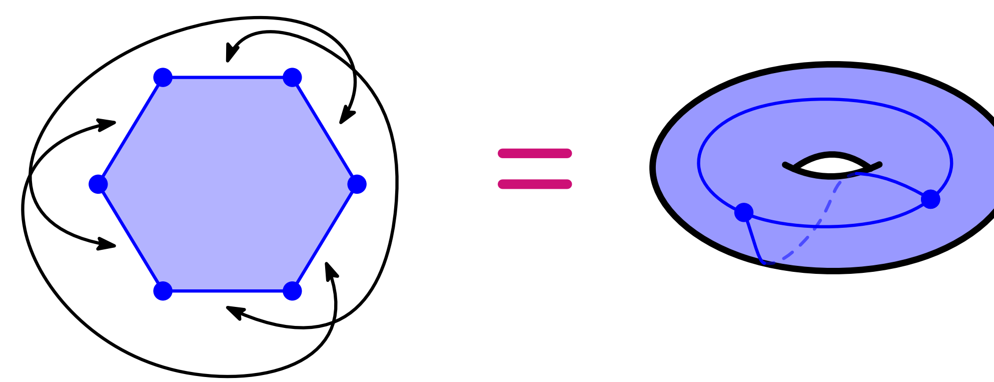
We will also encounter maps on other closed orientable surfaces: torus of genus g, disks, ...



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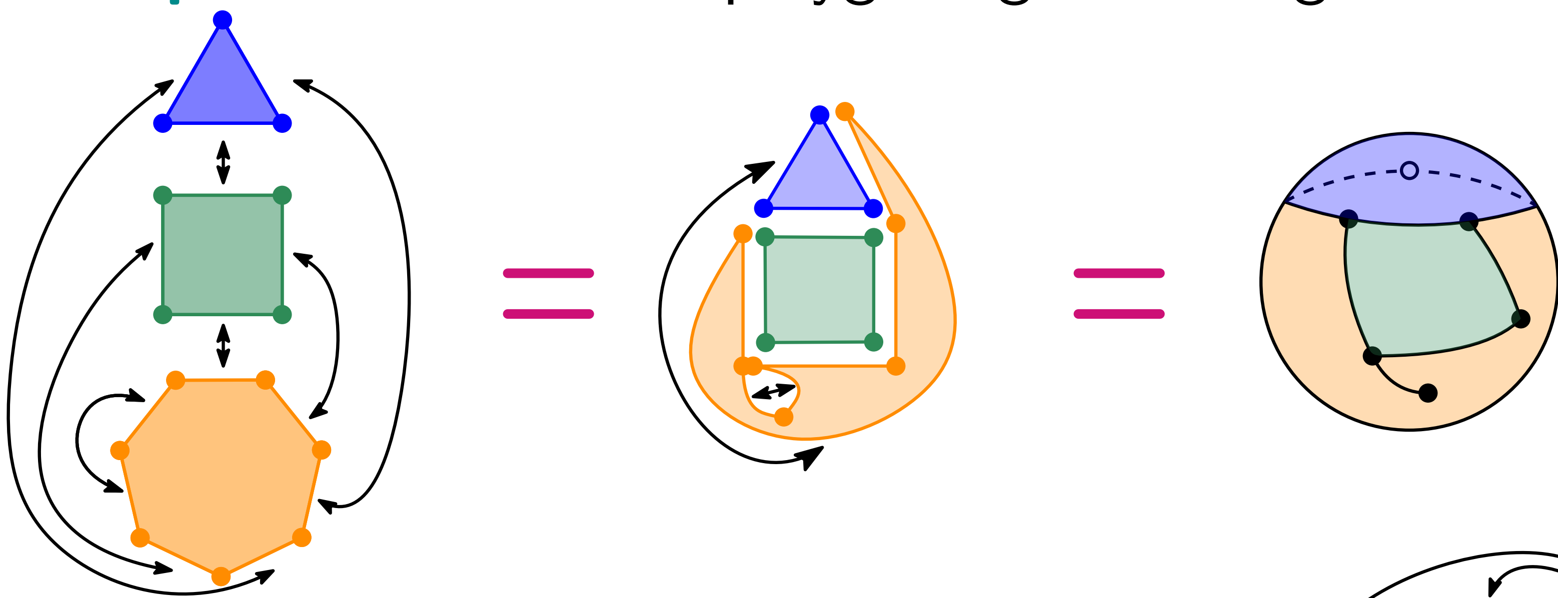

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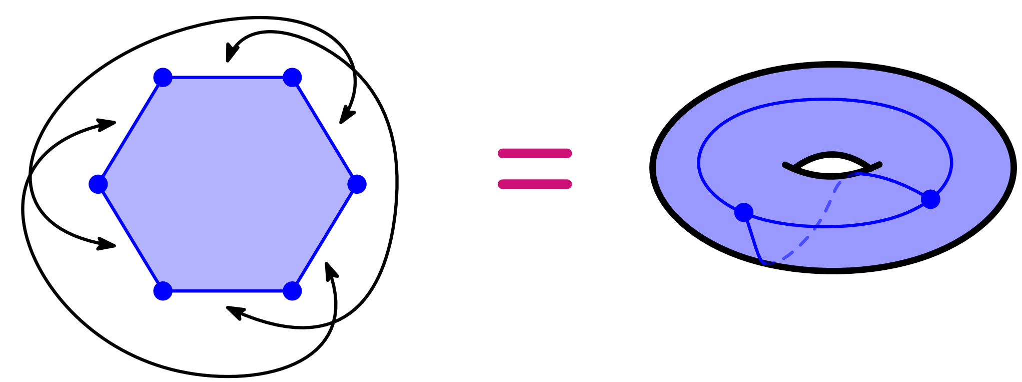

Euler's formula: for every map m (on a closed surface without boundary),

$$\boxed{|V(m)| + |F(m)| = 2 + |E(m)| - 2g(m)}$$

vertices faces edges genus

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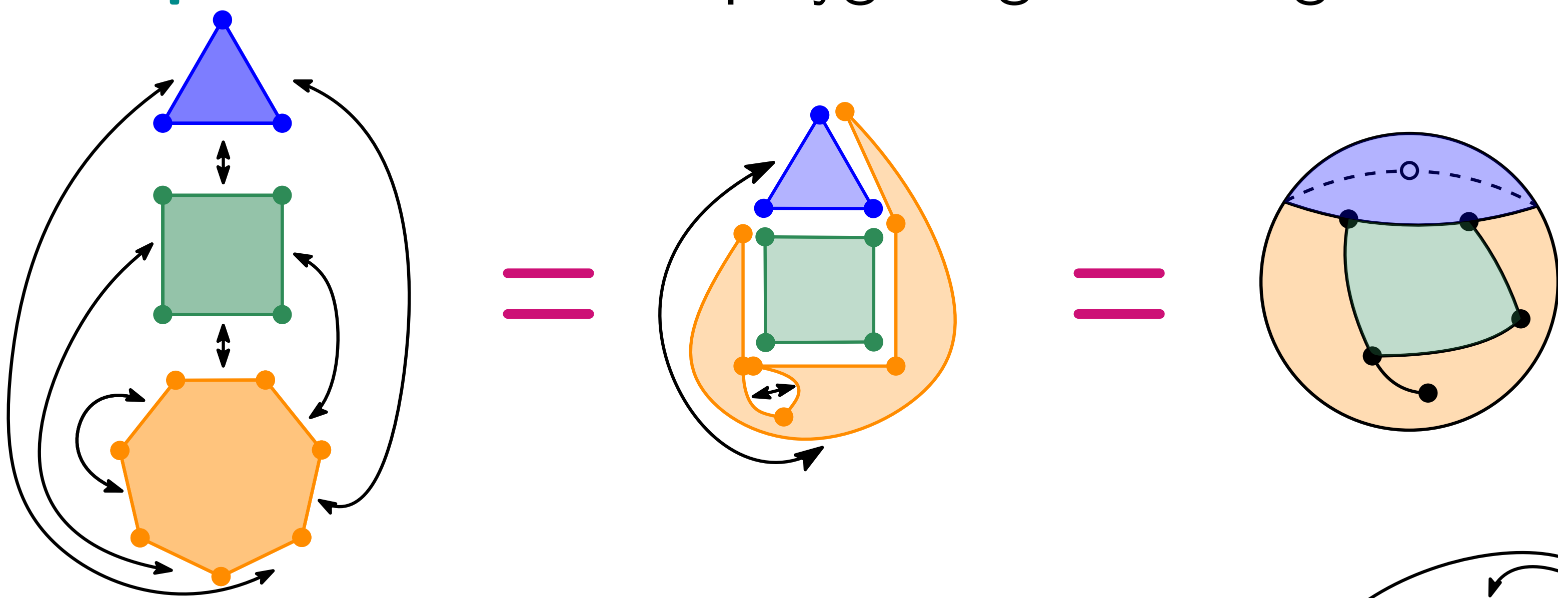

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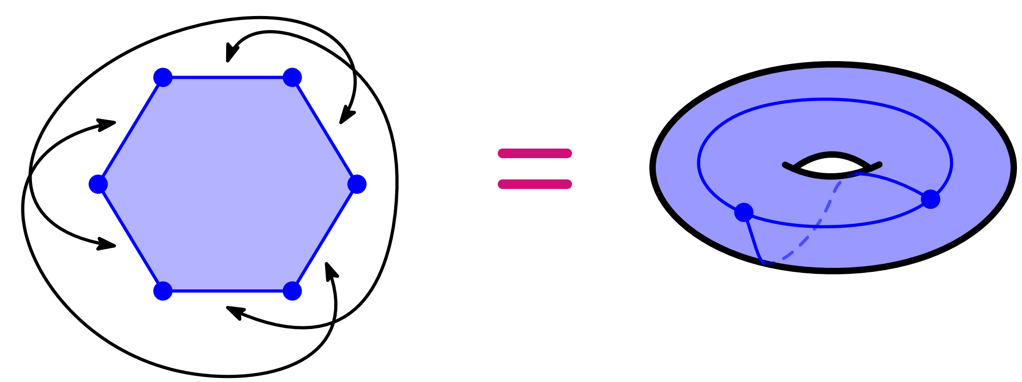

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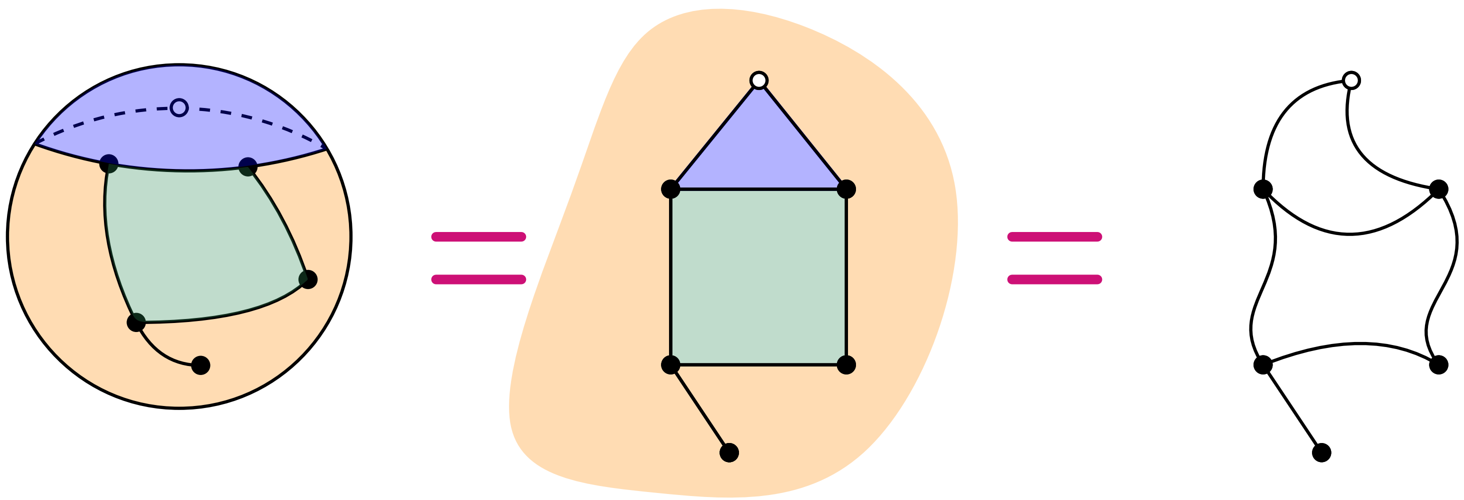
If all the polygons have p sides, the map is called a **p-angulation**

3-angulation = **triangulation**, 4-angulation = **quadrangulation**

Maps – Definition(s)

A **planar map** is a proper embedding of a planar connected graph in the 2-dimensional sphere (considered up to orientation-preserving homeomorphisms).



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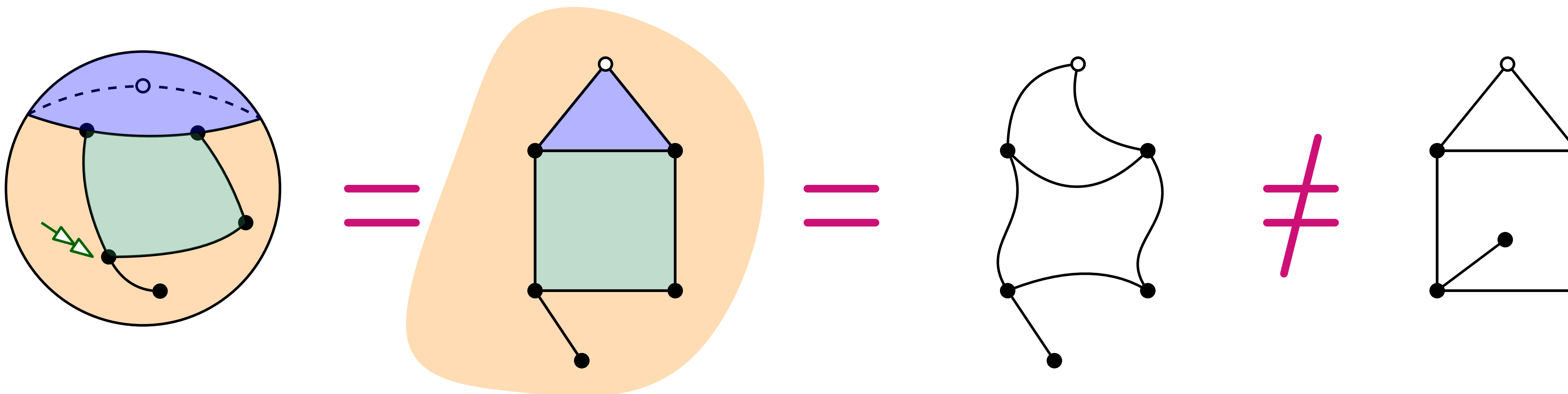

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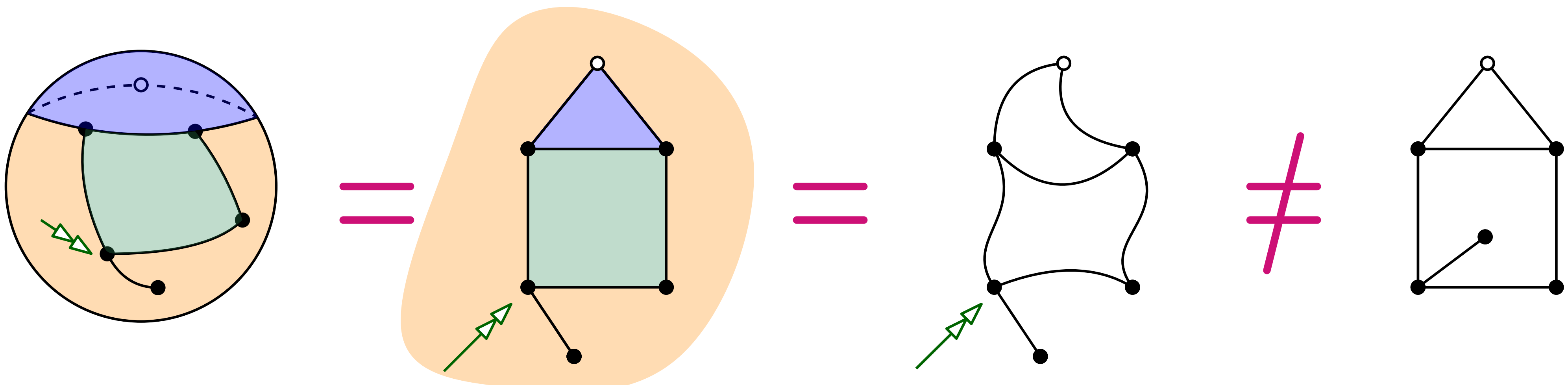

This is the root corner

Planar map = planar graph + cyclic order of edges around each vertex.

To avoid dealing with symmetries: maps are **rooted** (a corner is marked).

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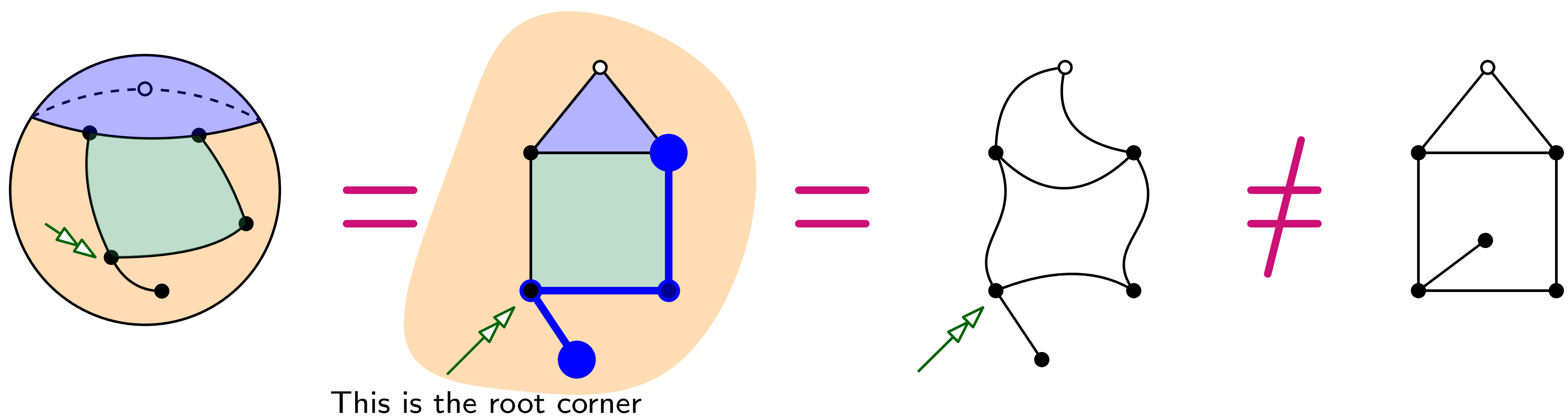

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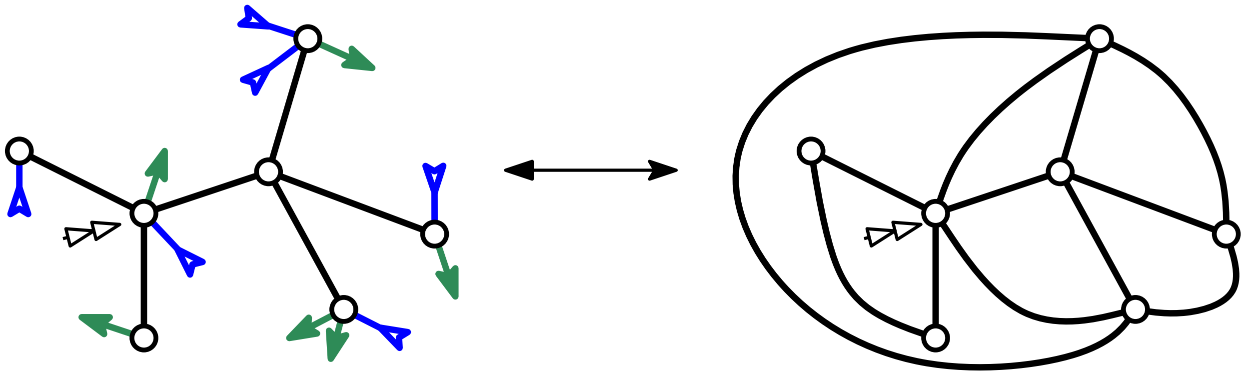
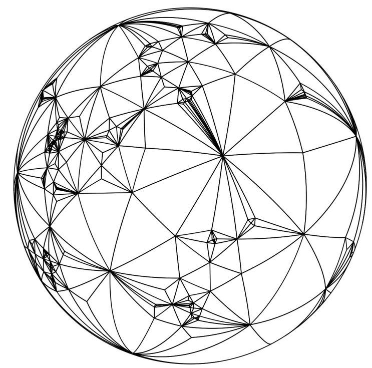
A map M defines a discrete **metric space**:

- points: set of vertices of $M = V(M)$.
- distance: graph distance $= d_{gr}$.

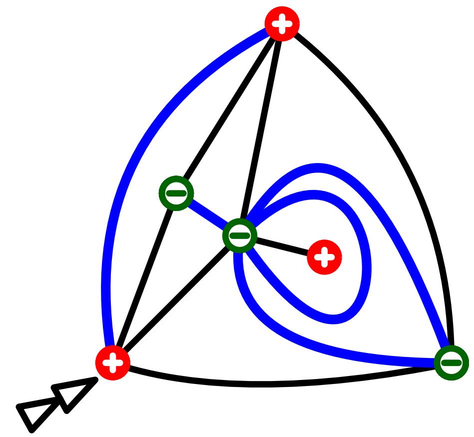
I - Bijective enumeration of maps



II - Scaling limits of random planar maps

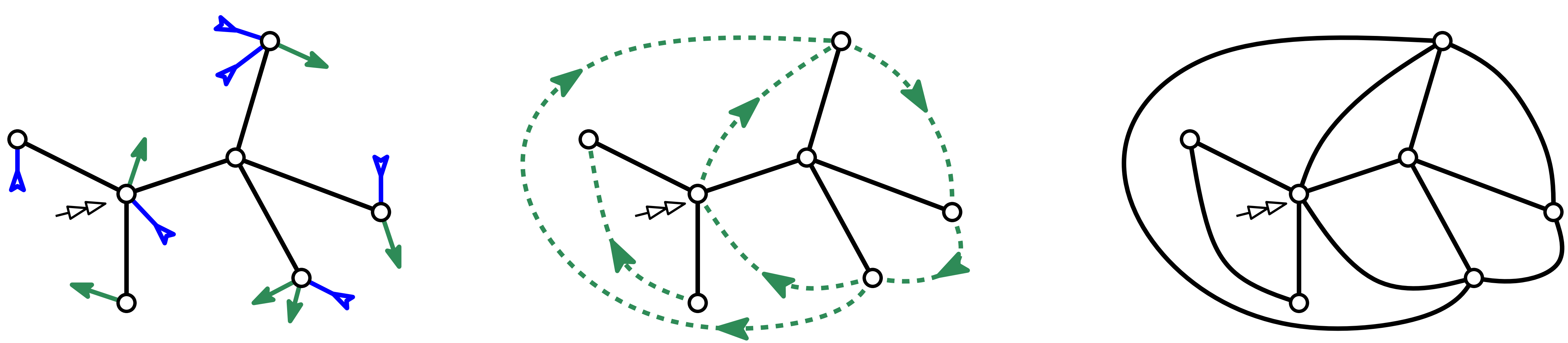


III - Local limit of Ising-weighted random triangulations



I - Bijective enumeration of maps

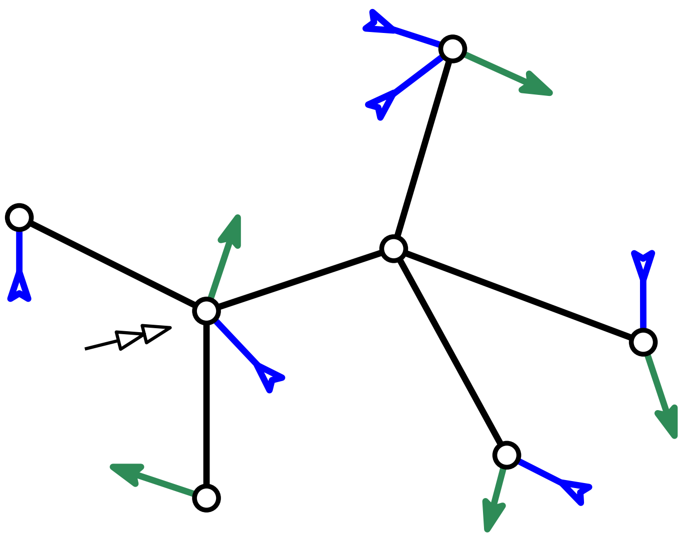
a tribute to blossoming bijections.



Bijections with blossoming trees

A **blossoming tree** is a plane tree where vertices can carry **opening stems** or **closing stems**:

$$\# \text{ closing stems} = \# \text{ opening stems}$$



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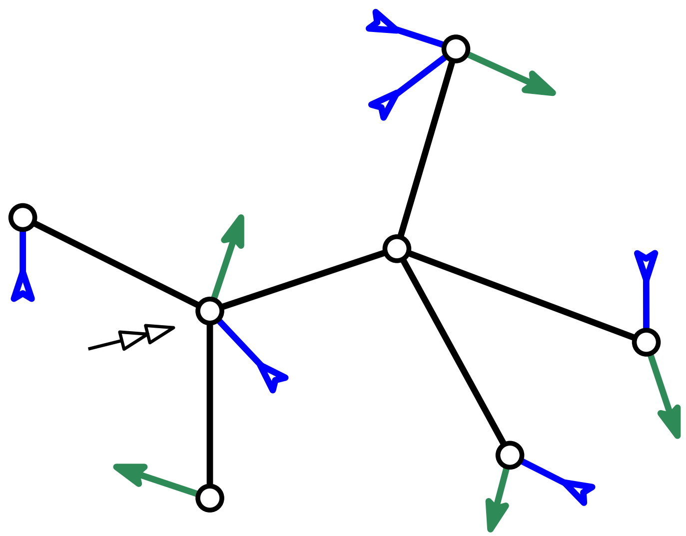

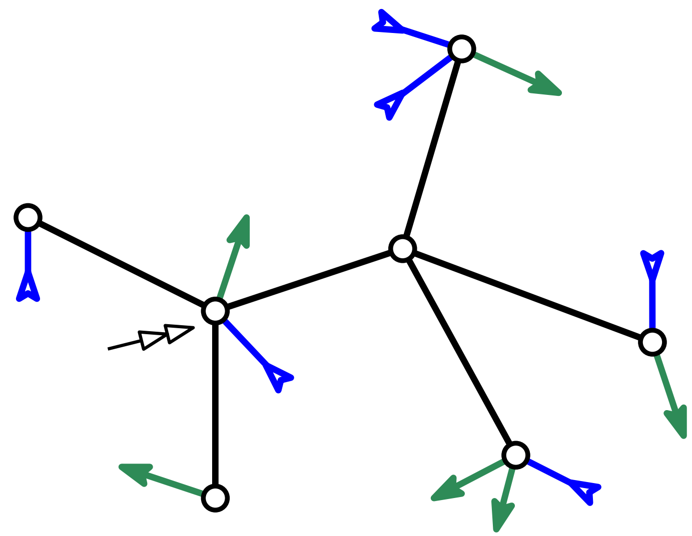

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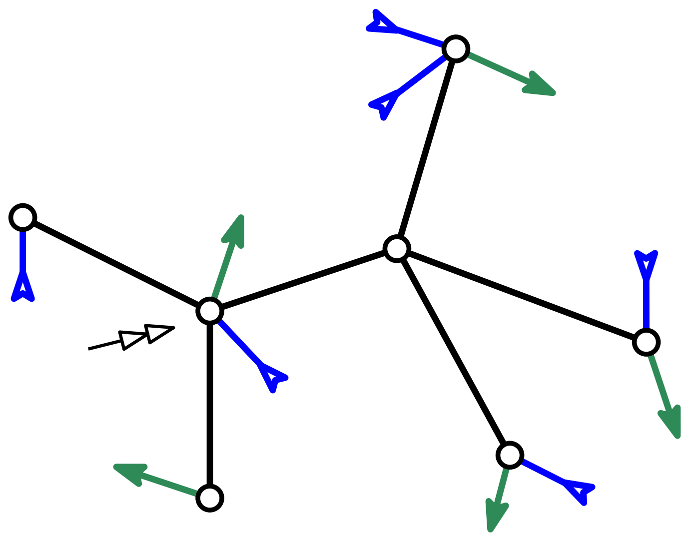

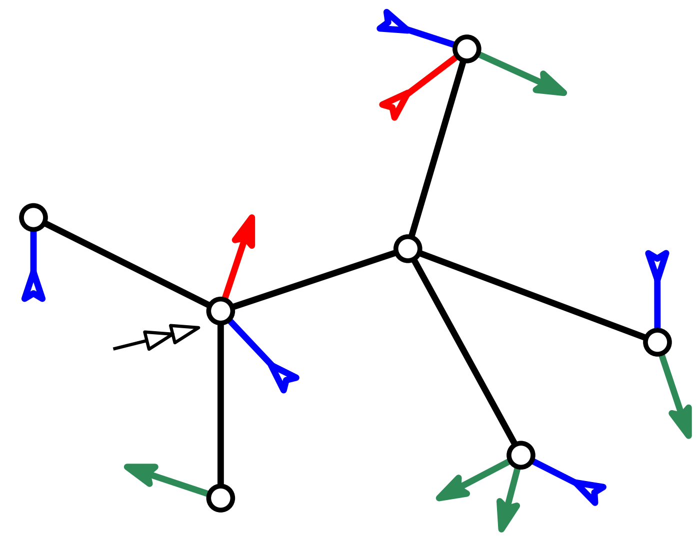

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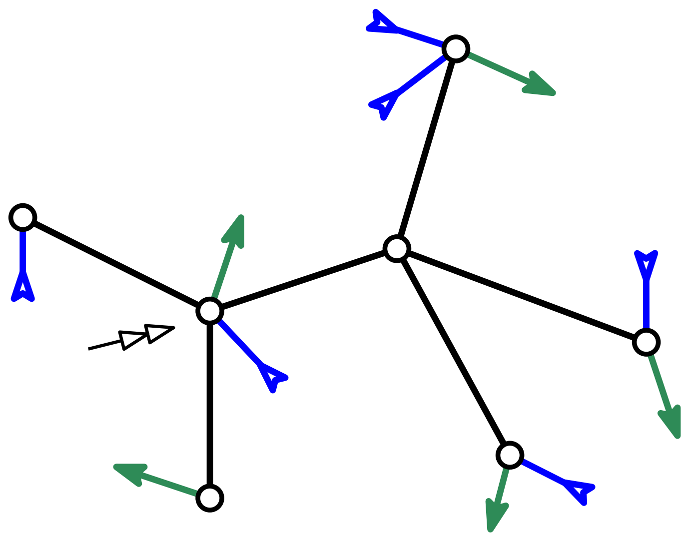

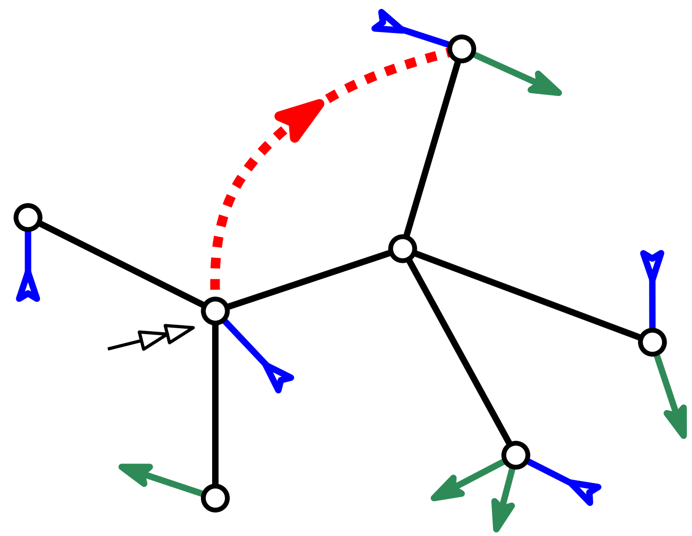

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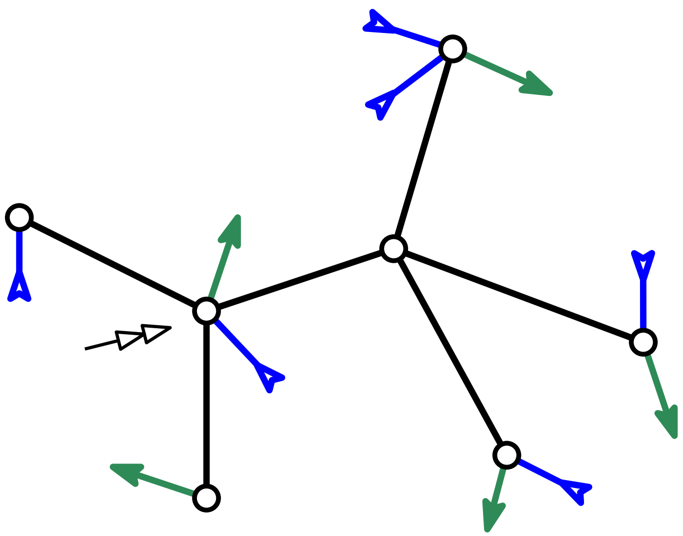

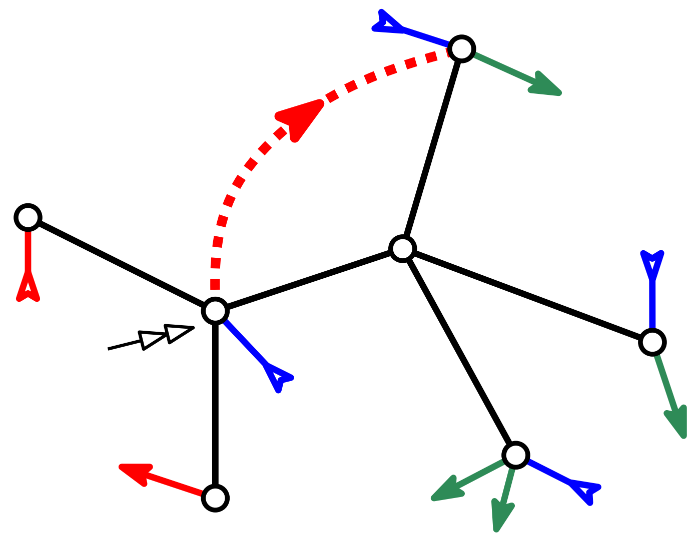

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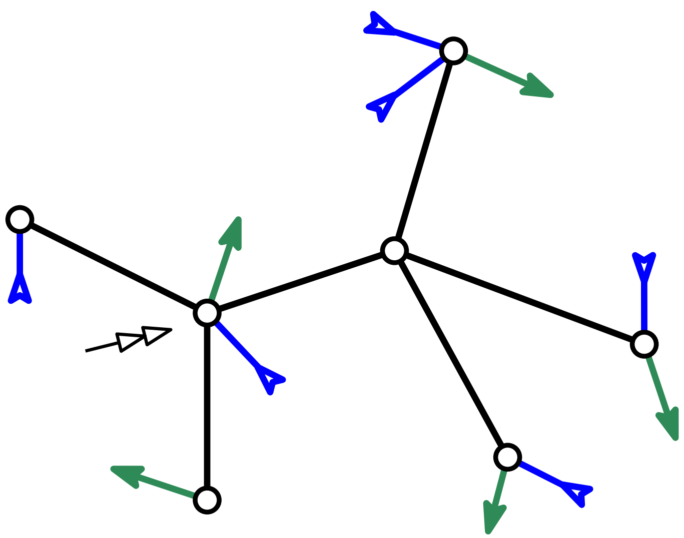

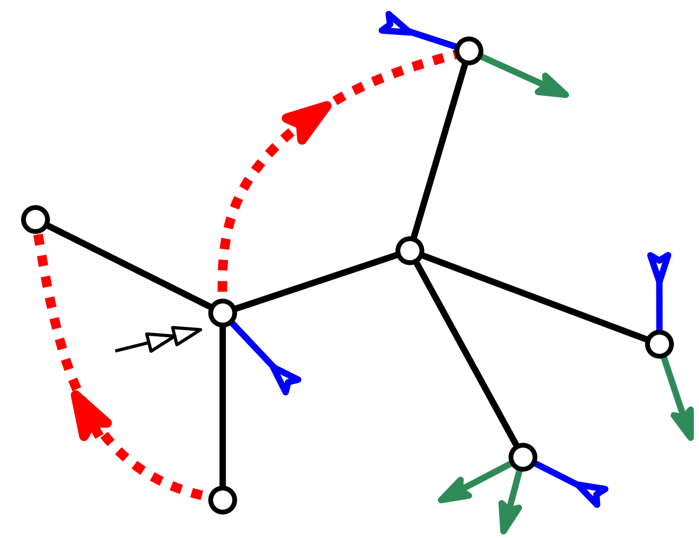

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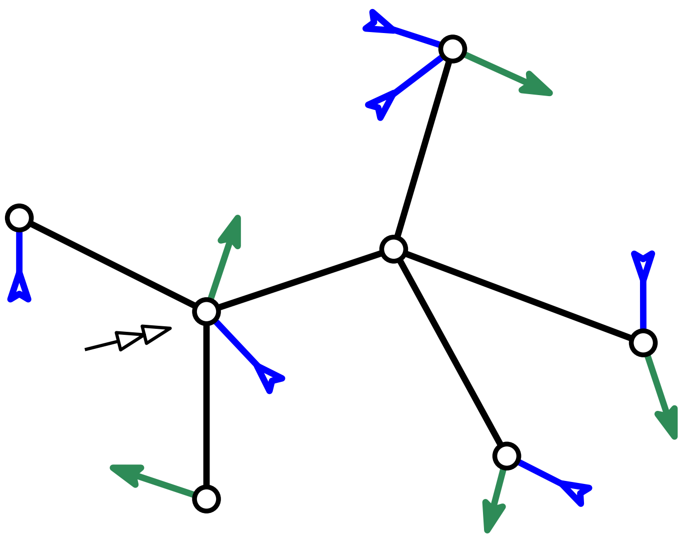

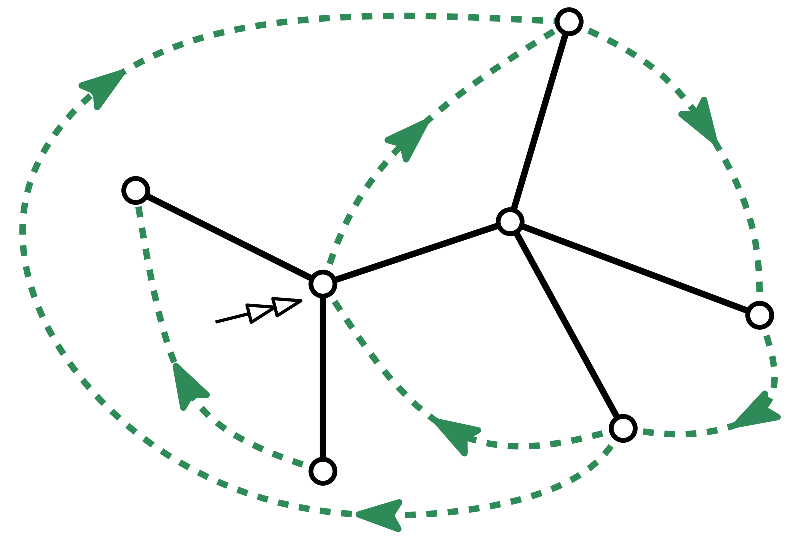

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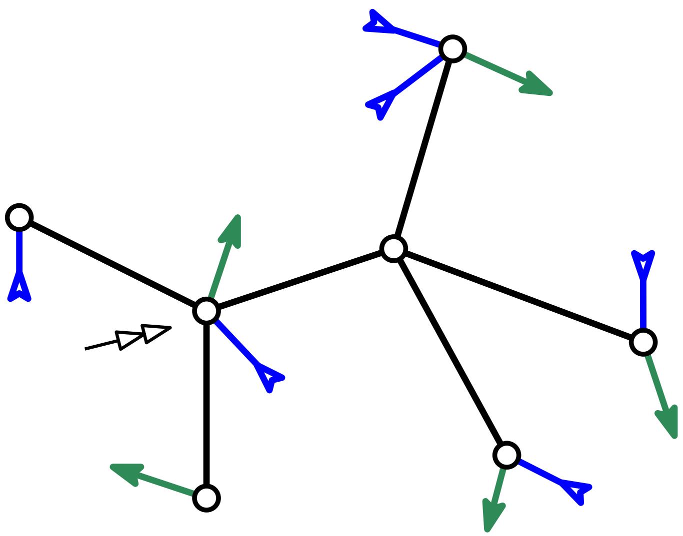

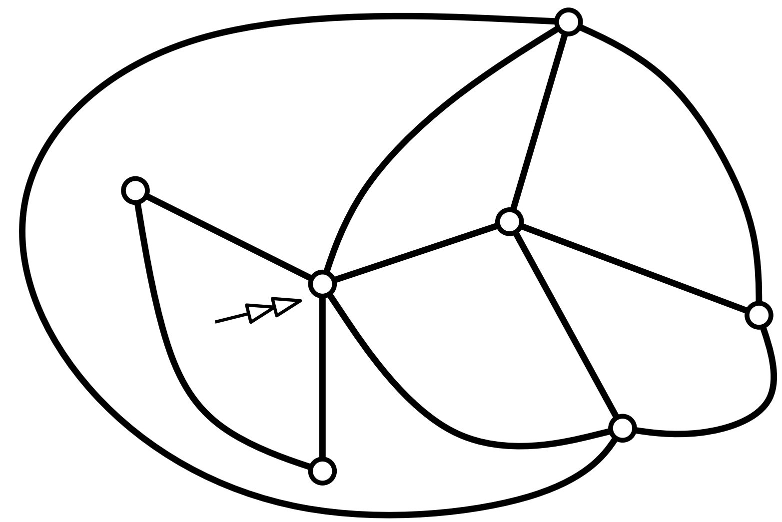

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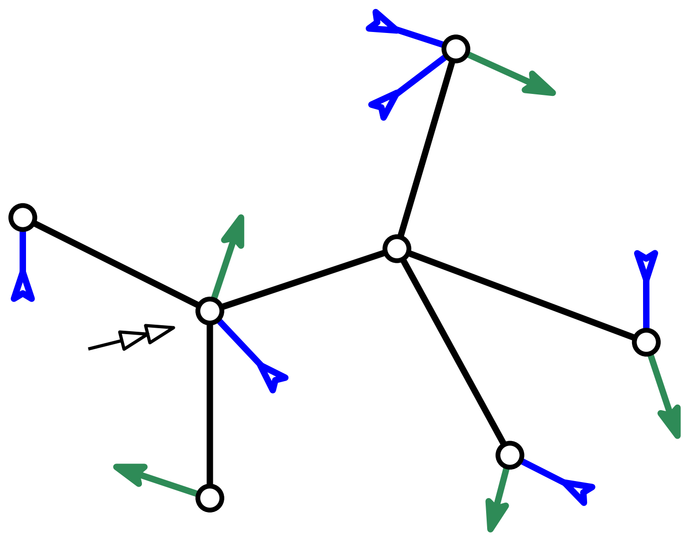

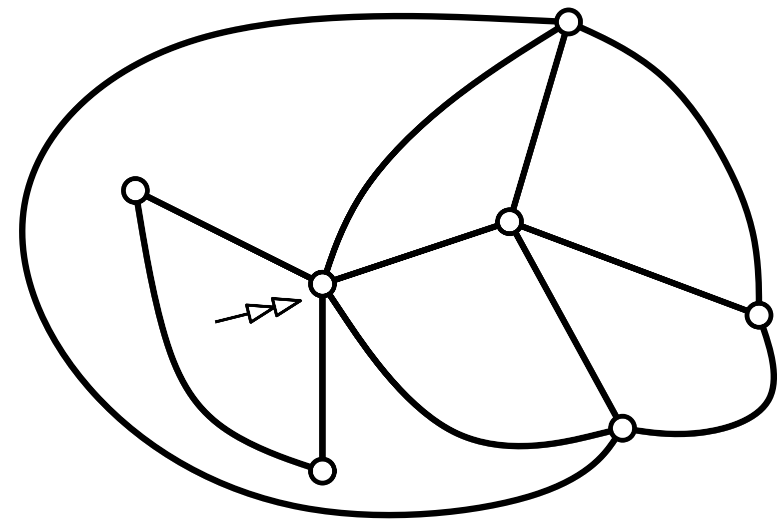

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i.e. can we determine a canonical spanning tree ?

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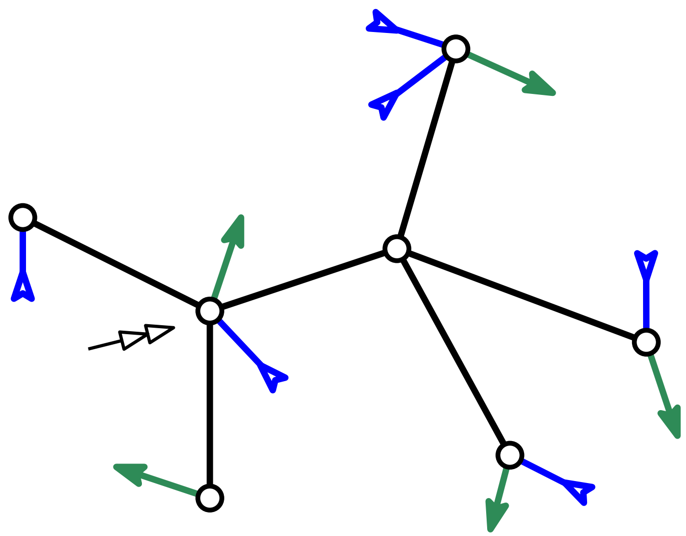

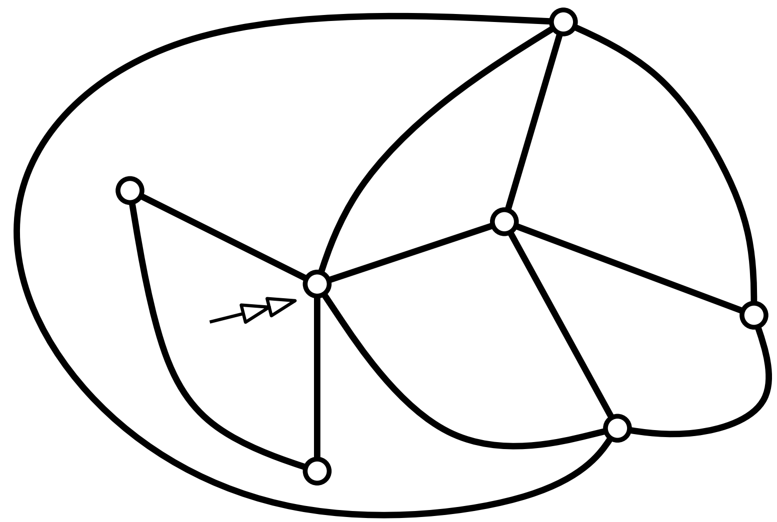

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Yes...

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e.g. $\boxed{\#\left\{\text{rooted planar maps with } n \text{ edges}\right\} = \frac{2 \cdot 3^n}{n+2}\text{Catalan}(n) \qquad \text{[Tutte 63]}}$

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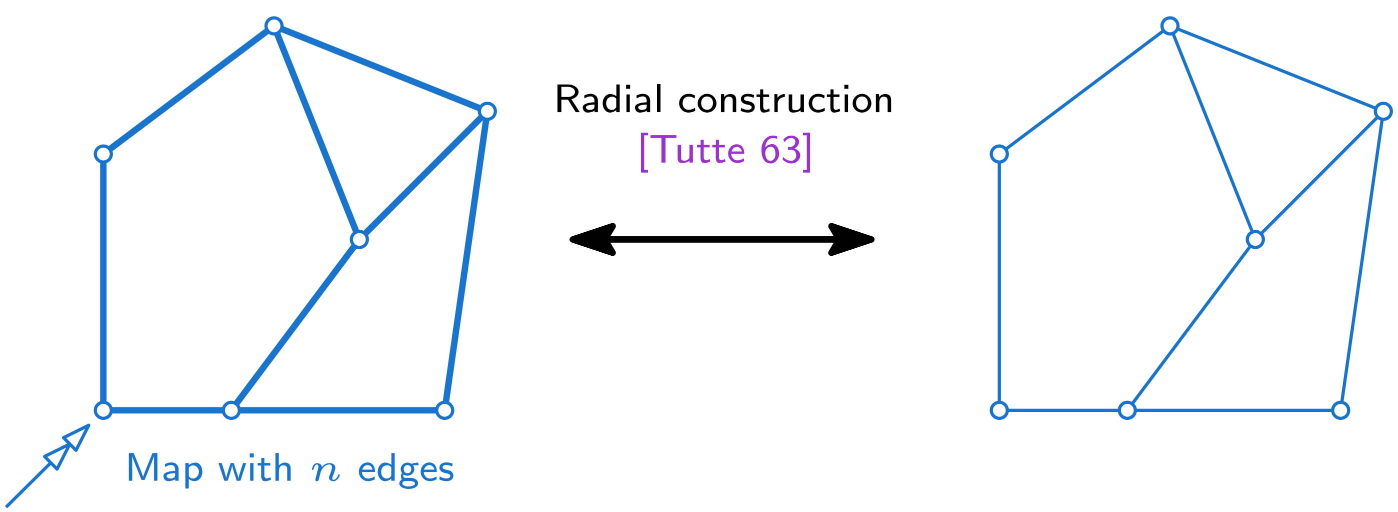

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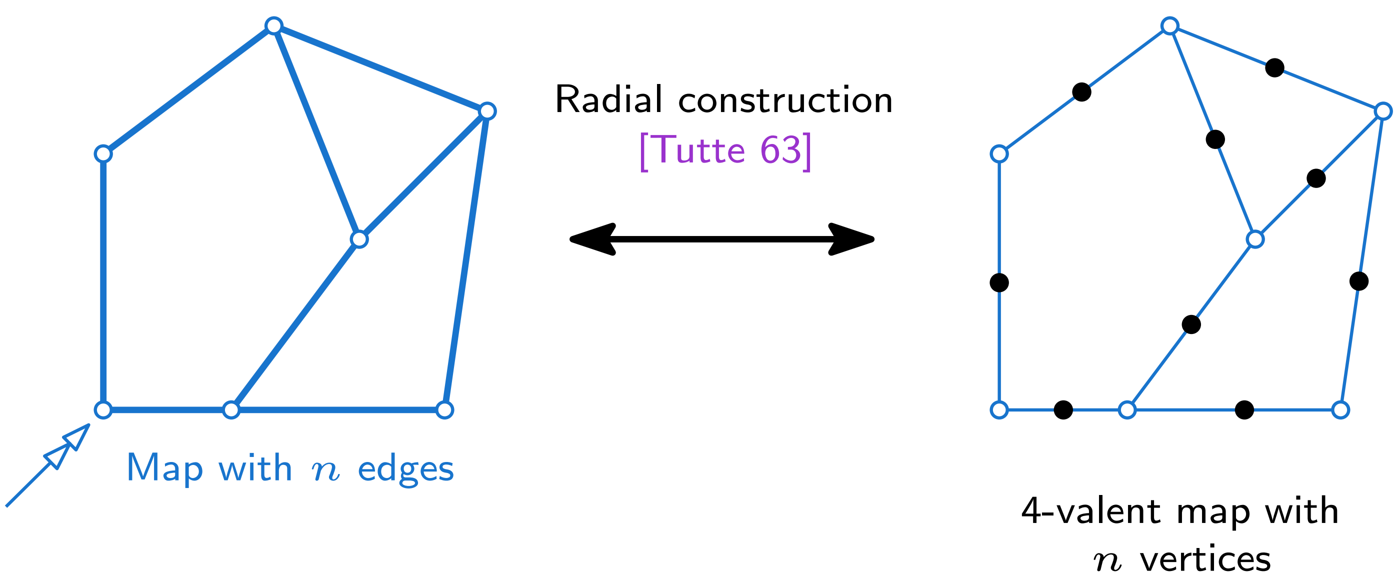

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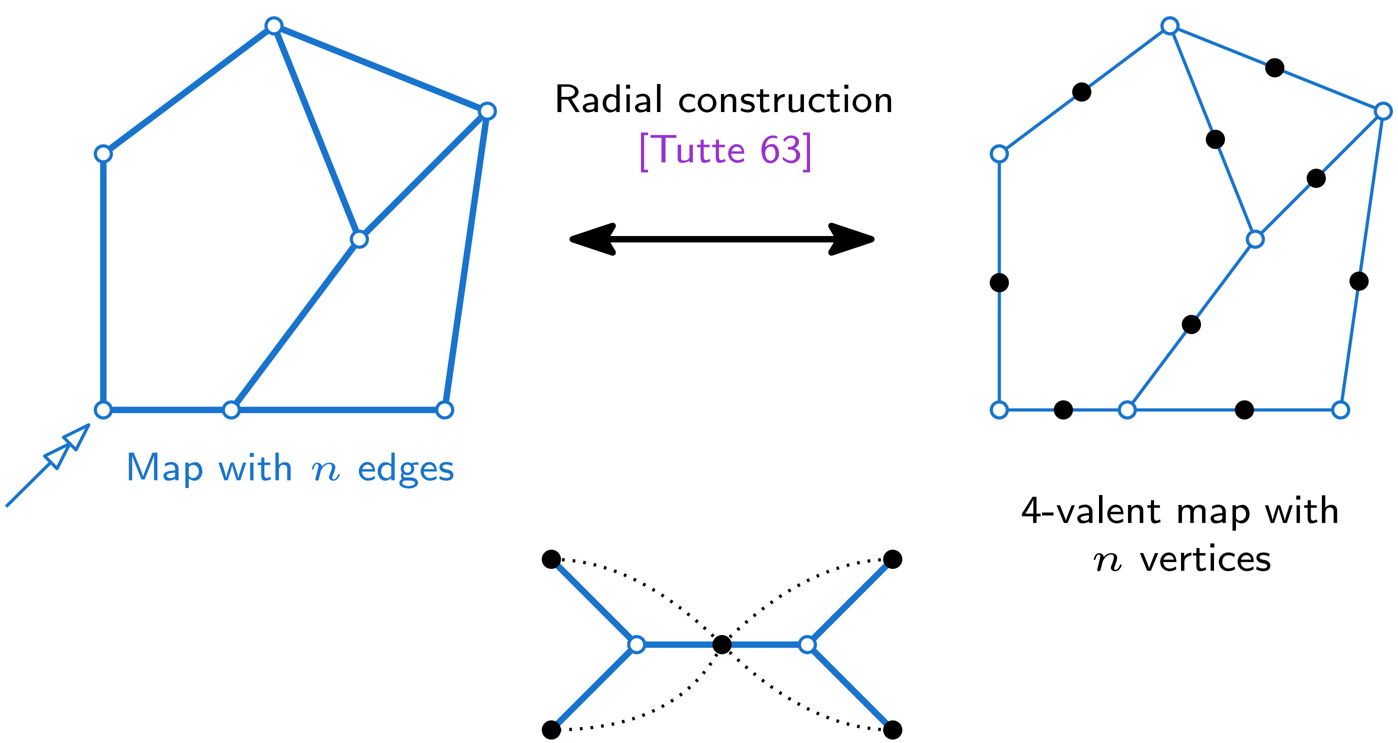

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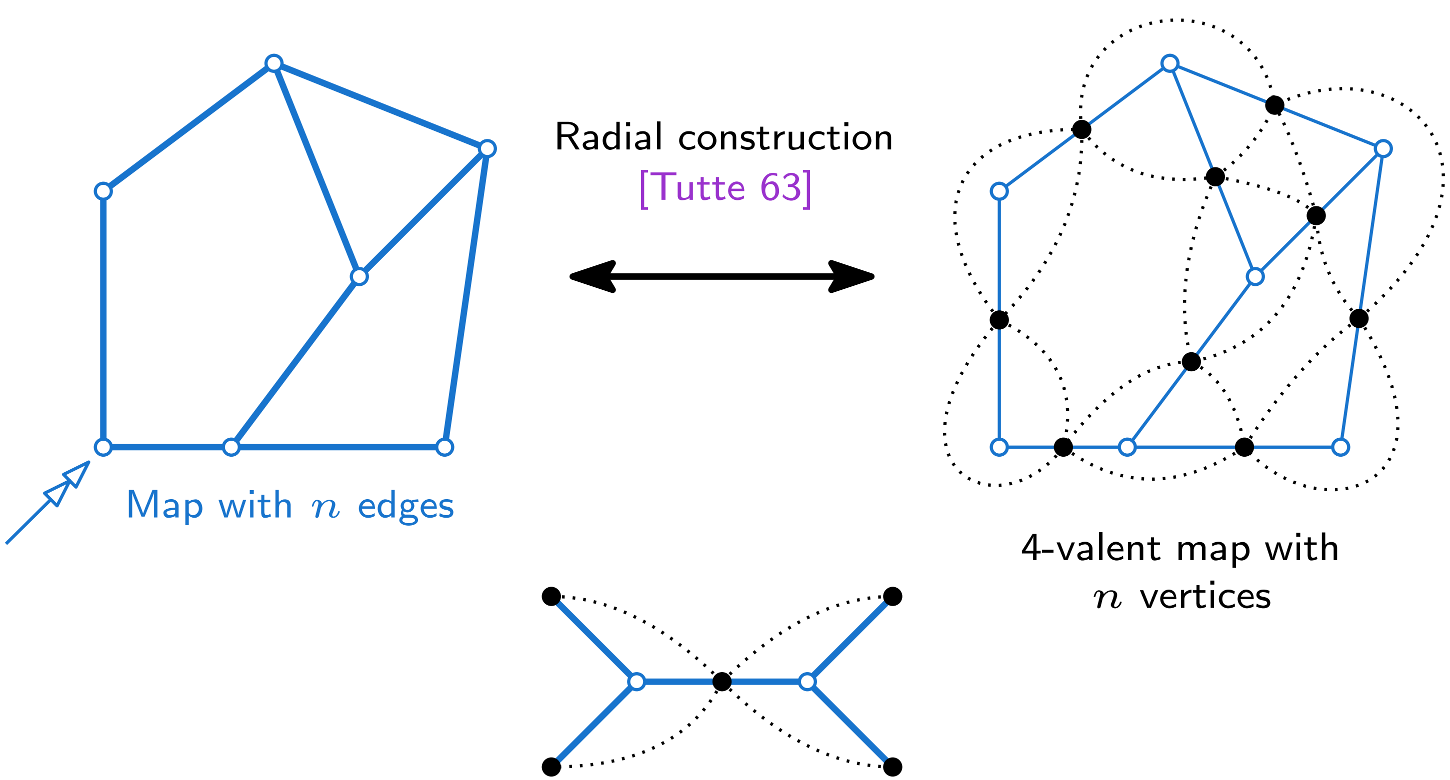

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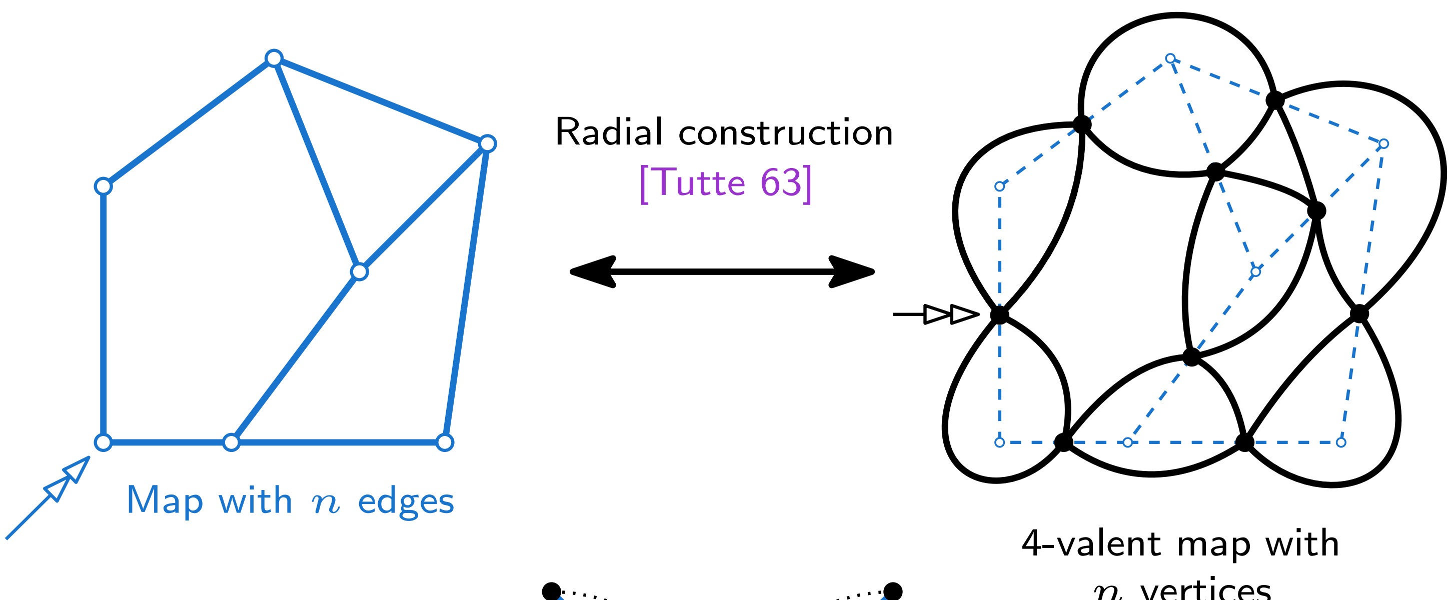

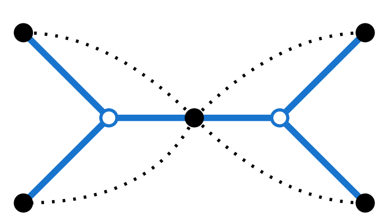

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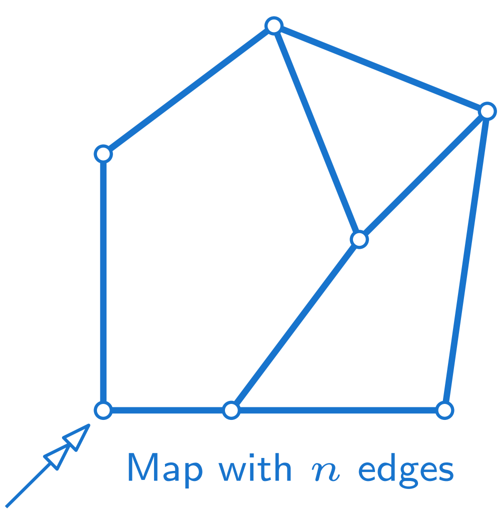
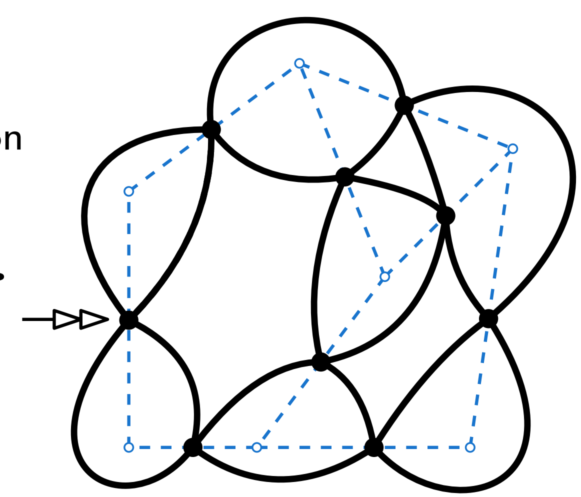


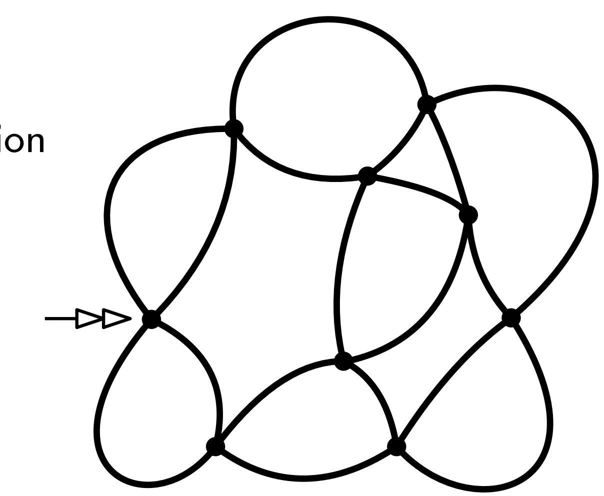
Map with n edges

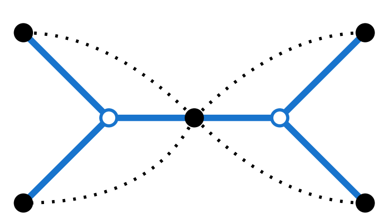
Radial construction
[Tutte 63]



4-valent map with
n vertices

Blossoming bijection
[Schaeffer 97]





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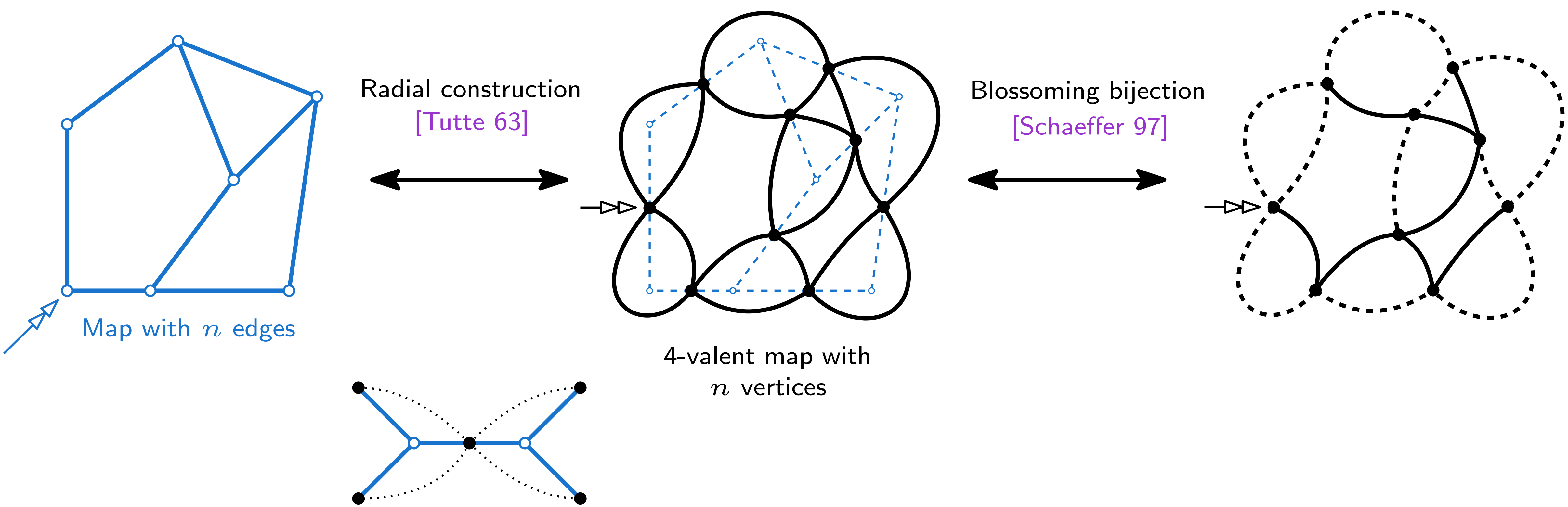

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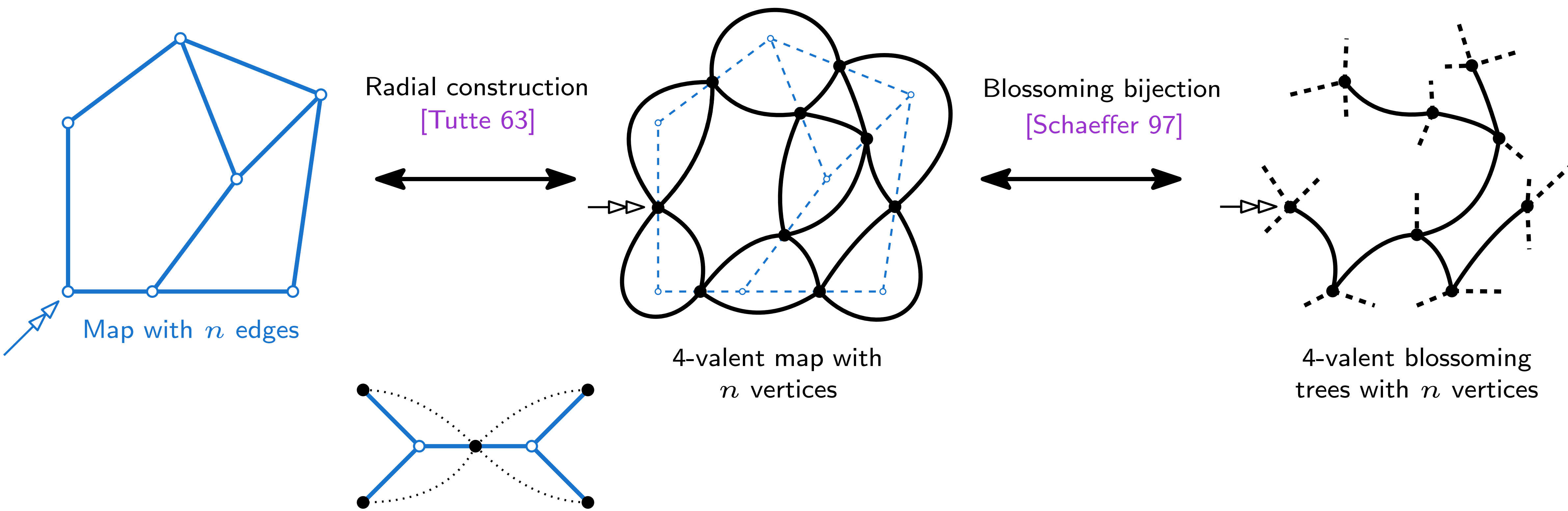

Enumeration of planar maps

In the 60's, **Tutte** obtained closed enumerative formulas for many families of planar maps.

e.g. $\boxed{\#\left\{\text{rooted planar maps with } n \text{ edges}\right\} = \dfrac{2 \cdot 3^n}{n+2}\text{Catalan}(n) \qquad \text{[Tutte 63]}}$

$\#\left\{\text{plane trees with } n \text{ vertices}\right\}$
$= \#\left\{\text{binary plane trees with } n \text{ inner vertices}\right\}$

Combinatorial proof ? Bijection ?

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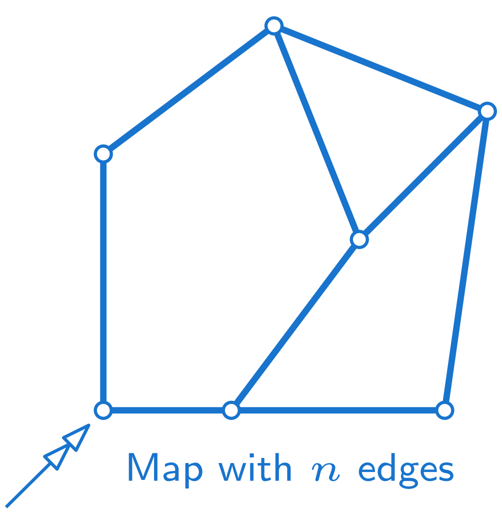
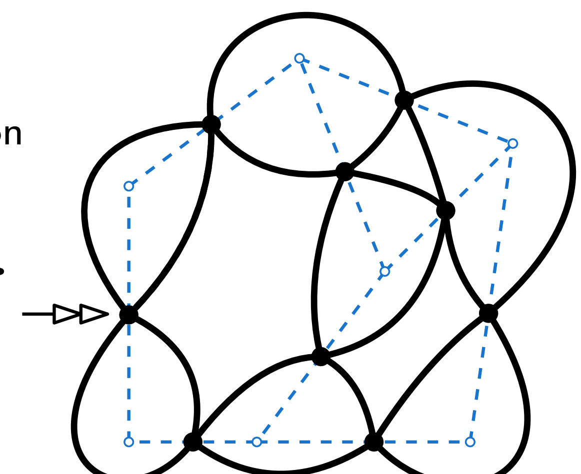


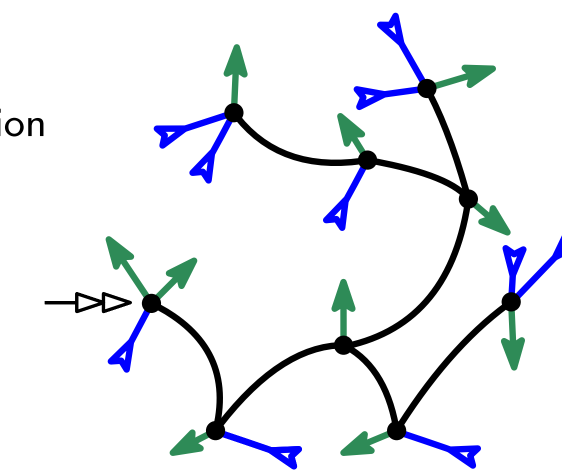
Map with n edges

Radial construction
[Tutte 63]

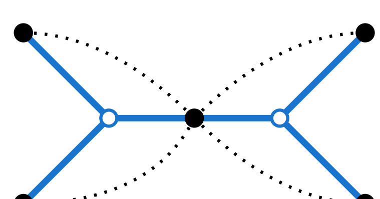


4-valent map with
n vertices

Blossoming bijection
[Schaeffer 97]

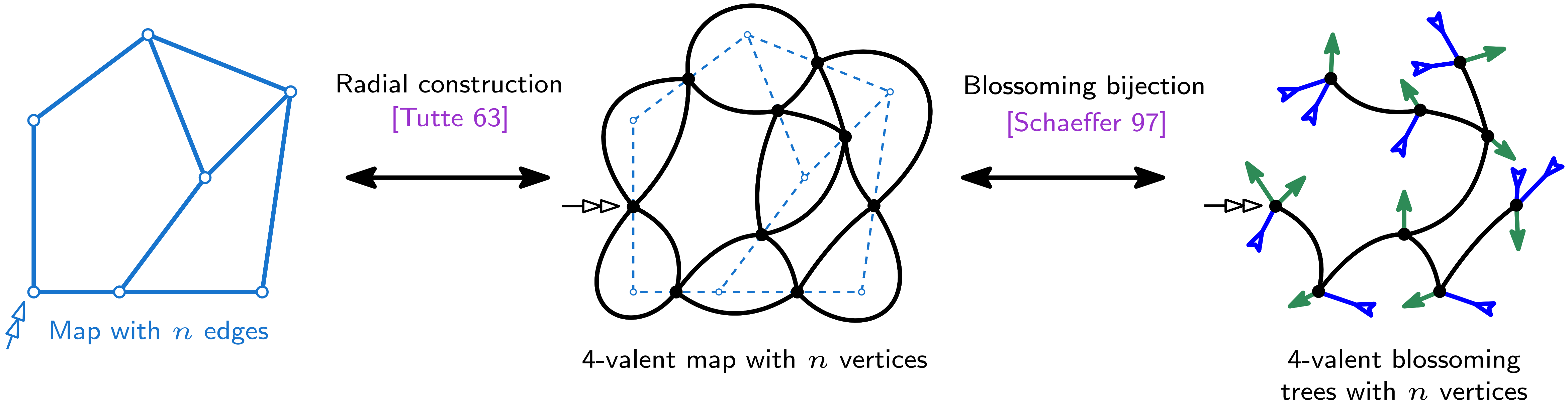


4-valent blossoming
trees with n vertices



As a corollary:
combinatorial proof of Tutte's formula.

Enumeration of planar maps: a dichotomy of bijections



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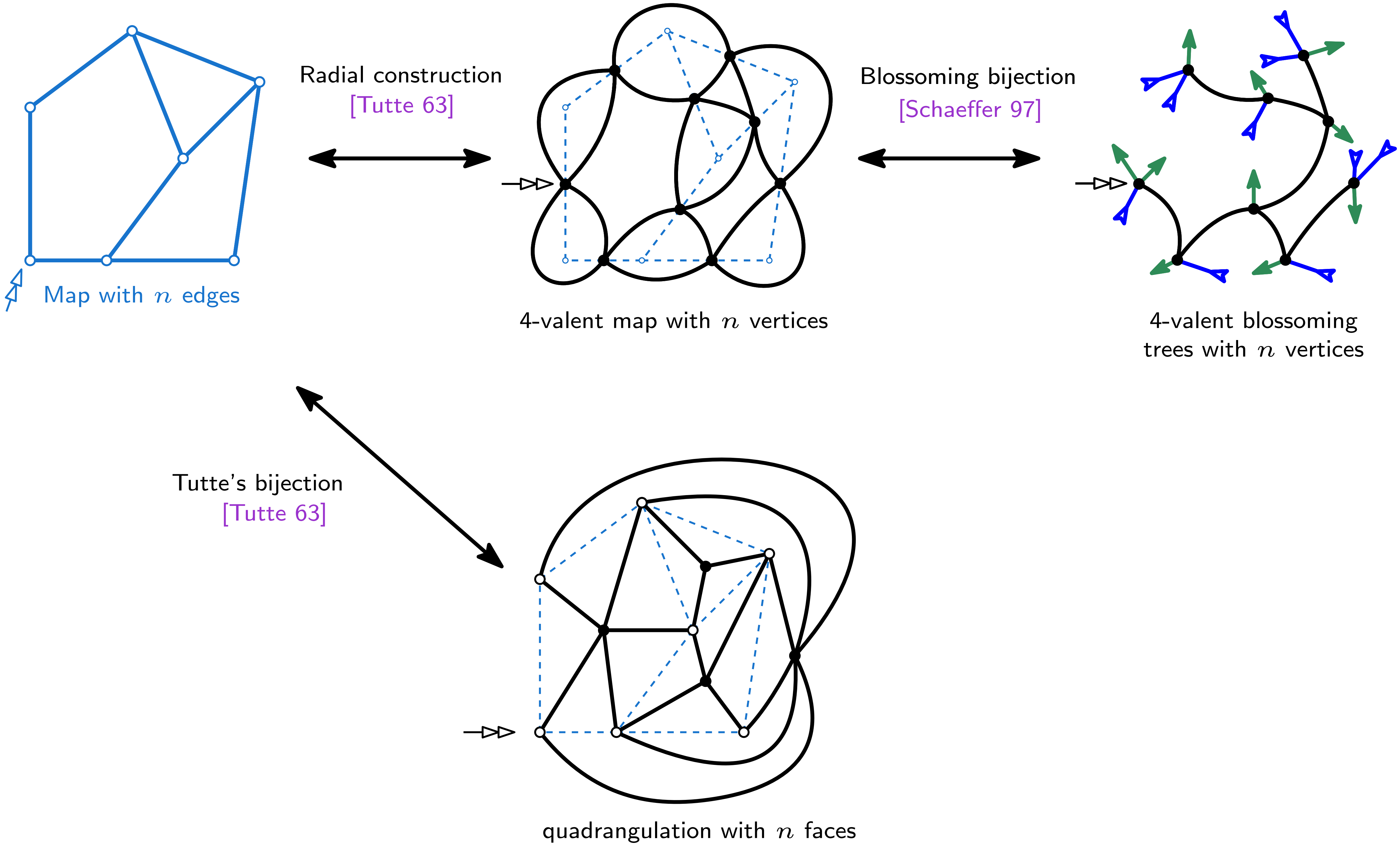

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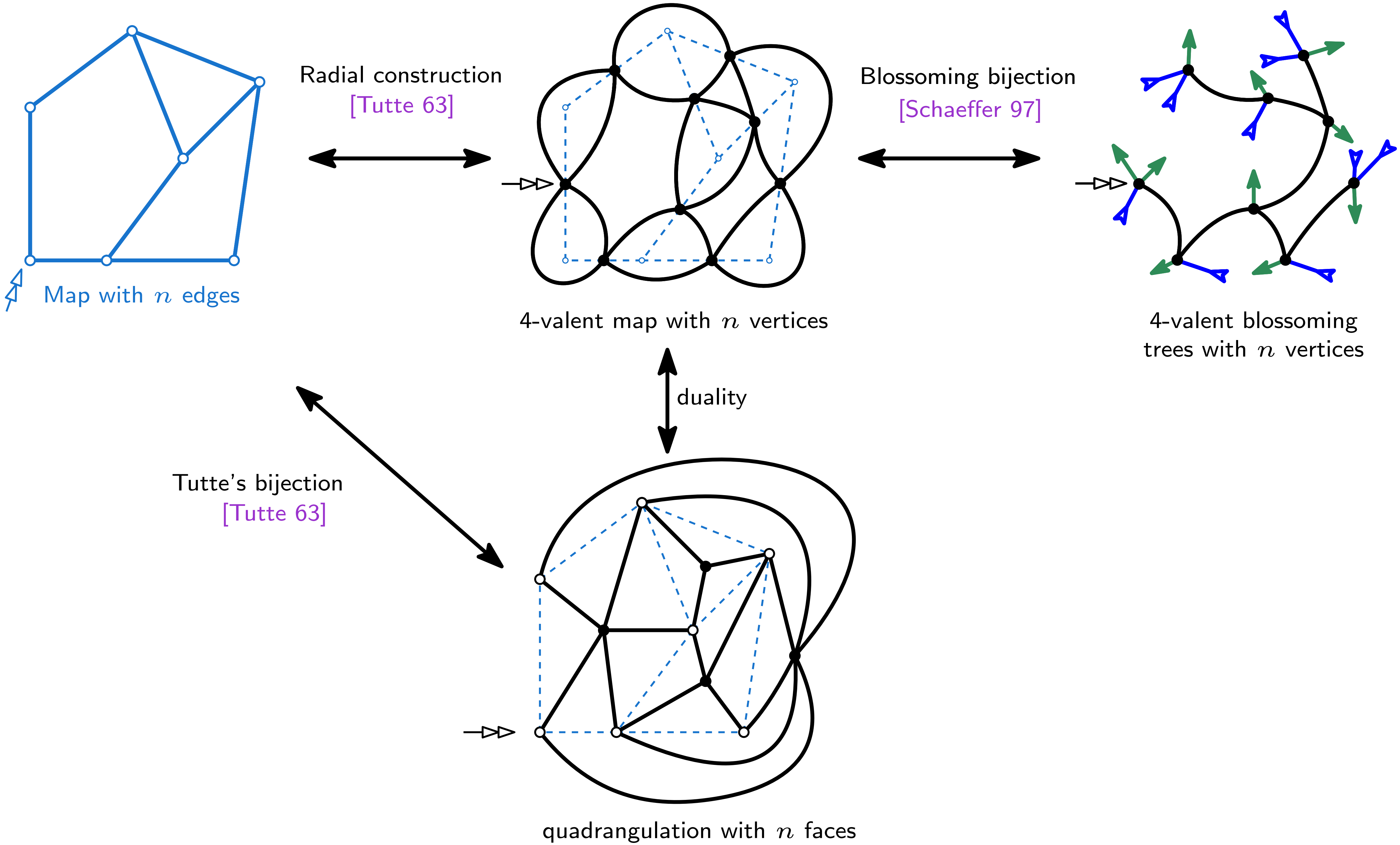

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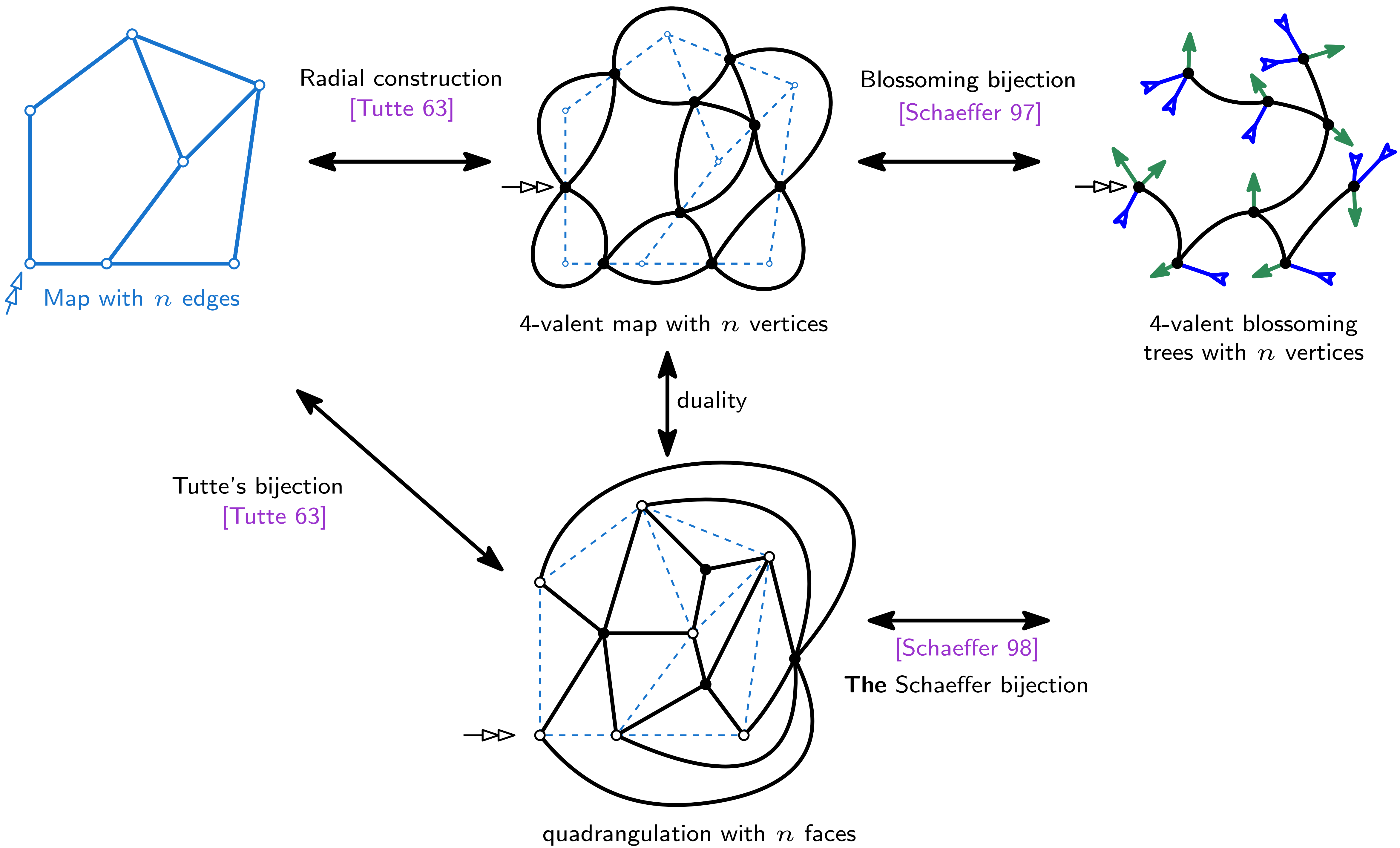

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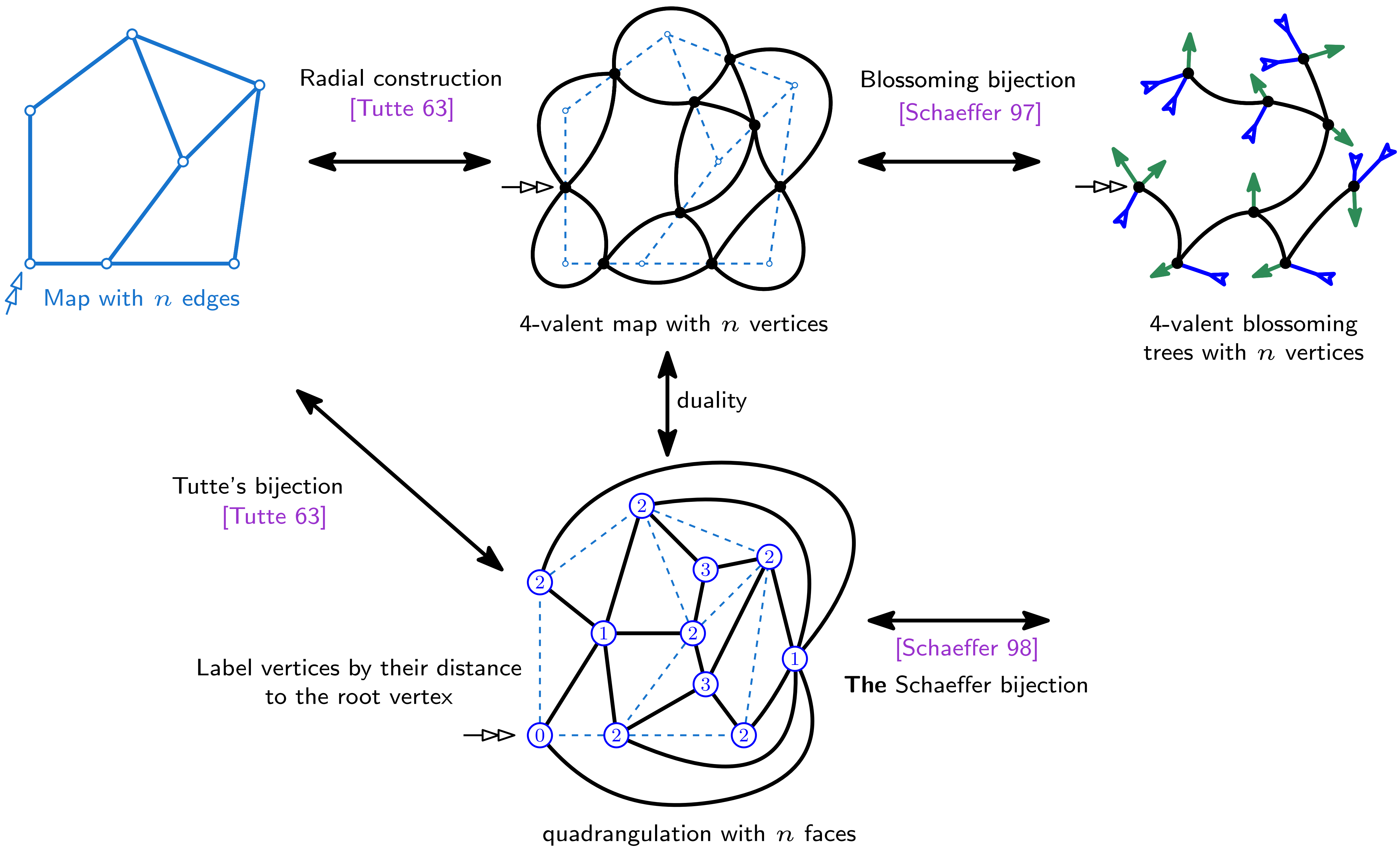

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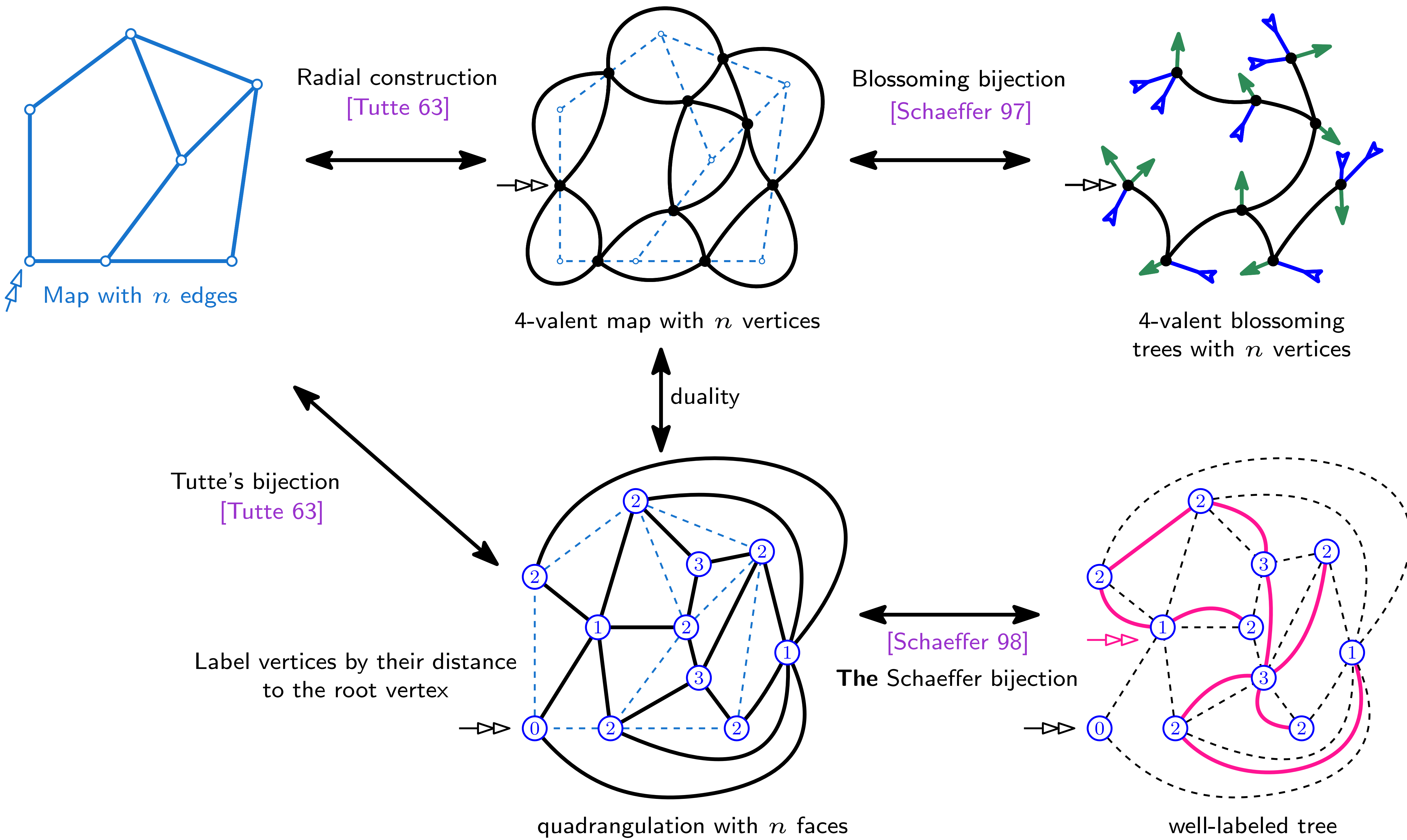

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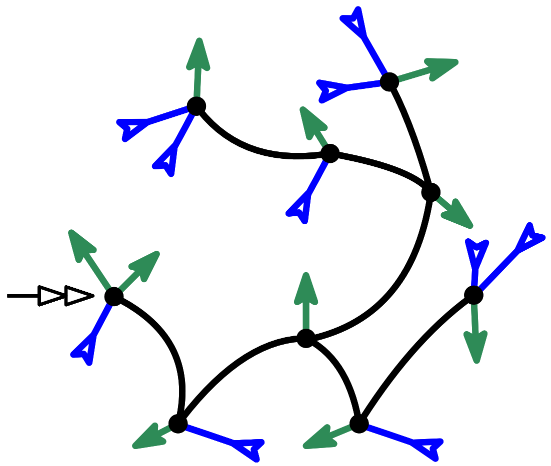

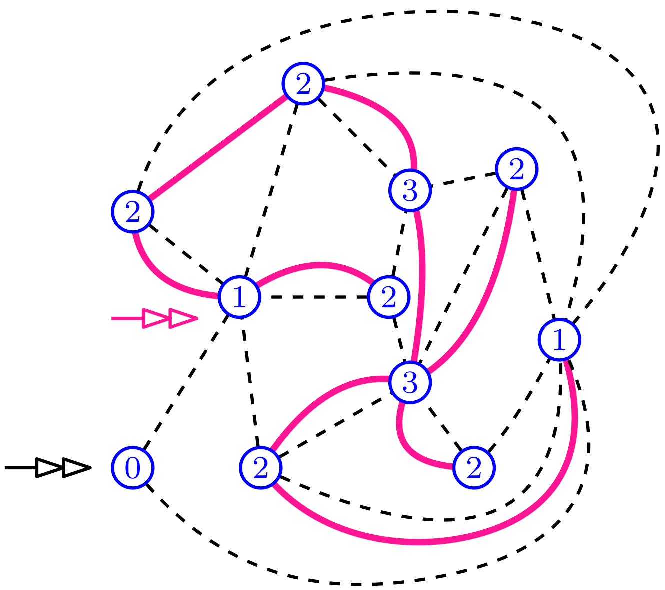

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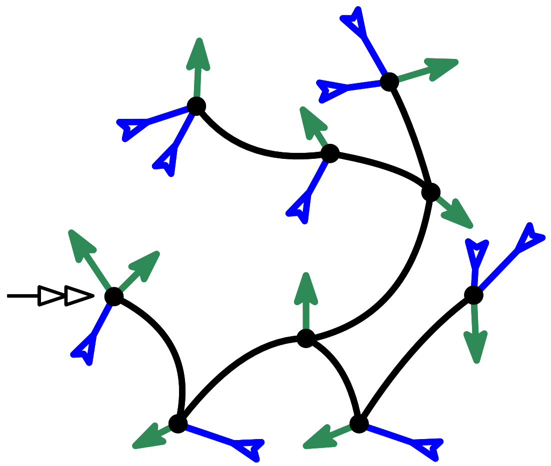
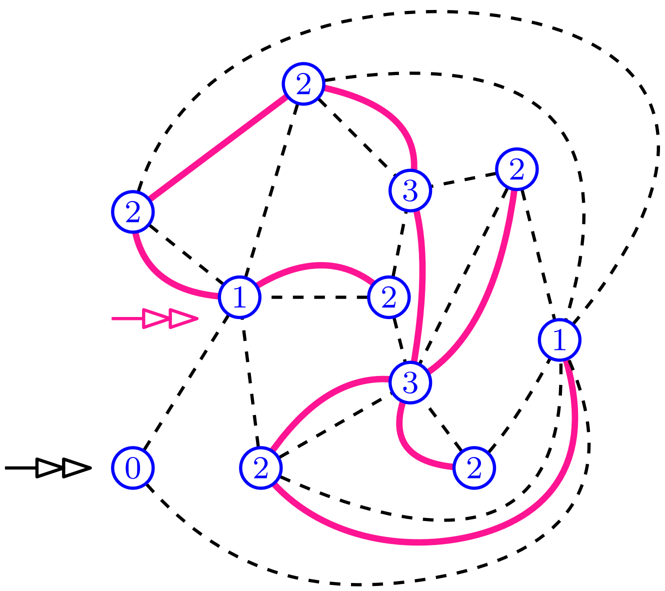


Blossoming bijections

- Eulerian maps [Schaeffer 97]
- General maps with prescribed vertices degree sequences [Bouttier, Di Francesco, Guitter 02]
- Constellations [Bousquet-Mélou, Schaeffer 00]
- Bipartite maps [Bousquet-Mélou, Schaeffer 02]
- Simple triangulations [Poulalhon, Schaeffer 05], simple quadrangulations [Fusy 07]

(without loops nor multiple edges)



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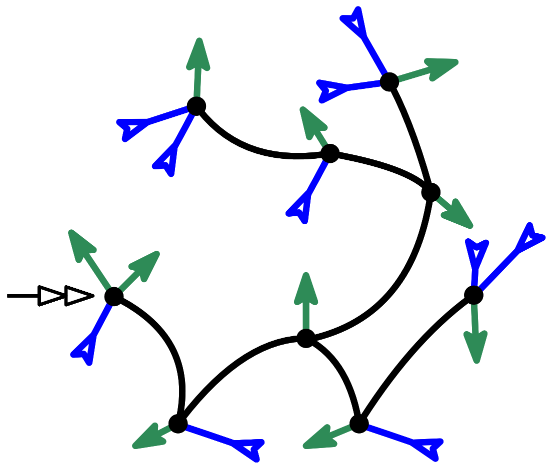

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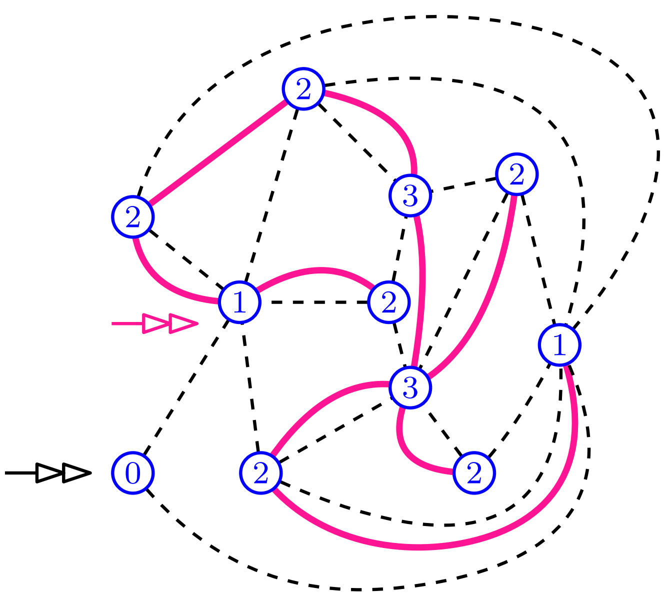
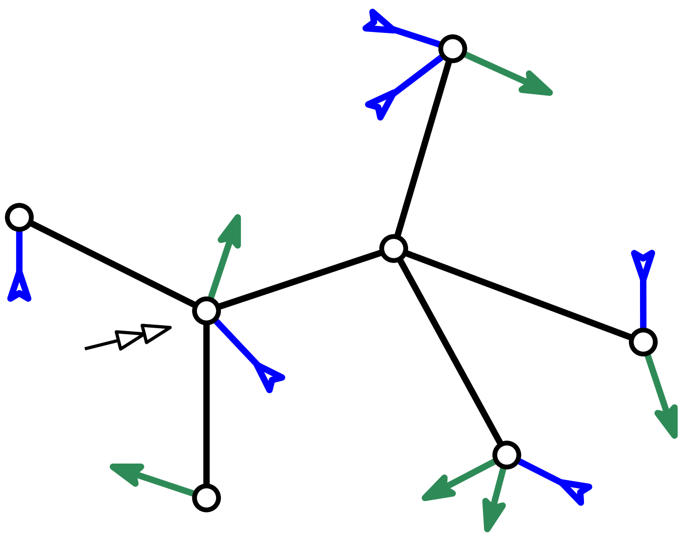


Mobile type bijections

- Quadrangulations [Schaeffer 98]
- General maps with prescribed faces degree sequences [Bouttier, di Francesco, Guitter 04] = BDG bijection
- Maps with sources and delays [Miermont 09], [Bouttier, Fusy, Guitter 14]
- Extension to higher genus [Chapuy, Marcus, Schaeffer 09],

Bijections with blossoming trees

Can we unify all the blossoming bijections ?



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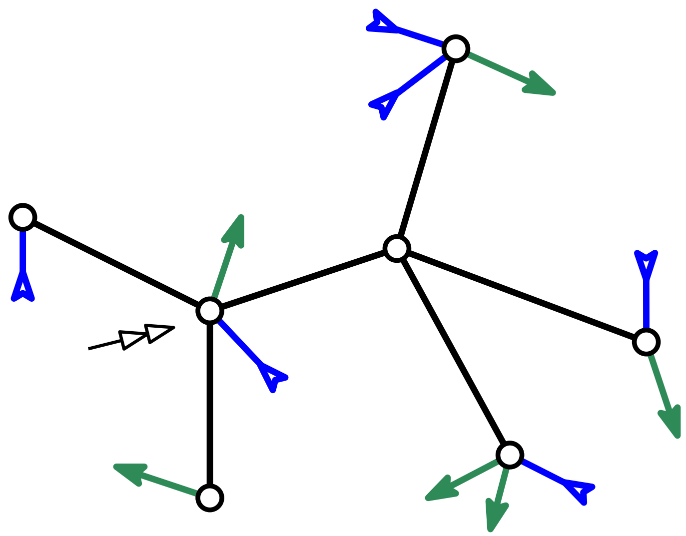

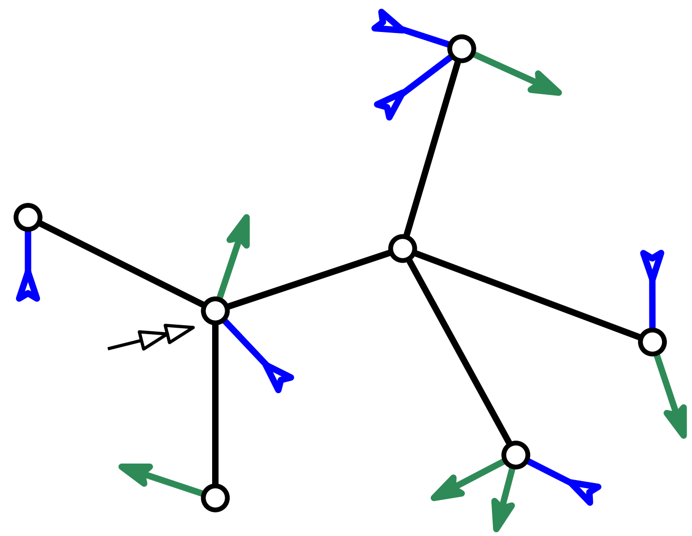

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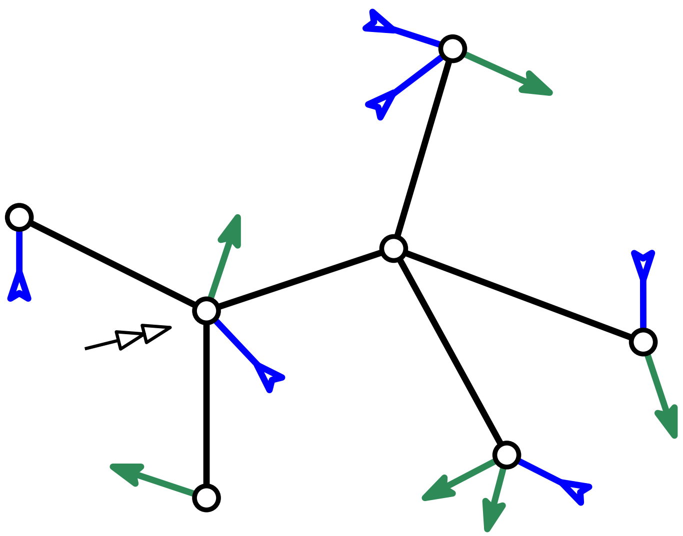

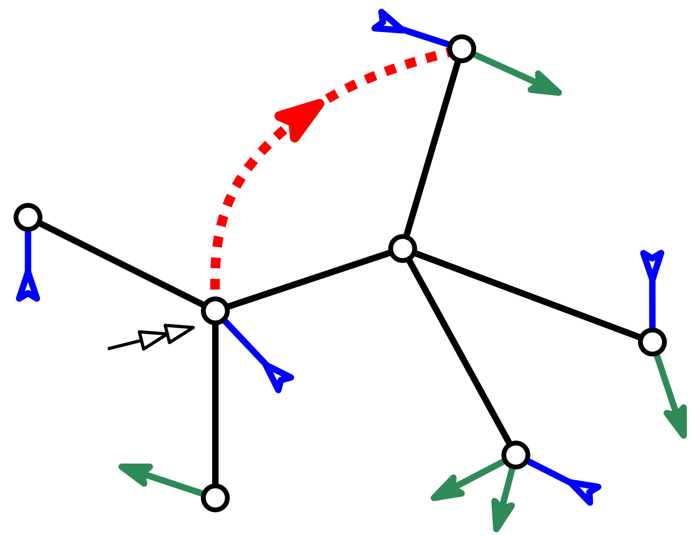

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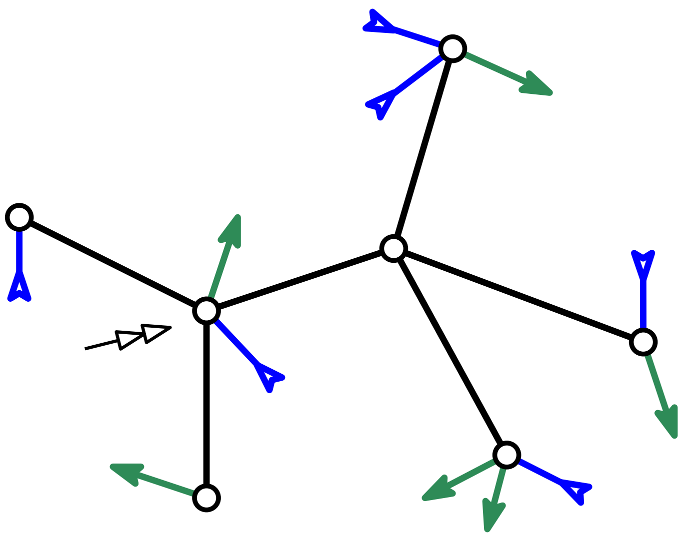

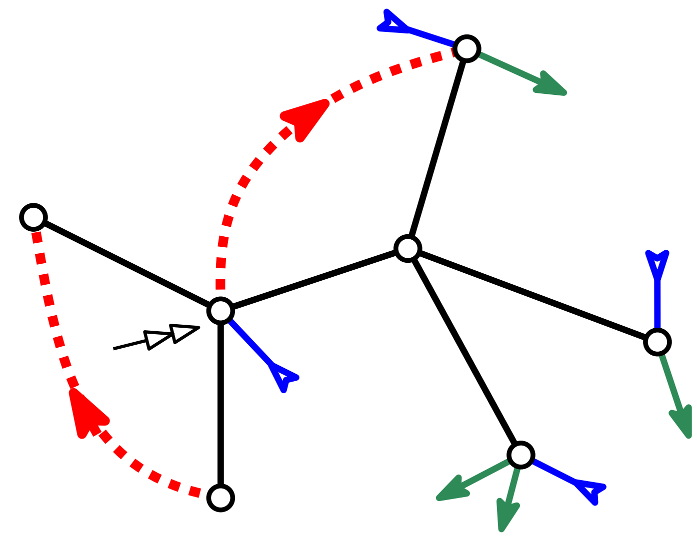

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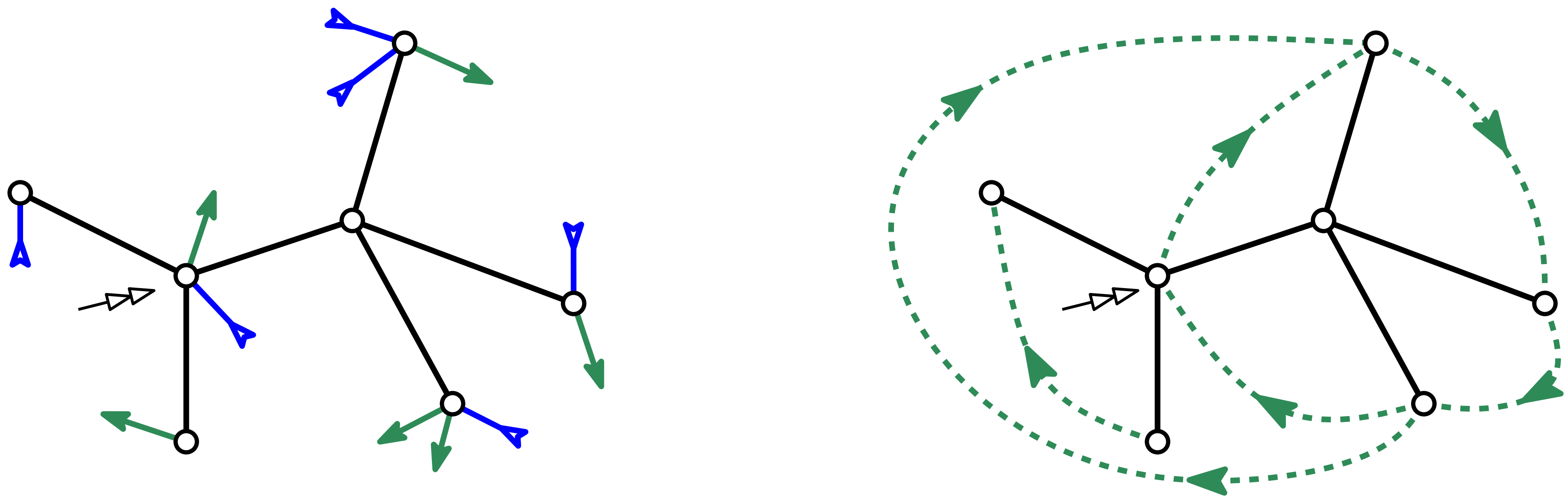

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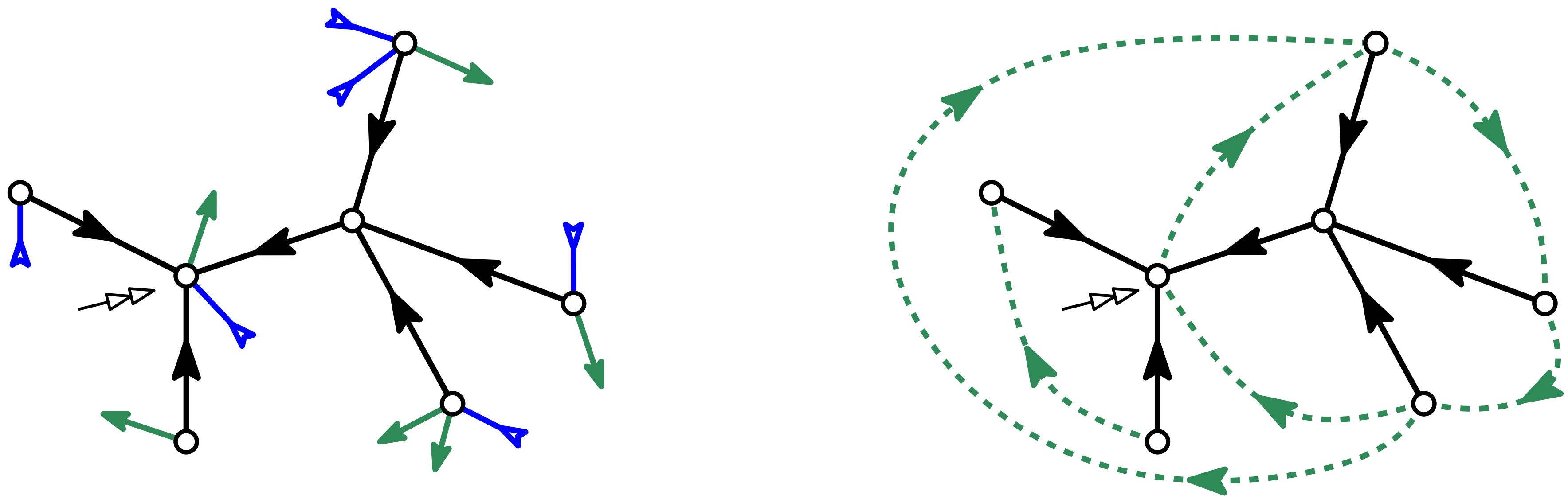

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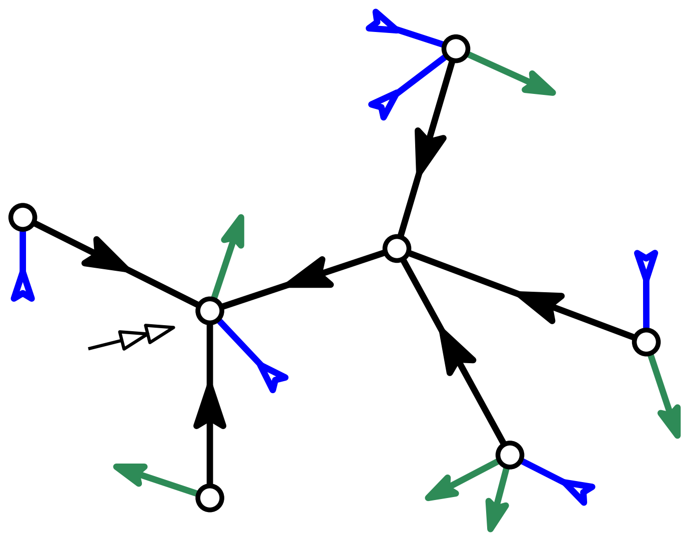

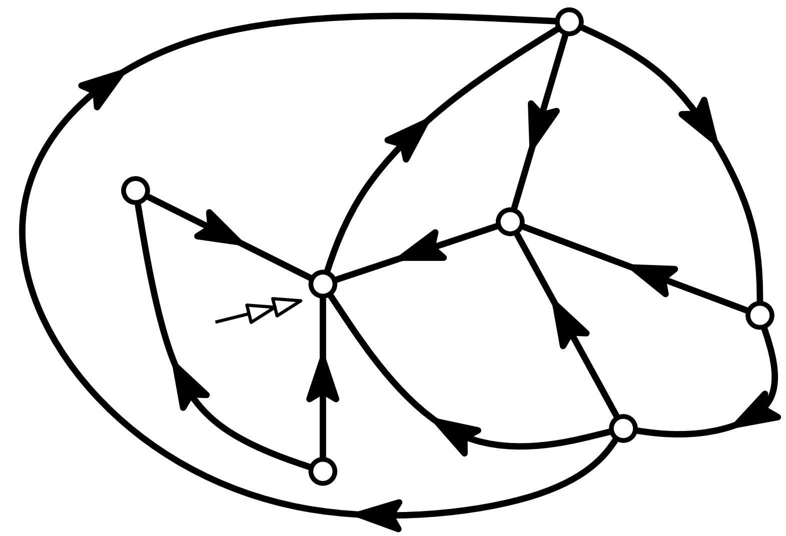

Via this construction, an **oriented** planar map is canonically associated to a blossoming tree.

Bijections with blossoming trees

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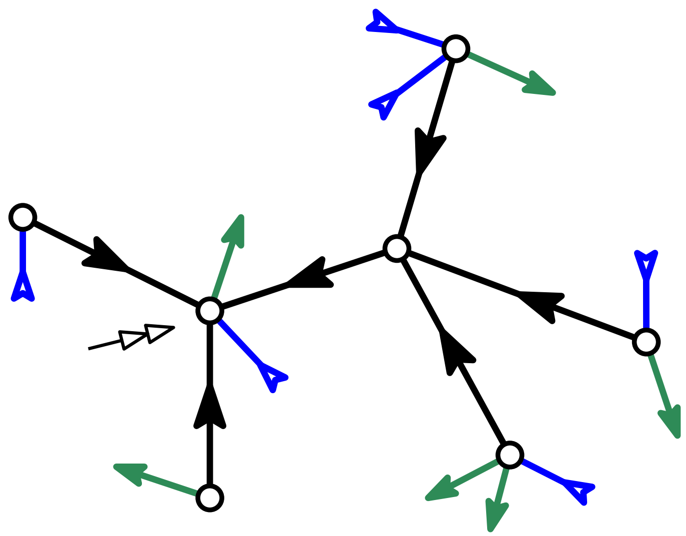

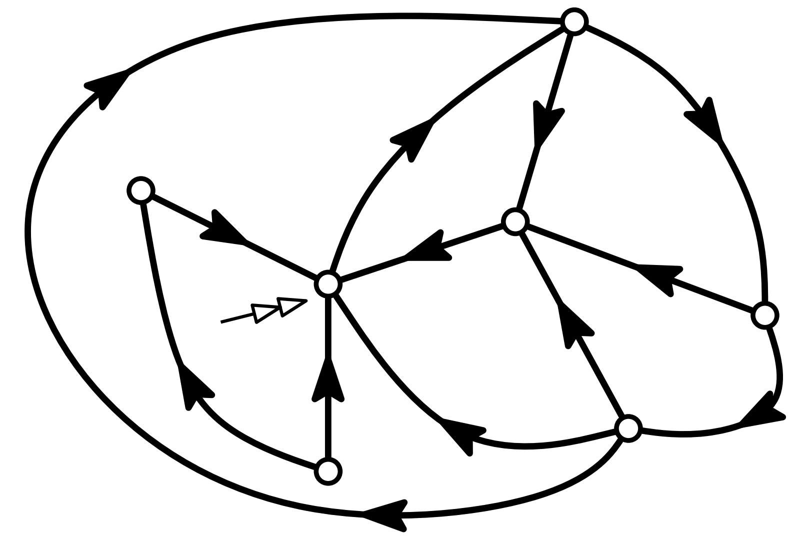

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Can we reverse the construction ??

Bijections with blossoming trees

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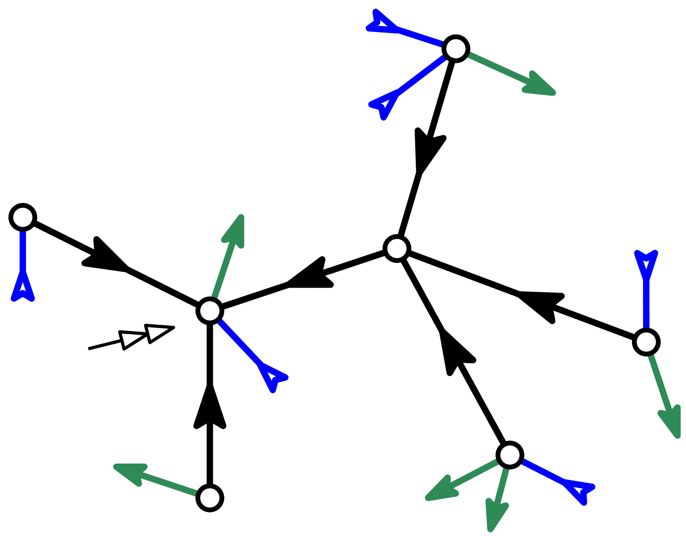

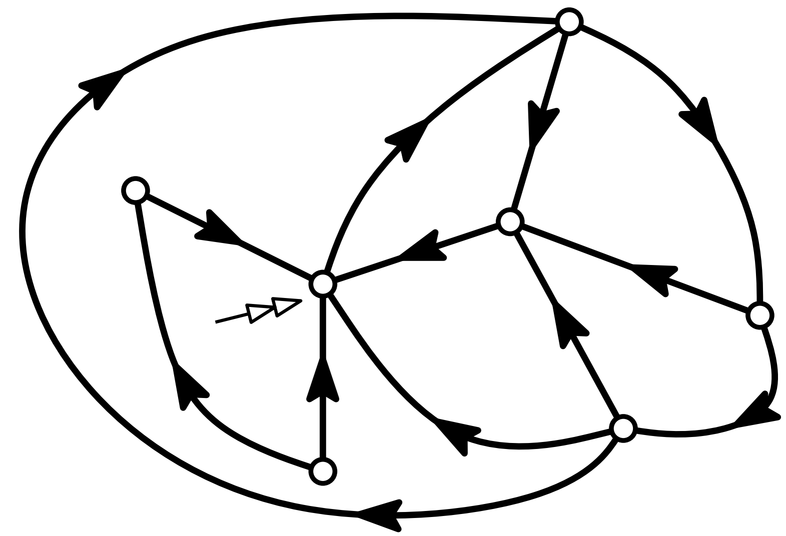

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Can we reverse the construction ?? Yes, by a **generic bijective scheme**:

Theorem: [A., Poulalhon 15] (generalization of results of [Bernardi '07])
If a planar map M is endowed with a "nice orientation" of its edges, then there exists a **unique** blossoming tree whose closure is M endowed with its orientation.

Bijections with blossoming trees

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Combined with the general theory of c-orientations [Propp 03] and/or α-orientations [Felsner 04], this allows to retrieve **all the bijections** mentioned above and to obtain **new bijections** for which no enumerative formulas are available (cf also [Bernardi, Fusy 12]).

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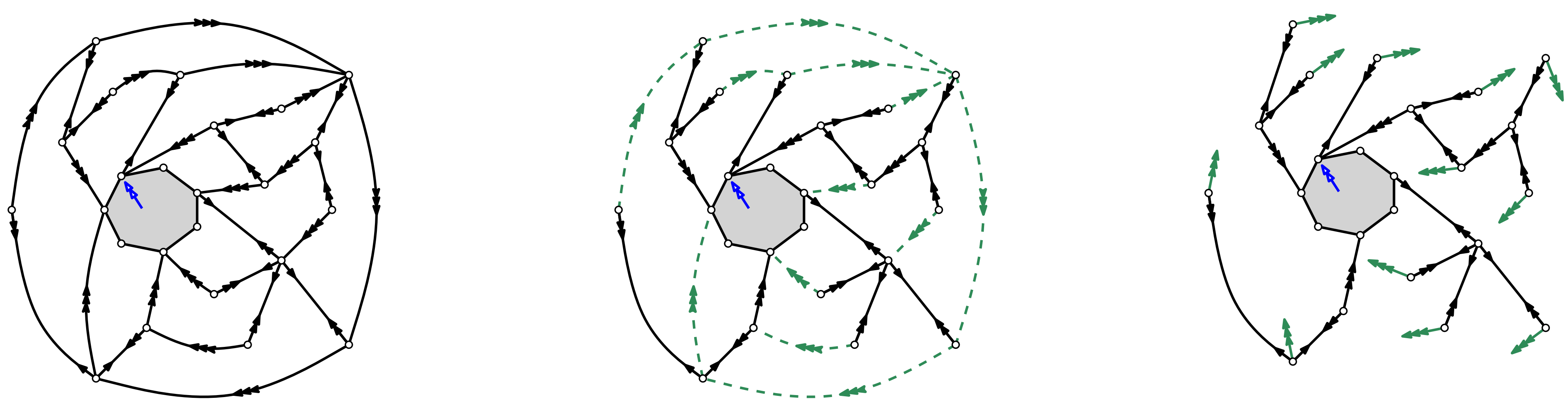
Often **easier to "guess" the right orientations** than the right families of trees.



Blossoming bijection for d-angulations of girth d with a boundary, [A., Poulalhon 15].

= length of the smallest cycle

Blossoming bijections in higher genus

Theorem: [Tutte 63], bijective proof in [Schaeffer 97]

$$M(z) = \sum_m z^{|E(m)|}, \text{ where } m \in \Big\{\text{planar maps}\Big\}.$$

Then: $M = T^2(1-4T)$ where T unique formal power series defined by $T = z + 3T^2$

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For any $g \geq 1$, let $M_g(z) = \sum_m z^{|E(m)|}$, where $m \in \Big\{\text{maps of genus } g\Big\}$.

Then M_g is a rational function of T.

Idea of proof: Generalization of Schaeffer's blossoming bijection to higher genus.
Careful analysis of the blossoming **unicellular** maps

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Result not available with the "mobile-type" bijection of [Chapuy – Marcus – Schaeffer]

Blossoming bijections in higher genus

Theorem: [Tutte 63], bijective proof in [Schaeffer 97]

$$M(z_\bullet, z_\circ) = \sum_m z_\bullet^{|V(m)|} z_\circ^{|F(m)|}, \text{ where } m \in \Big\{\text{planar maps}\Big\}.$$

Then $\quad M = T_\circ T_\bullet(1 - 2T_\circ - 2T_\bullet) \quad$ where $\quad \begin{cases} T_\bullet & = z_\bullet + T_\bullet^2 + 2T_\circ T_\bullet \\ T_\circ & = z_\circ + T_\circ^2 + 2T_\bullet T_\circ \end{cases}$

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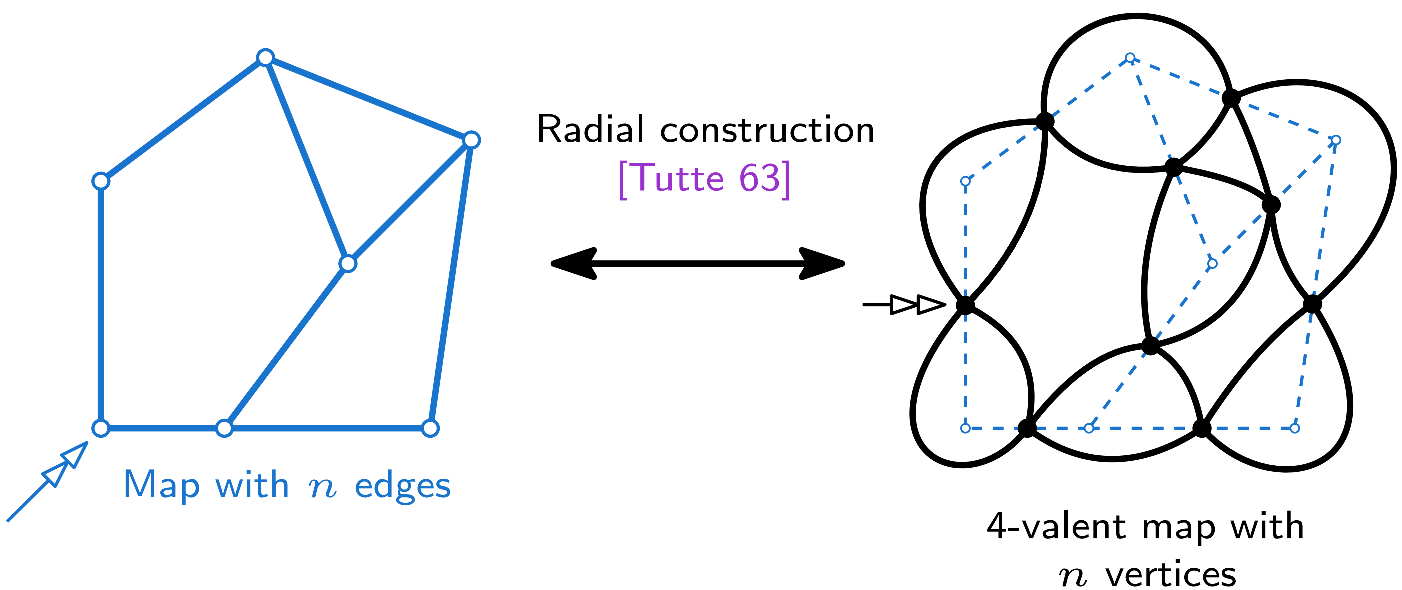

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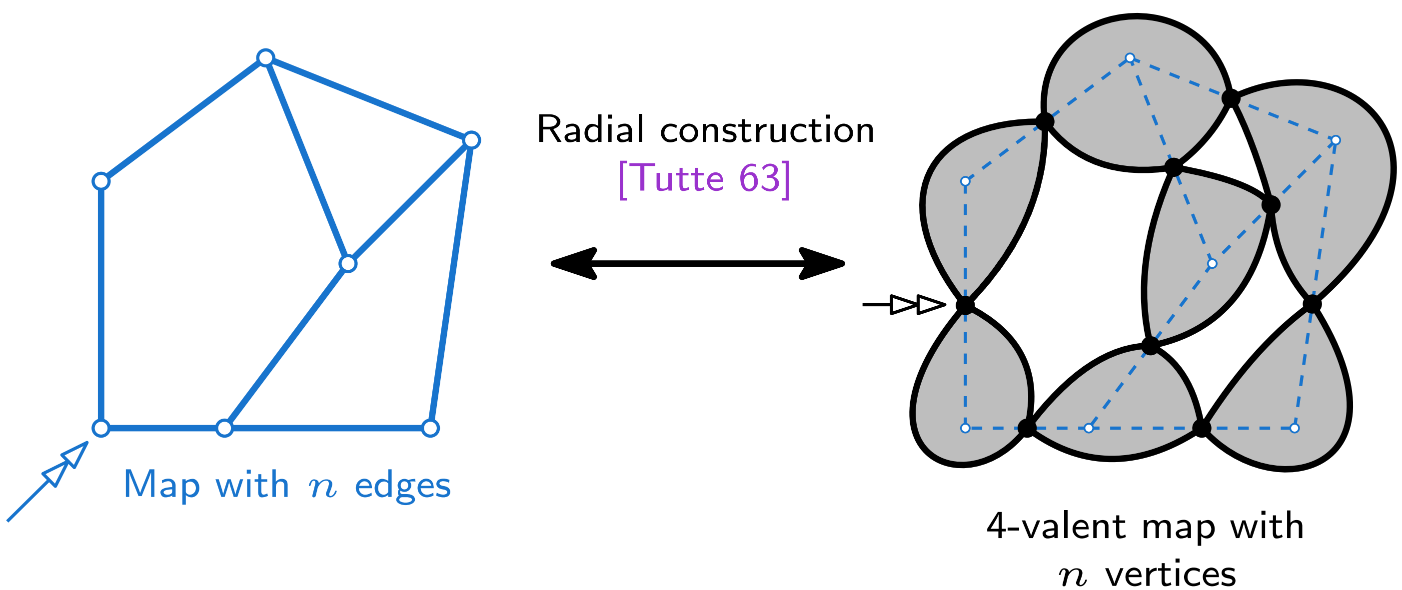

Already for planar maps, this result is not accessible with mobile-type bijections.

Blossoming bijections in higher genus

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Theorem: [Bender, Canfield, Richmond 95], bijective proof in [A.,Lepoutre 20+]
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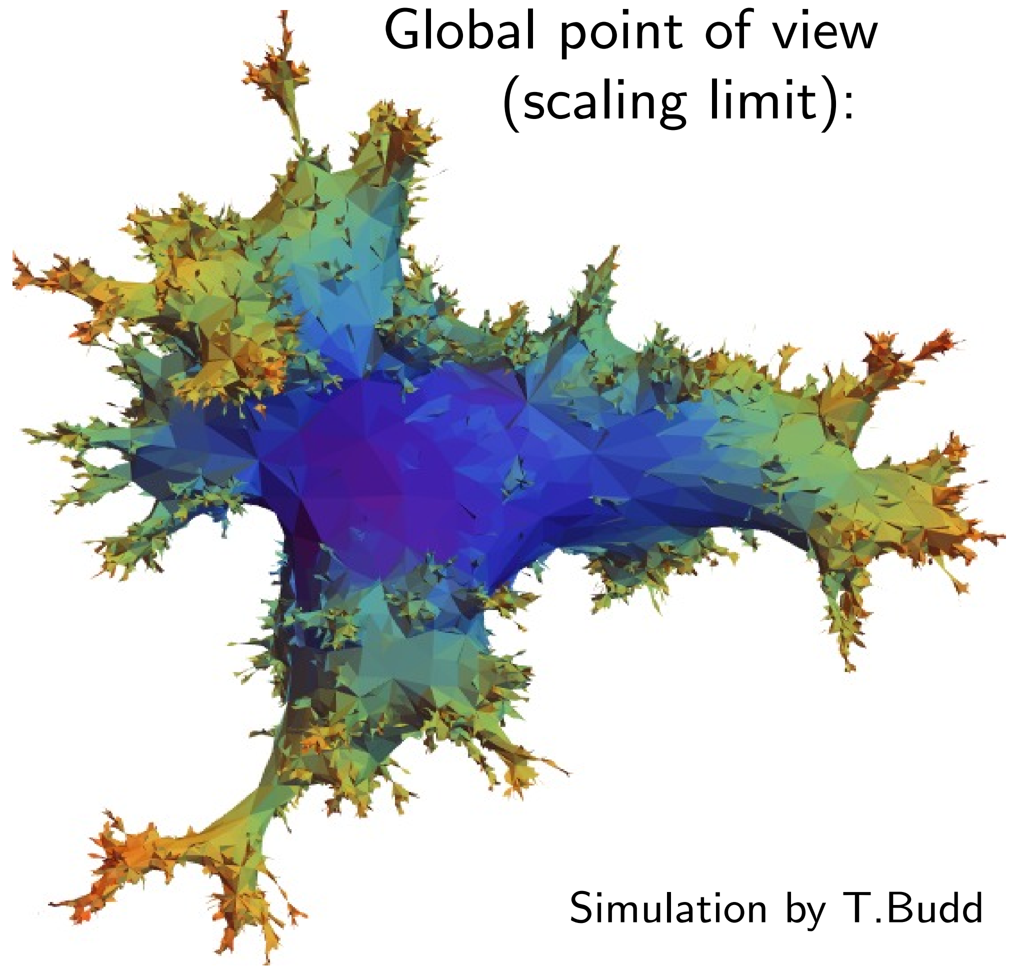
Then M_g is a rational function of $T_\bullet$ and $T_\circ$.

Idea of proof: Same bijection but different proof for the analysis of the unicellular blossoming maps (gives also a simpler proof of the univariate case).

II - Scaling limits of random maps

Global point of view
(scaling limit):



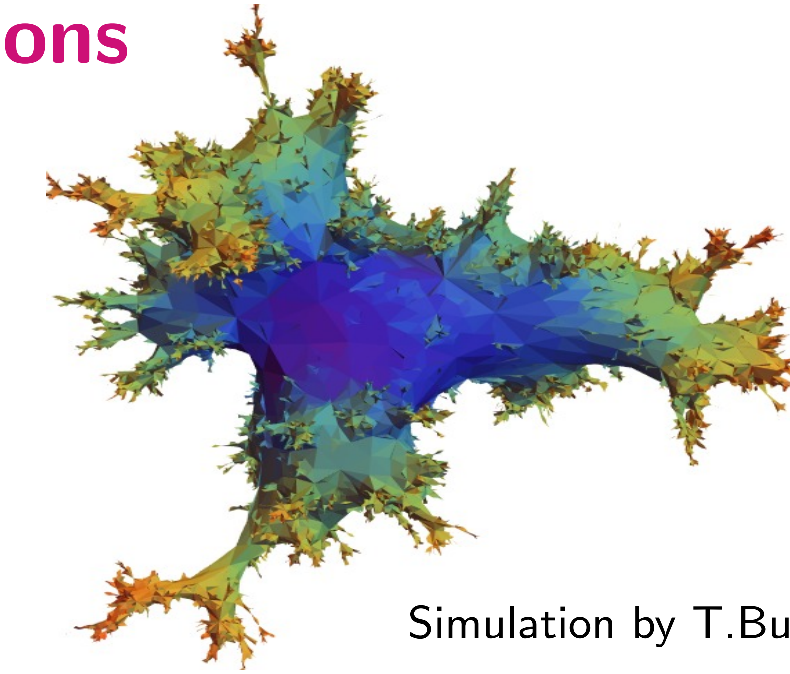
Simulation by T.Budd

Scaling limit of random quadrangulations

$\mathcal{Q}_n = \{\text{Quadrangulations of size } n\}$

$= n + 2$ vertices, n faces, $2n$ edges

$Q_n =$ Uniform random element of $\mathcal{Q}_n$



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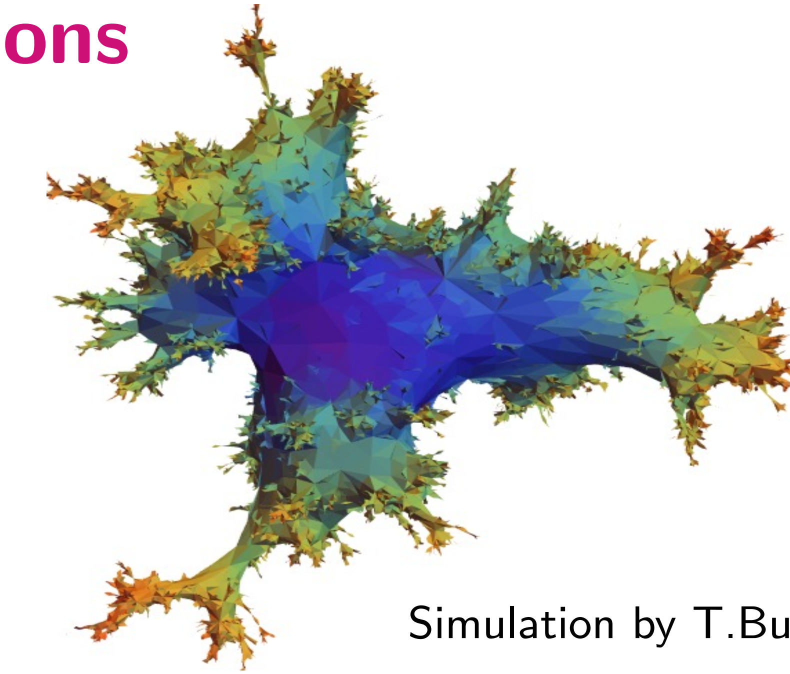
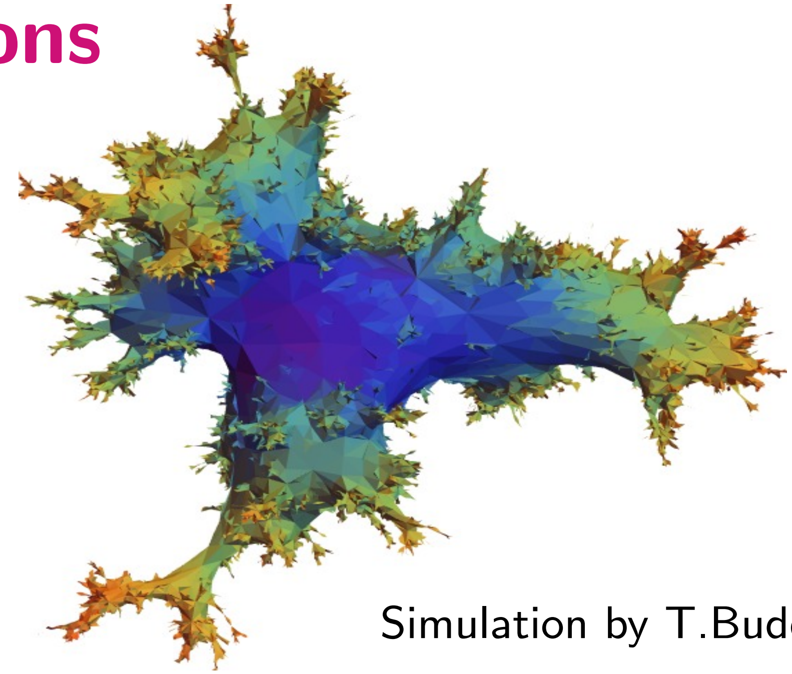


Simulation by T.Budd

When the size of the map goes to infinity, so does the typical distance between two vertices.

Idea: "scale" the map = length of edges decreases with the size of the map.
Goal: obtain a limiting (non-trivial) compact object

Scaling limit of random quadrangulations



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Motivations:

- Natural random discretization of a continuous surface.
- Construction of a 2-dim. analogue of the Brownian motion: **The Brownian Map** [Miermont 13],[Le Gall 13].
- Link with Liouville Quantum Gravity,
[Duplantier, Sheffield 11], [Duplantier, Miller, Sheffield 14], [Miller, Sheffield 16,16,17]

Scaling limit of uniform quadrangulations

Idea: "scale" the map = length of edges decreases with the size of the map.
Goal: obtain a limiting (non-trivial) compact object

For quadrangulations : well understood

- **The** bijection of Schaeffer: quadrangulations $\leftrightarrow$ labeled trees.
Labels in the trees = distances between the vertices and the root.

- distance between two random points $\sim n^{1/4}$ + law of the distance [Chassaing-Schaeffer '04]

- cvgence of normalized quadrangulations + properties of the limit [Marckert-Mokkadem '06], [Le Gall '07], [Le Gall, Paulin '08] [Miermont '08]

Hausdorff dimension = 4

topology of the limit = sphere

Theorem: [Miermont 13], [Le Gall 13]

Let (Q_n) be a sequence of random quadrangulations of size n. Then:

$$\left(V(Q_n), \left(\frac{9}{8n}\right)^{1/4} d_{gr}\right) \xrightarrow[\text{Gromov–Hausdorff topology}]{(d)} \text{The Brownian Map}$$

Universality of the scaling limit

what if quadrangulations are replaced by triangulations, maps, simple triangulations, ...?

Idea : The Brownian map is a **universal** limiting object.

All "reasonable models" of maps (properly rescaled) are expected to converge towards it.

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Theorem: [Le Gall 13]

Fix $p \in \{3\} \cup 2\mathbb{N}$, let (M_n) be a sequence of random p-angulations of size n. Then:

$$\left(V(M_n), \frac{C_p}{n^{1/4}} d_{gr}\right) \xrightarrow[\text{Gromov–Hausdorff topology}]{(d)} \text{The Brownian Map}$$

Idea of the proof:

Replace Schaeffer's bijection by the bijection of [Bouttier, Di Francesco, Guitter 04].

Le Gall's magic trick:

Since uniform quadrangulations are invariant by rerooting, the fact that they converge to the Brownian map, implies that the **Brownian map is invariant by rerooting**.

$\rightarrow$ Use this invariance to prove the convergence of others models of maps.

Universality of the scaling limit

To prove that another model of maps converges to the Brownian map:

1. encode the maps by some labeled trees,
2. **study the limits of the labeled trees**,
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Theorem: [Addario–Berry, A. 15] Let (Δ_n) = random **simple** triangulations:

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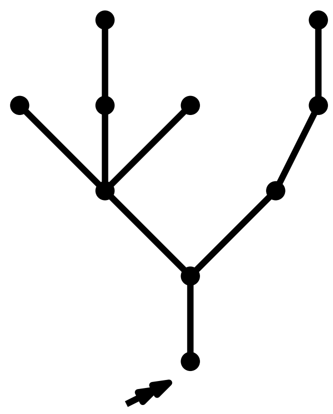
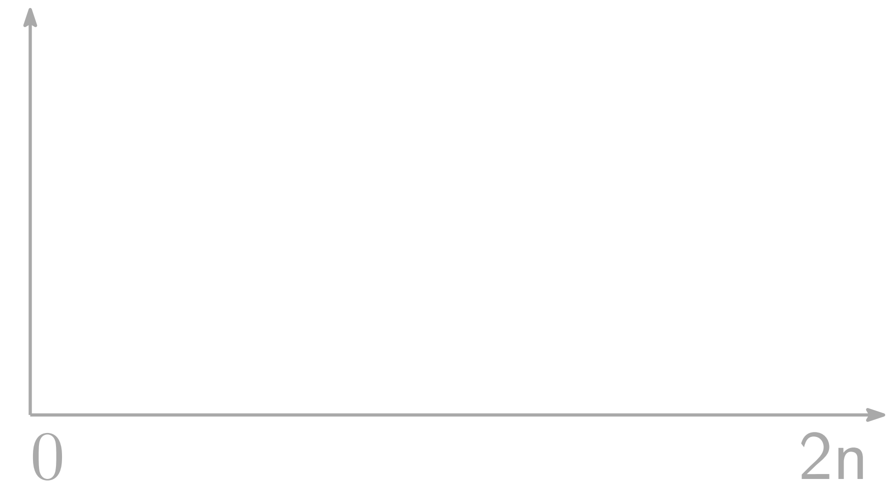
Difficulty: 3. Track distances in blossoming bijections.

Convergence of trees





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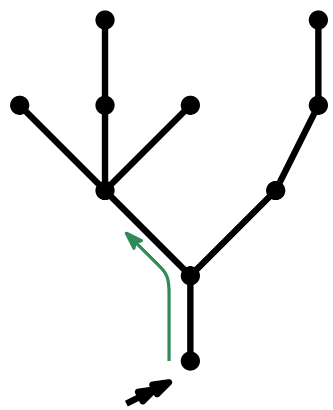

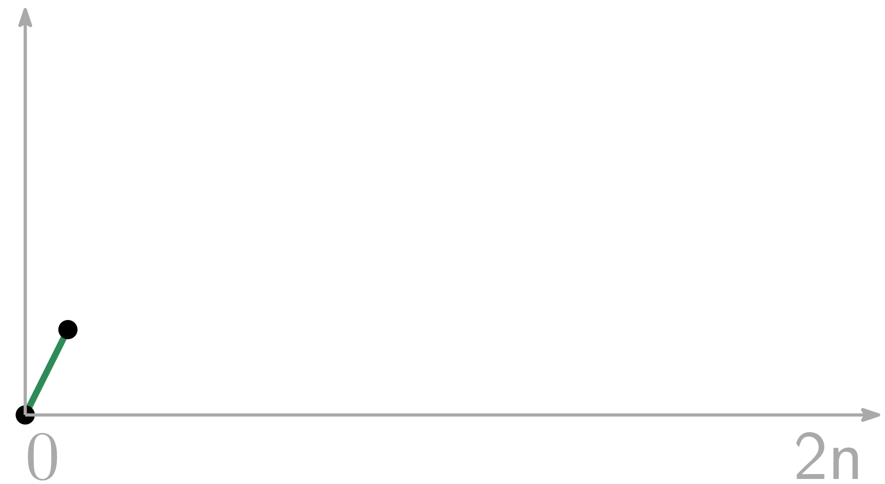

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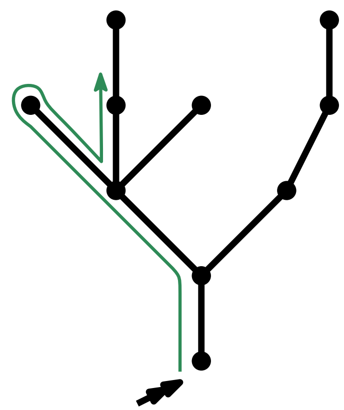

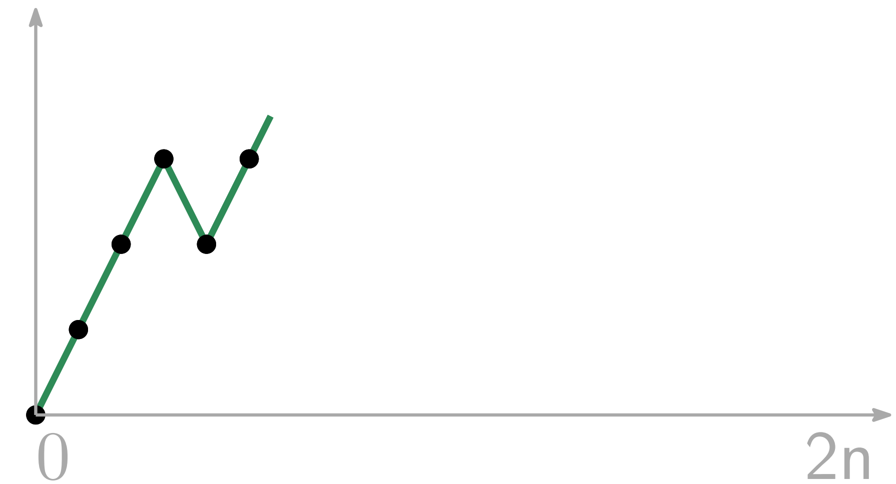

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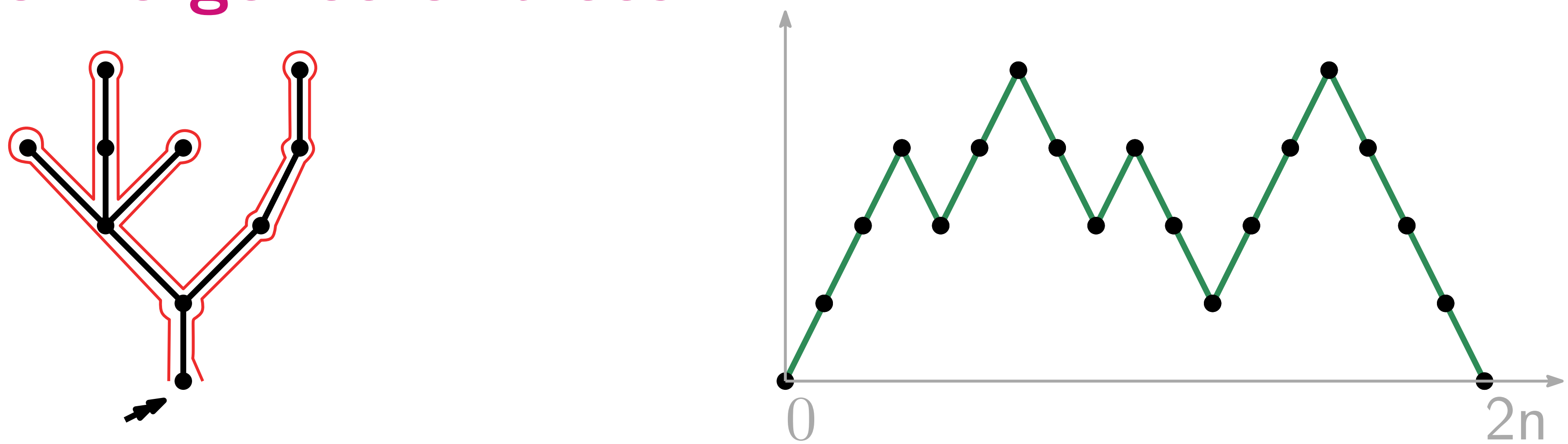

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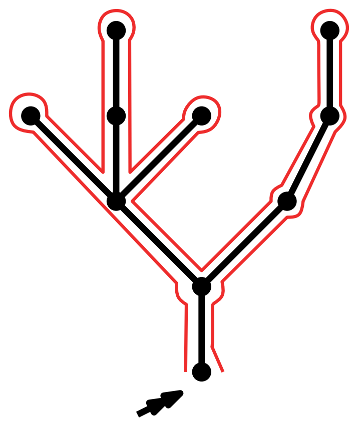
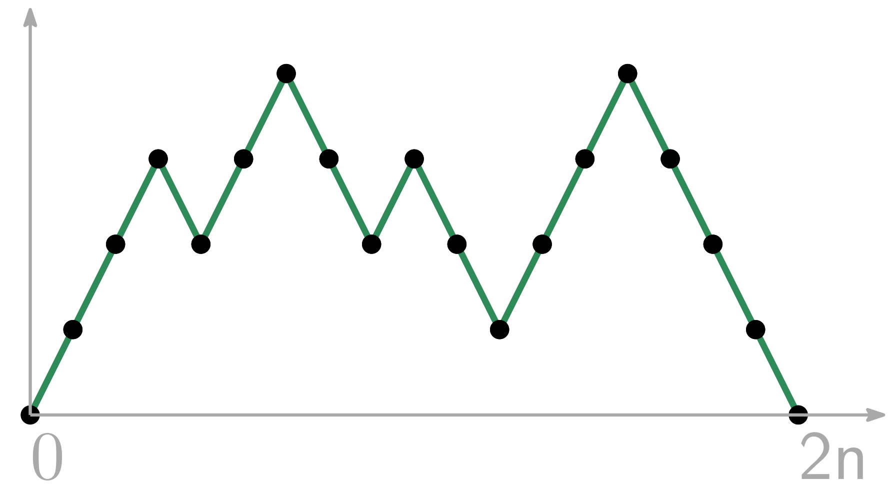
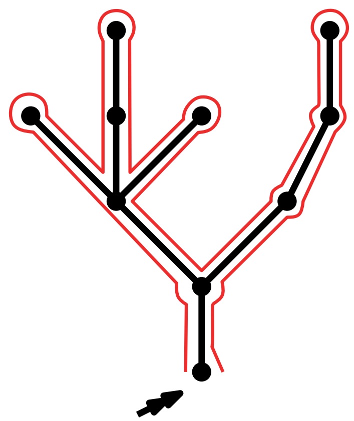


T ⟶



$C_n(2n \cdot t) =$ contour process

Convergence of trees



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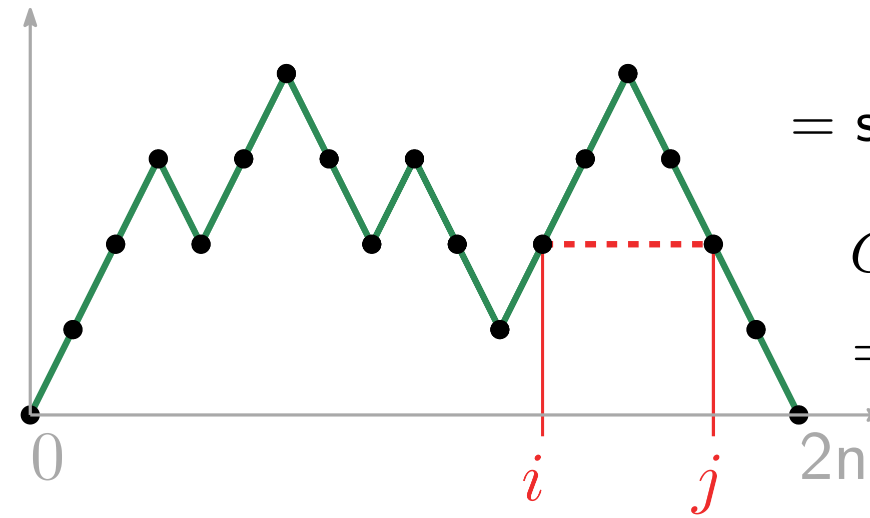

i and j

$=$ same vertex of T

$\Leftrightarrow$

$C_n(i) = C_n(j)$

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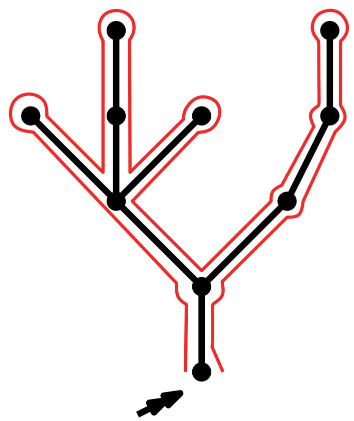

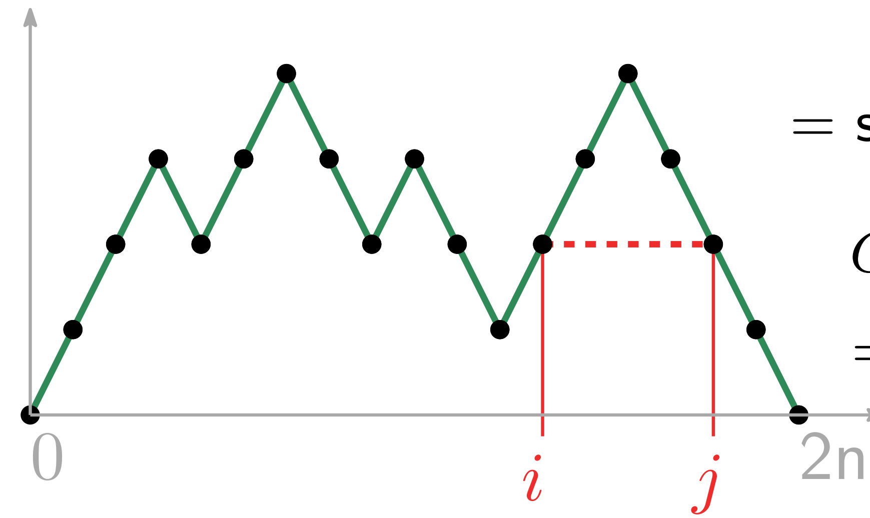

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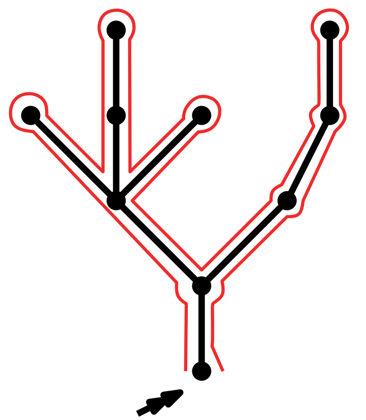
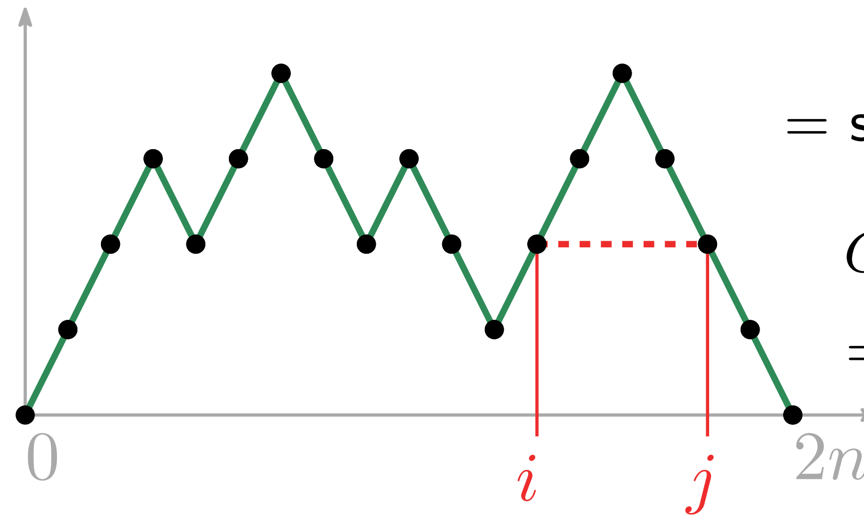
$T \longrightarrow C_n(2n \cdot t) =$ contour process

$T =$ labeled tree, $\longrightarrow (C_n(2n \cdot t), Z_n(2n \cdot t)) =$ contour and label processes

Convergence of trees



T



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scaling limit (rescaled by $n^{-1/2}$)

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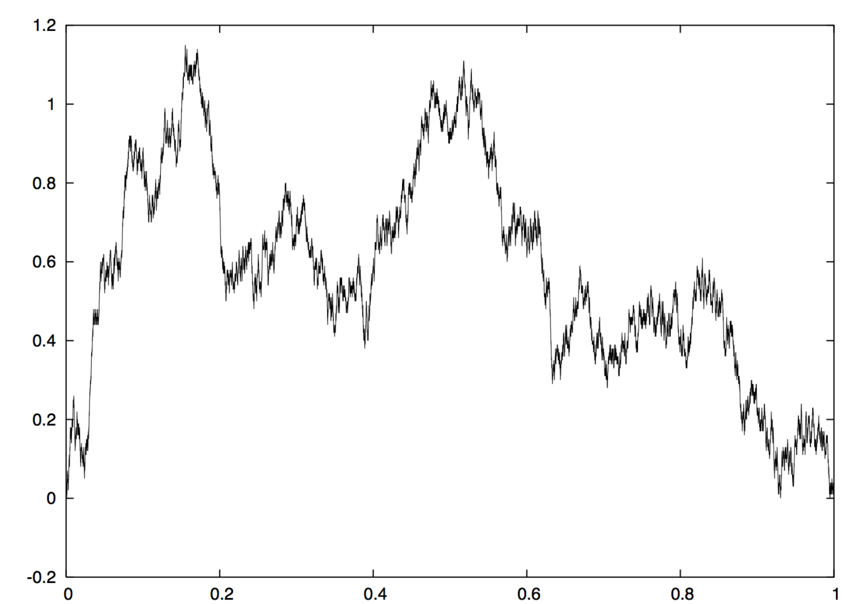
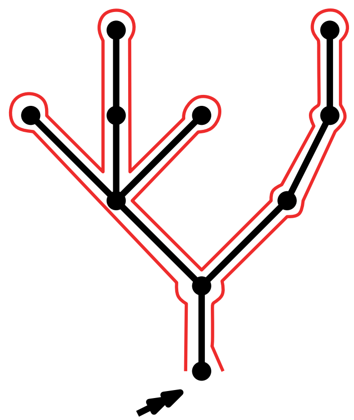
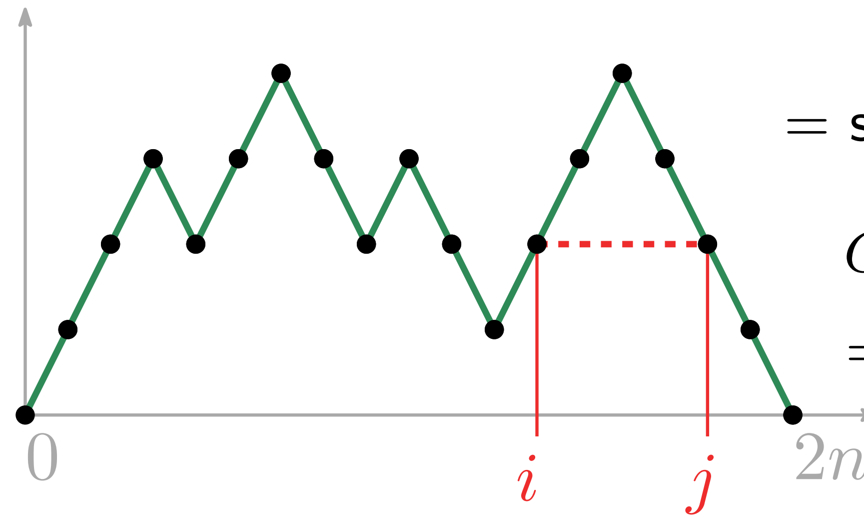


Simulations by I. Kortchemski

Convergence of trees



T $\longrightarrow$



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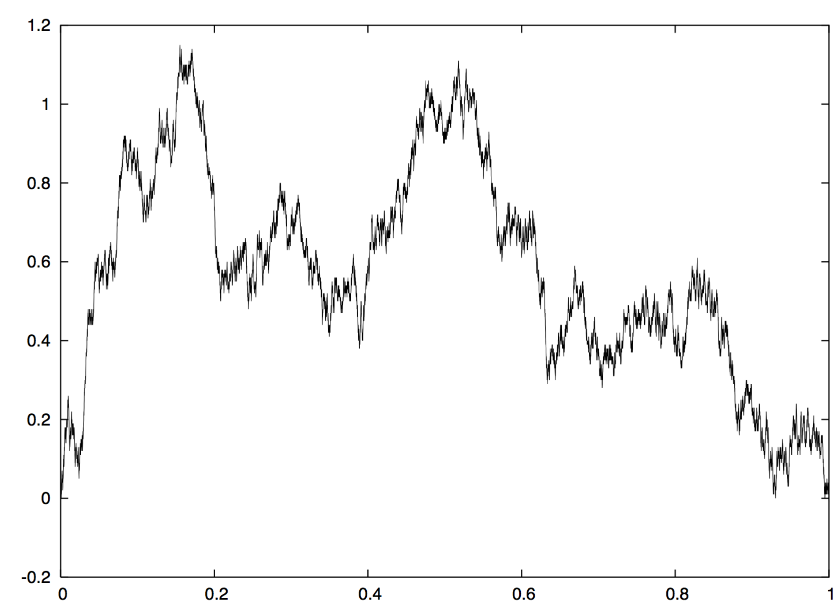

$\longleftarrow$

Continuum Random Tree

$\mathcal{T}_e$, [Aldous'91]

Simulations by I. Kortchemski

Convergence ~~of trees of~~ labeled trees

1st step : the Brownian tree

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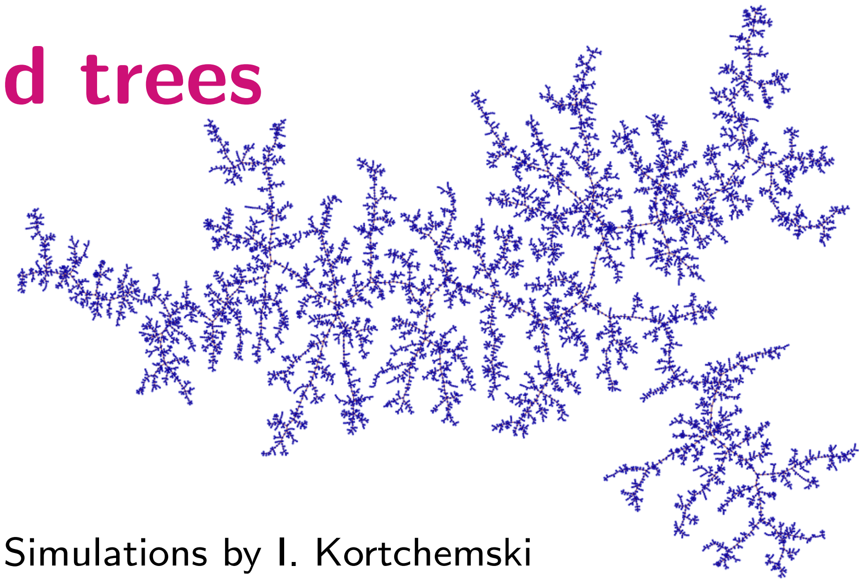
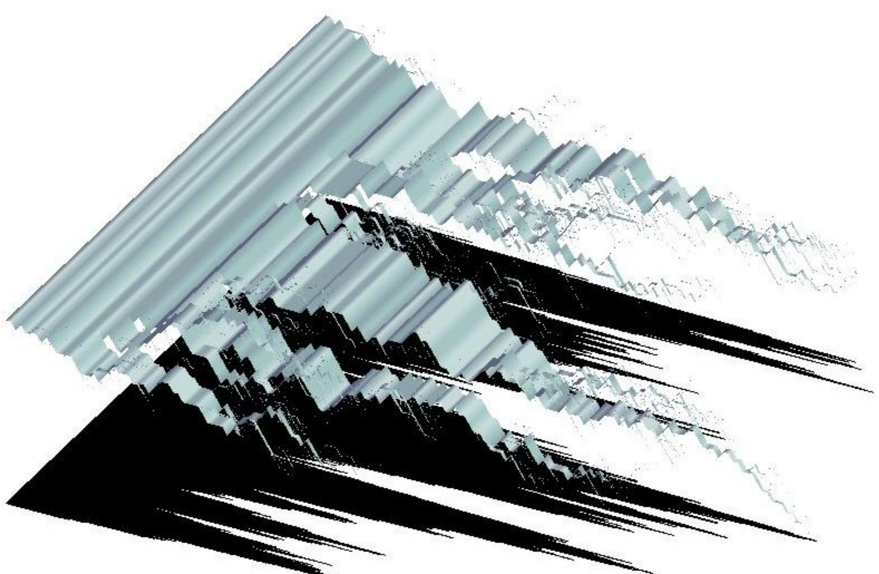
Convergence ~~of trees of~~ labeled trees

1st step : the Brownian tree



Simulations by I. Kortchemski

2nd step : Brownian snake



Simulation by J. Bettinelli

$(e_t, Z_t) =$ **Brownian snake**

[Le Gall 93]

Theorem: [Janson, Marckert 04],[Miermont 08],...

For a sequence (T_n) of "nice" random labeled trees:

$$\left(\frac{aC_n(2nt)}{\mathbf{n^{1/2}}}, \frac{bZ_n(2nt)}{\mathbf{n^{1/4}}}\right) \xrightarrow{(d)} (e_t, Z_t)$$

for the uniform topology of $\mathcal{C}([0,1],\mathbb{R})^2$,

Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_s - Z_t)^2] = d_e(s,t)$.

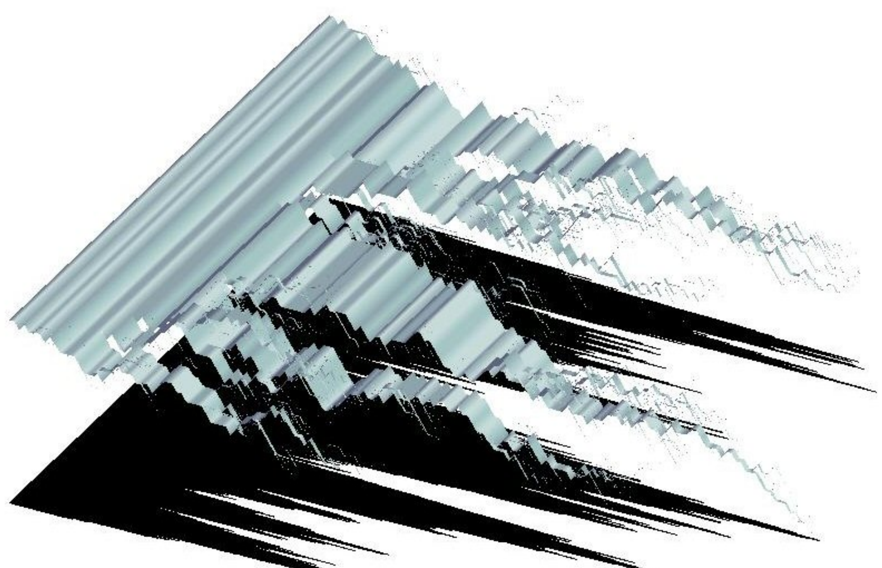
$Z \sim$ **Brownian motion on the tree**

Convergence ~~of trees of~~ labeled trees

1st step : the Brownian tree

Simulations by I. Kortchemski

2nd step : Brownian snake



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Nice = typically Galton-Watson trees, with **centered** increments of labels along edges.

Convergence of odd-angulations

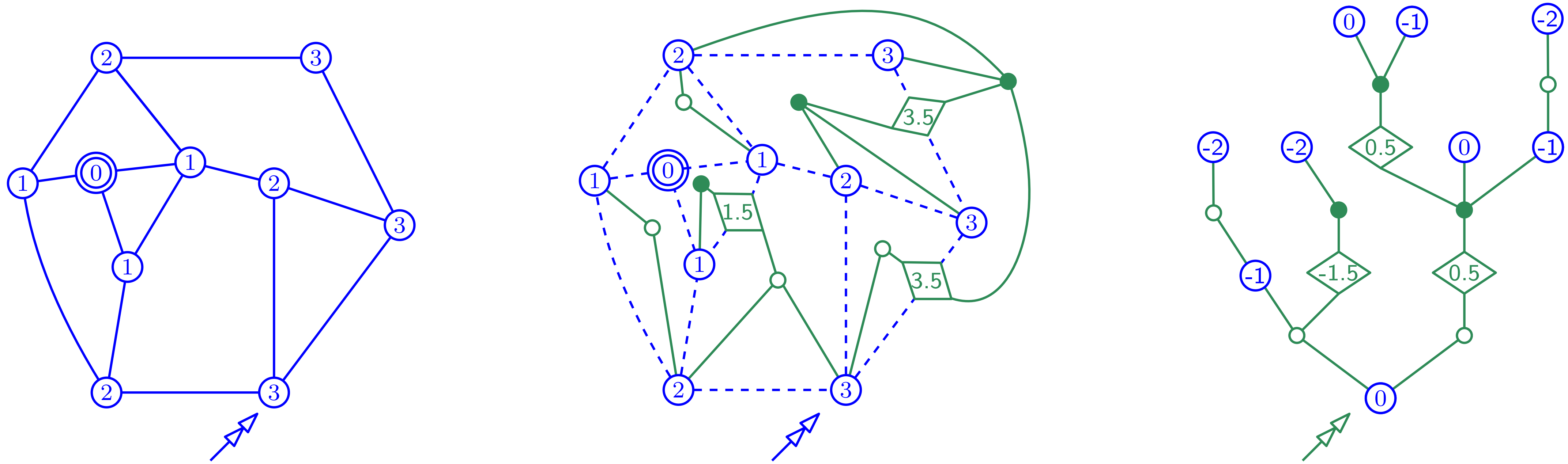

Illustration of the Bouttier – Di Francesco – Guitter bijection for a **non-bipartite** map.

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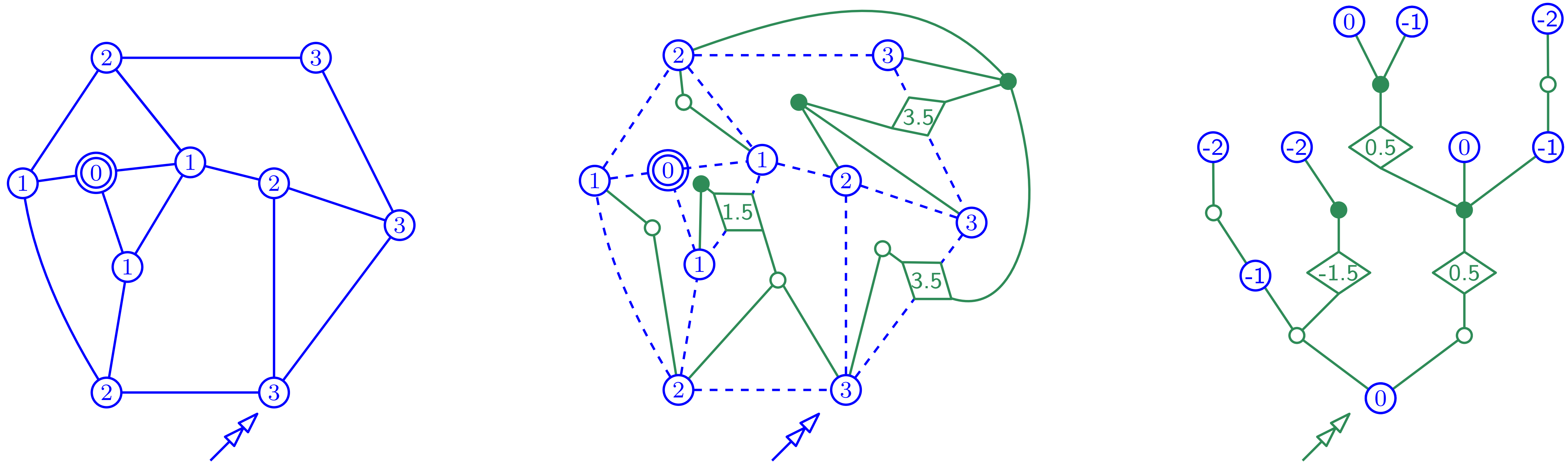

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Labeled tree obtained = 4-type Galton-Watson tree T + random label increments along edges.

Problem: For e an edge of T, $\mathbb{E}[\text{label increments along } e] \neq 0$

i.e. the **the label increments are not centered**.

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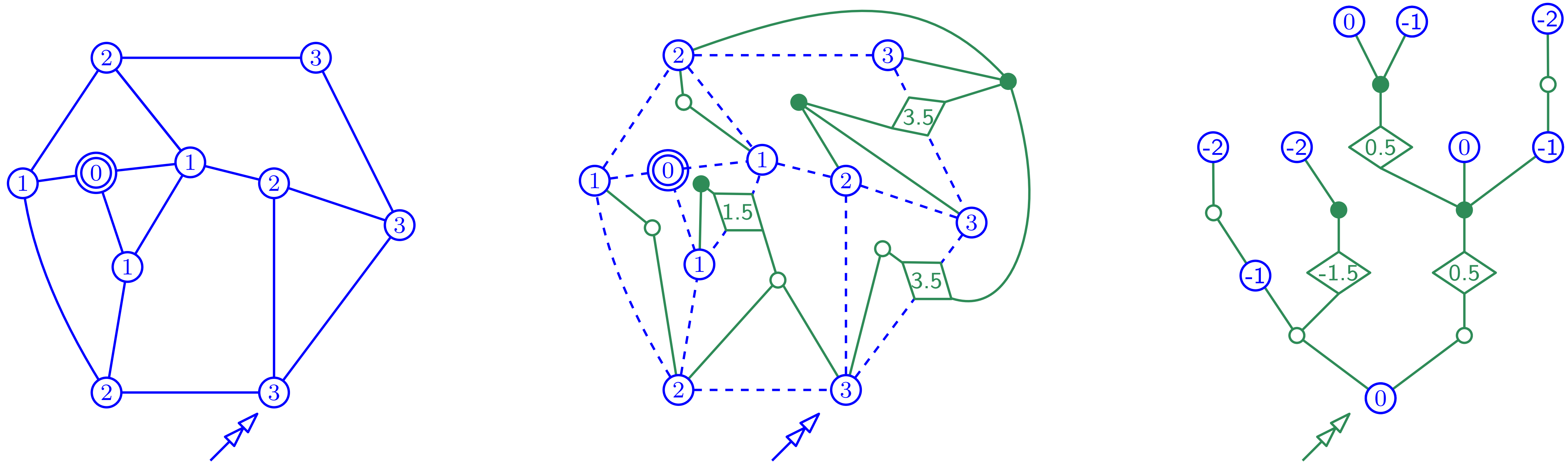

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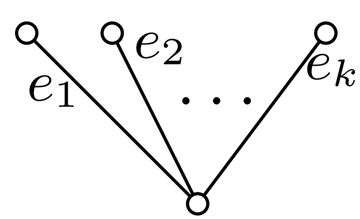
$\Rightarrow$ Known results of convergence do not apply.

A source of hope:



$\sum_{i=1}^{k} \mathbb{E}[\text{label increment along } e_i] = 0$

Convergence of odd-angulations

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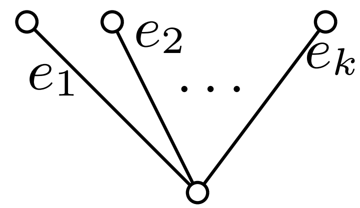

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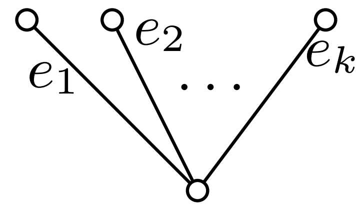

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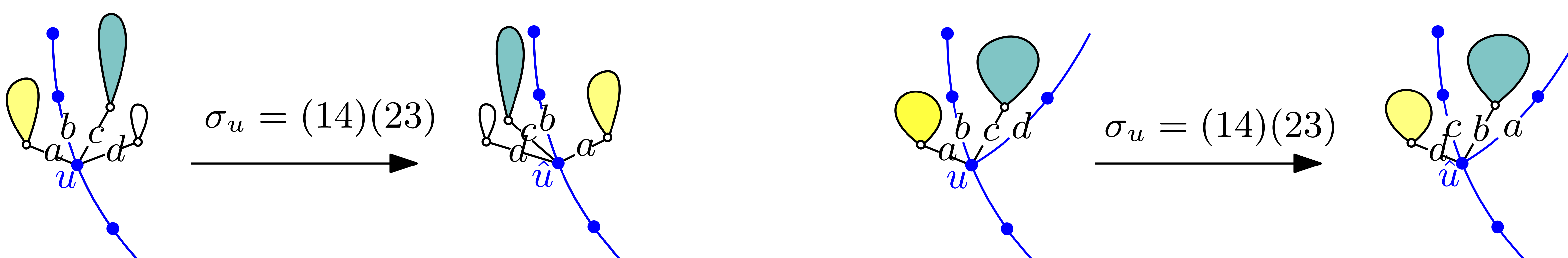
[Marckert 07] convergence in this setting (with even weaker "centering assumption") but requires **monototype** GW trees + **bounded** number of children.

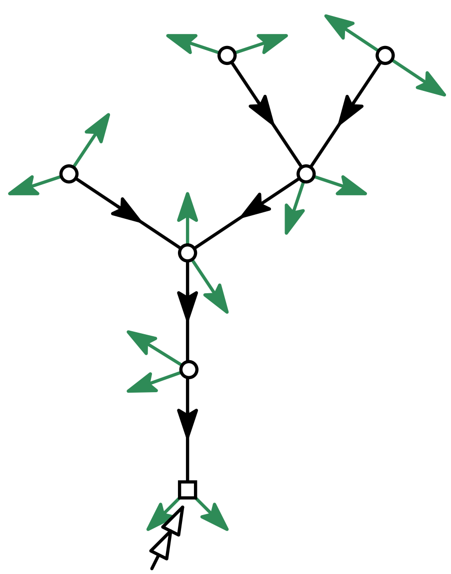
Strategy of proof: Randomly shuffle "our" trees to get a coupling with a "nice" model.

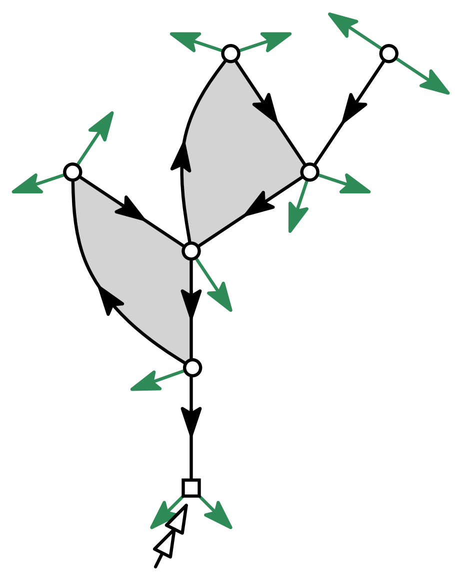
in our case [Miermont 08] is the nice model but it gives a general **bootstrapping principle**.

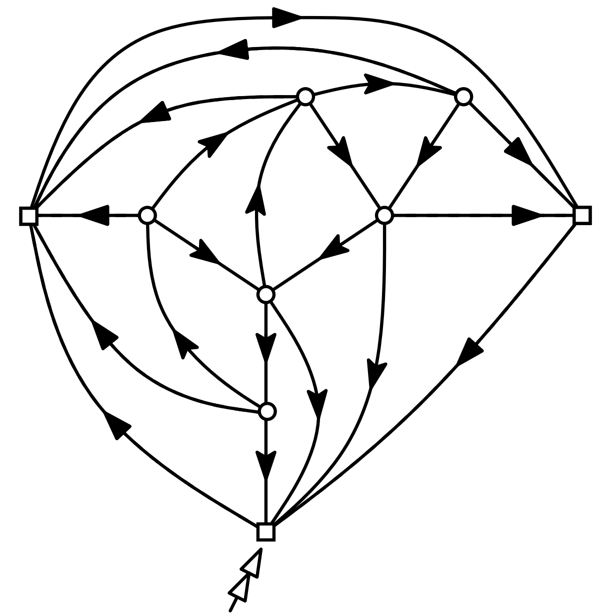


Convergence of simple triangulations

First step: blossoming bijection of [Poulalhon, Schaeffer 05] for simple triangulations.







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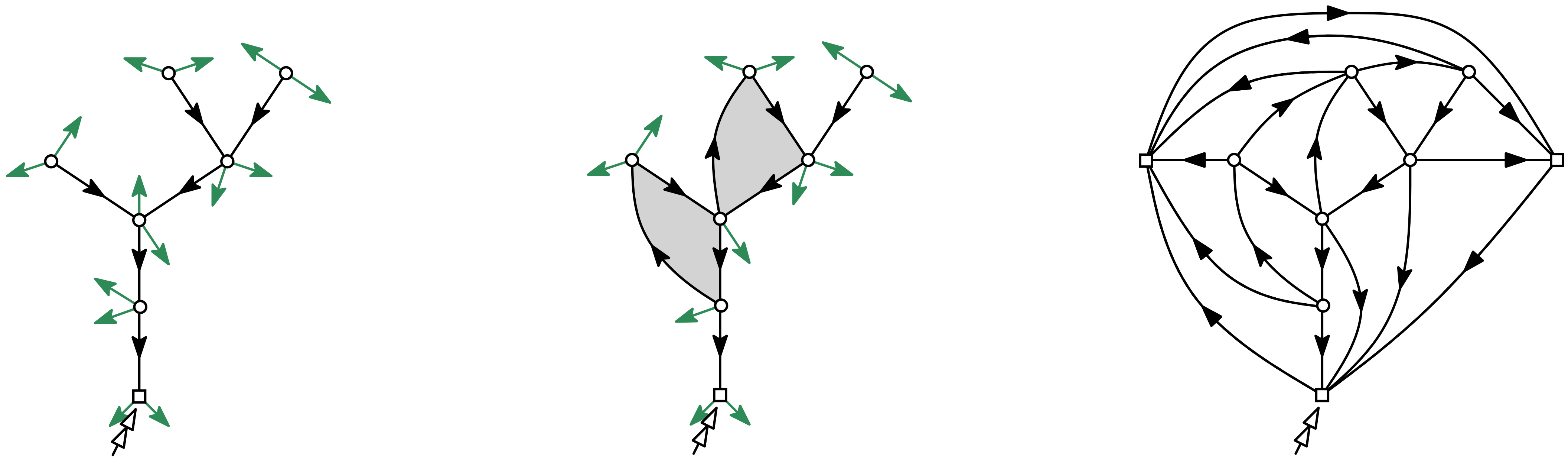

Problem: How to track distances in the map in the blossoming tree ?

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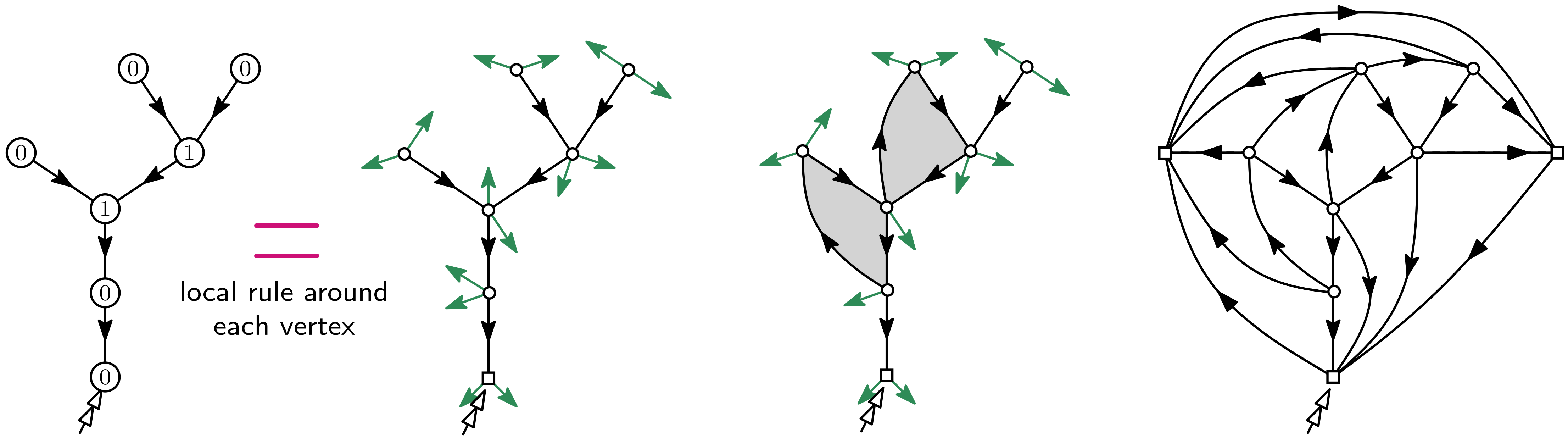

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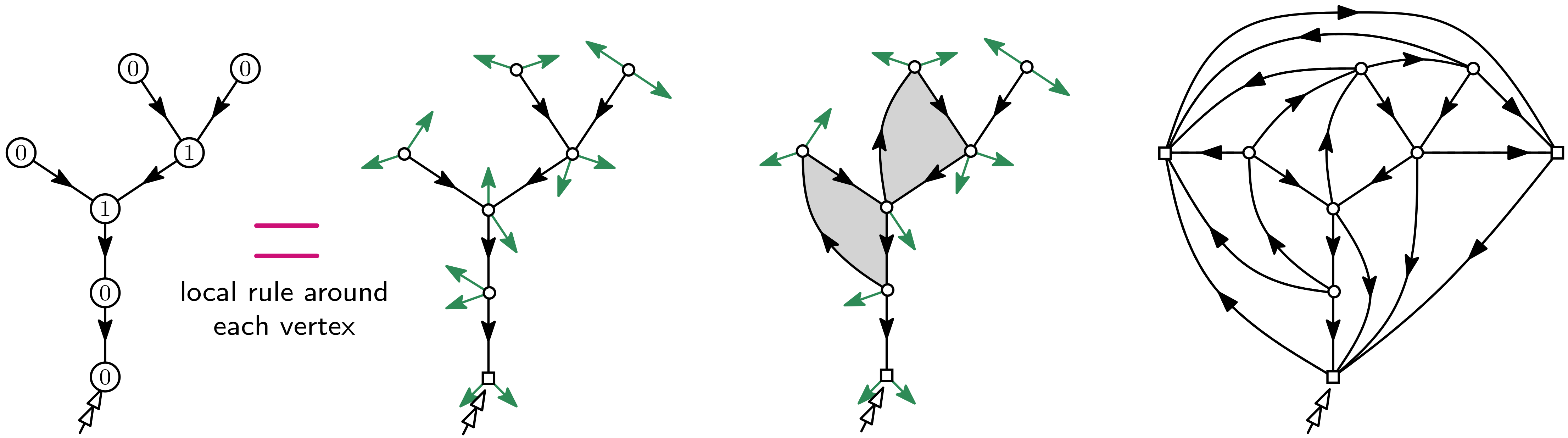

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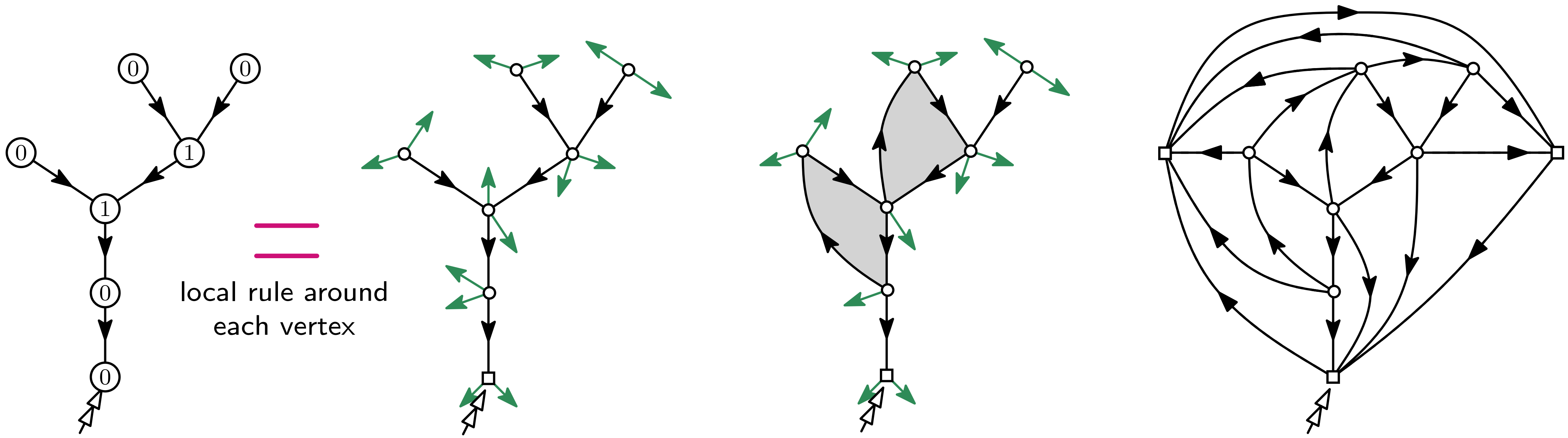

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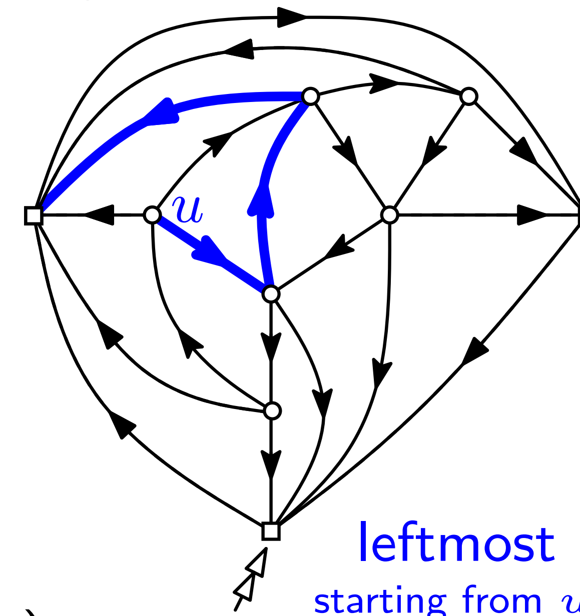

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$\rightarrow$ Two key combinatorial observations:

Labels in the **tree** = length of **leftmost path** in the **map**

Leftmost paths are **almost geodesic** (up to $o(n^{1/4})$ error term).

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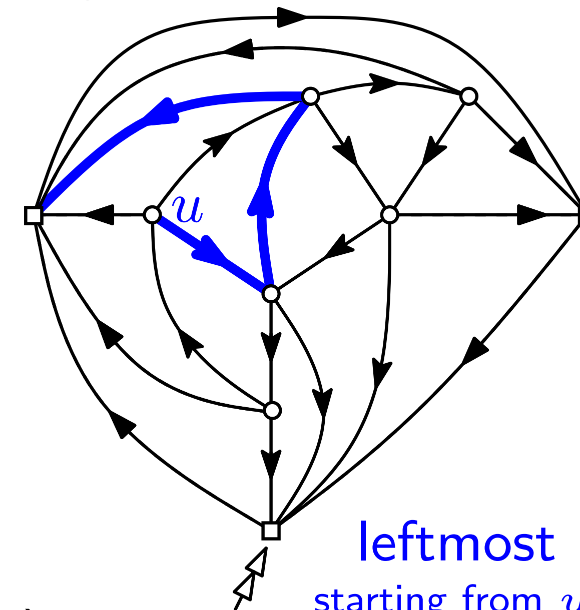

leftmost path:
starting from u,
ending at the root face

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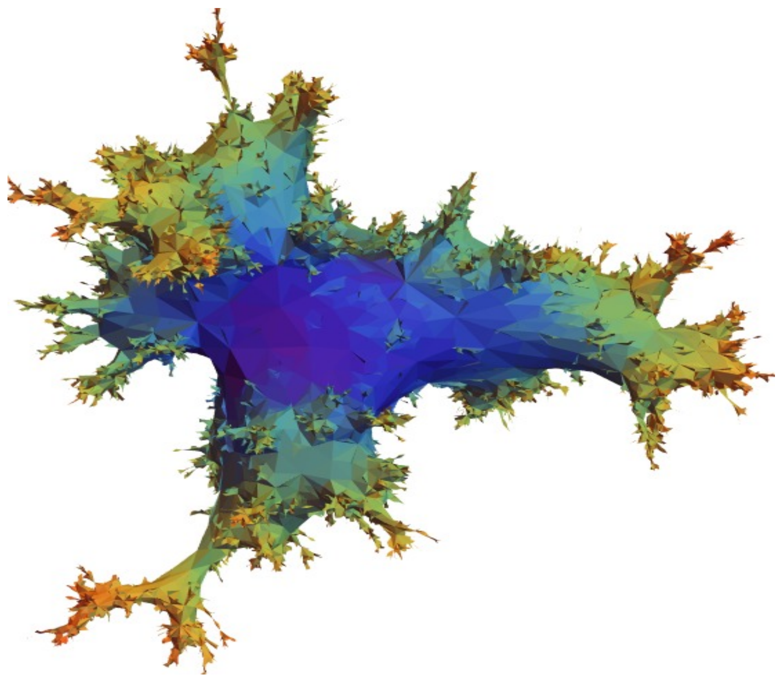
$$\left(V(\Delta_n), \left(\frac{3}{4n}\right)^{1/4} d_{gr}\right) \xrightarrow[\text{Gromov–Hausdorff topology}]{(d)} \text{The Brownian Map}$$

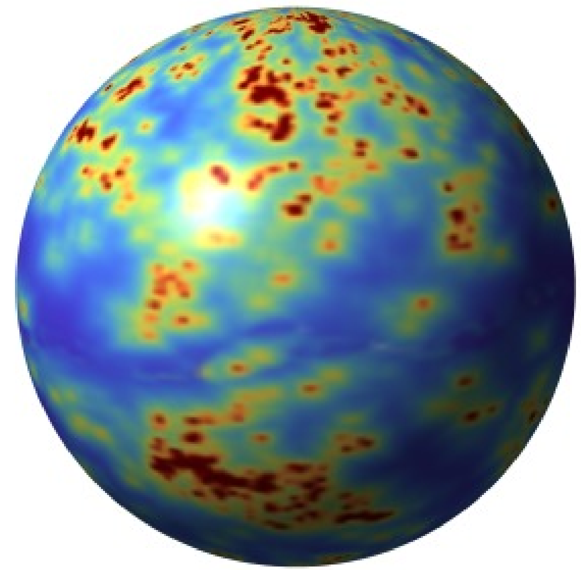
Same ideas sucessfully applied to study simple maps [Bernardi, Collet, Fusy 14], simple triangulations on the torus [Beffara, Huynh, Lévêque 20], simple triangulations with a boundary [A., Holden, Sun 20]

Link with Liouville Quantum Gravity

$\gamma \in (0,2)$, γ-Liouville Quantum Gravity = measure on a surface [Duplantier, Sheffield 11].



Simulation of the Brownian map by T.Budd



Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd

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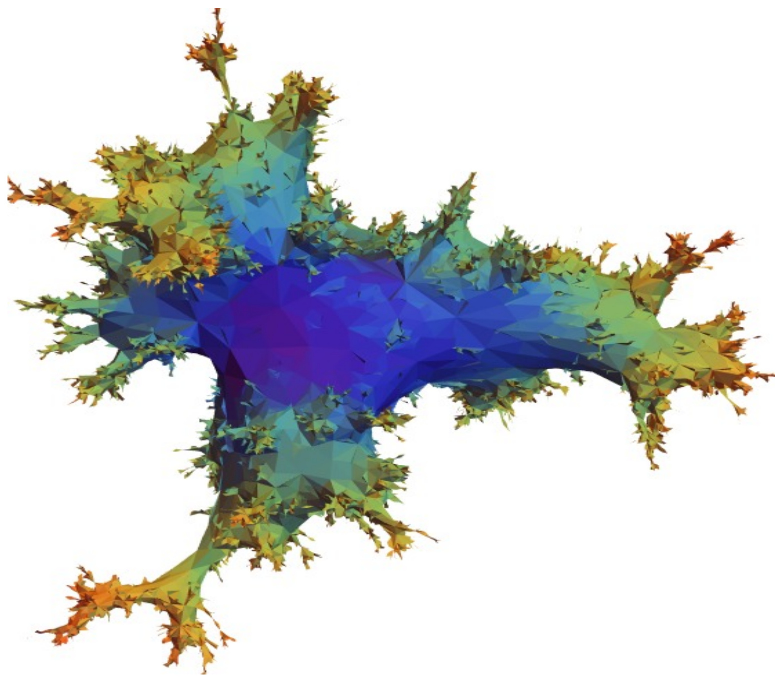


Simulation of the Brownian map by T.Budd

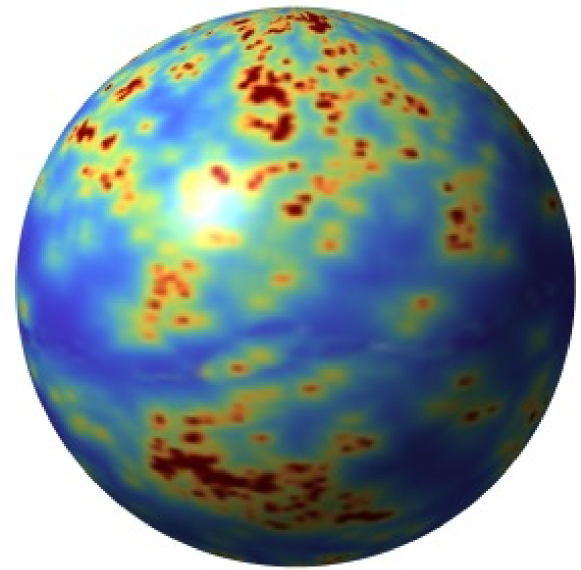
[Duplantier, Miller, Sheffield 14]

[Miller, Sheffield 16+16+17]

$\longleftrightarrow$

Construction in the continuum.

in the discrete setting ?



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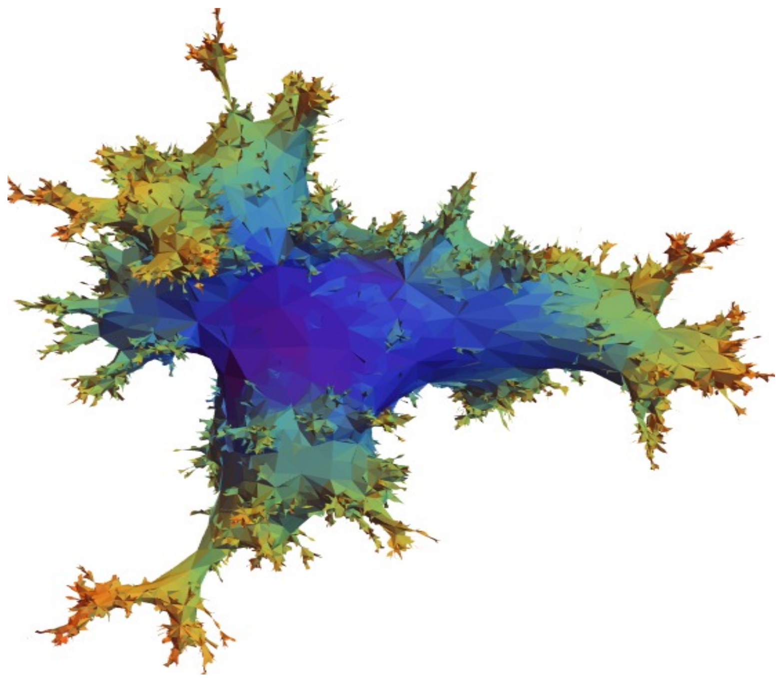
Link with Liouville Quantum Gravity

$\gamma \in (0,2)$, γ-Liouville Quantum Gravity $=$ measure on a surface [Duplantier, Sheffield 11].



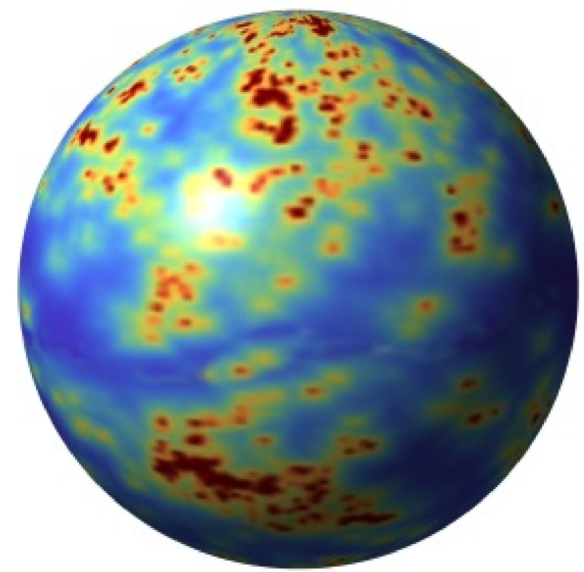
Simulation of the Brownian map by T.Budd

[Duplantier, Miller, Sheffield 14]
[Miller, Sheffield 16+16+17]

$\longleftrightarrow$

Construction in the continuum.

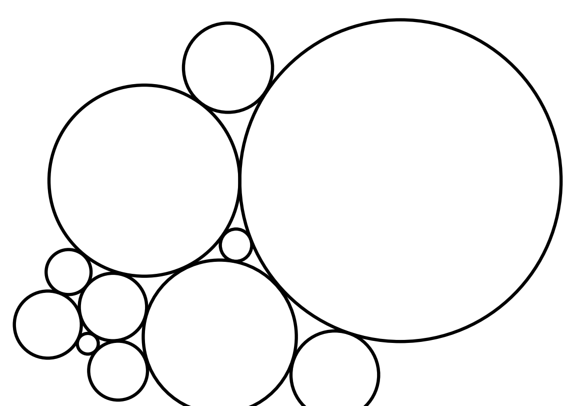
in the discrete setting ?

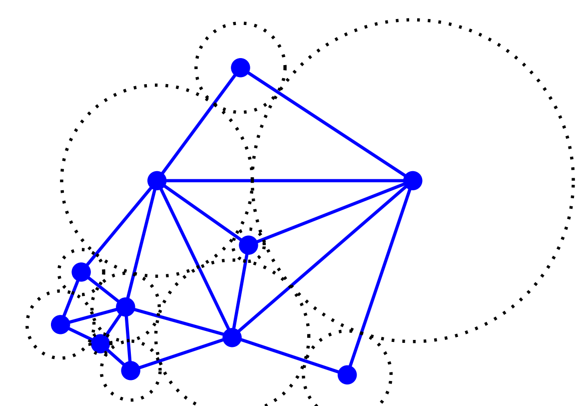


Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd

A priori , there is no canonical way to embed a planar map in the sphere.

But, for simple triangulations:
the **circle packing theorem**
gives a canonical embedding.
(Unique up to Möbius transformations.)





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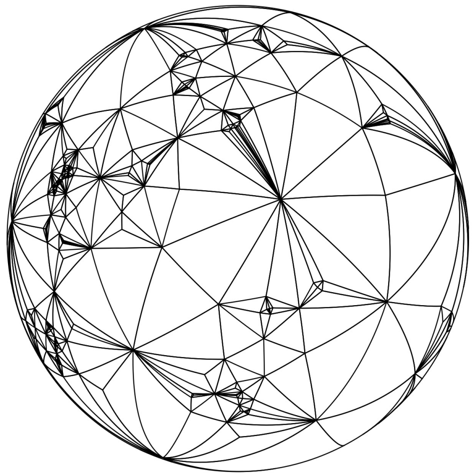
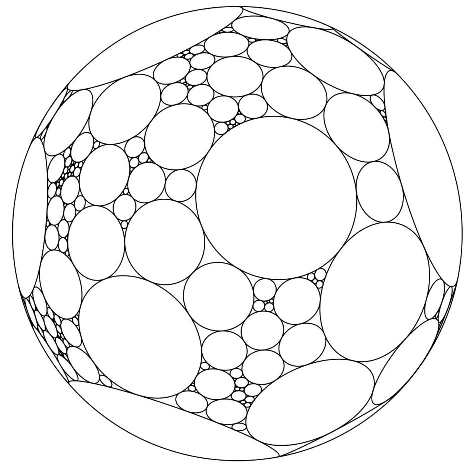
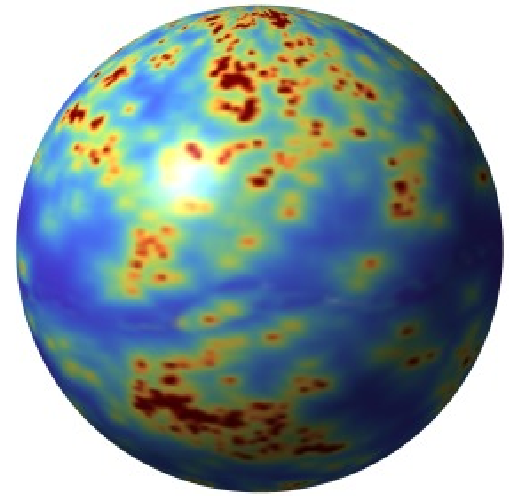




Simulation of a large simple triangulation
embedded in the sphere by circle packing.

Software CirclePack by K.Stephenson.



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Motivation to study simple triangulations, but so far no results for random circle packings.

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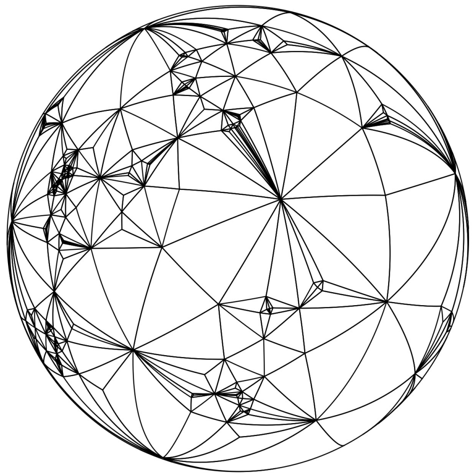

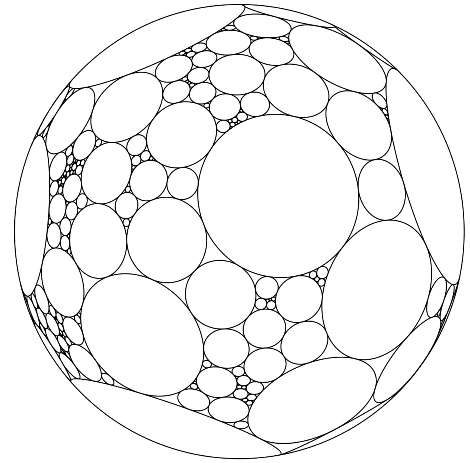

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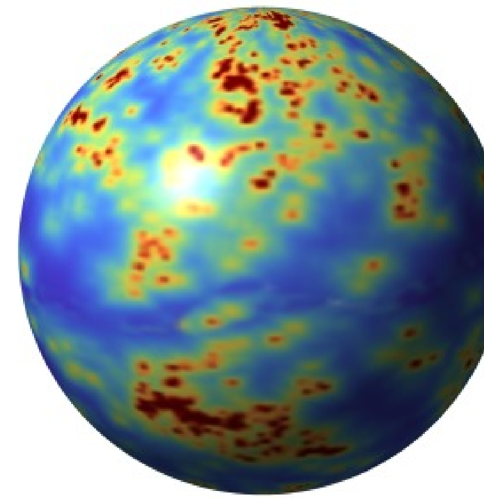


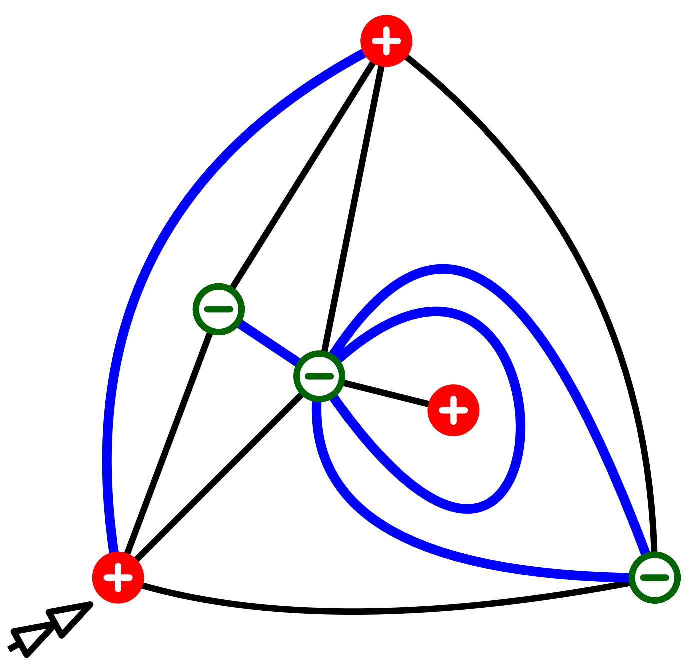
Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd

Motivation to study simple triangulations, but so far no results for random circle packings.

However, [Holden, Sun 19] proved that uniform triangulations (without multiple edges) embedded via the **Cardy embedding** converge towards $\sqrt{8/3}$-LQG.

Proof is built on many results, among which [A., Holden, Sun '20] : the scaling limit of triangulations without multiple edges and with a boundary is the Brownian disk.

III - Local limit of Ising-weighted random triangulations



Local point of view
(Benjamini-Schramm topology):

Simulation by I.Kortchemski

Local limit of large uniformly random triangulations

Take a random triangulation with n edges. What does it look like when $n \to \infty$?

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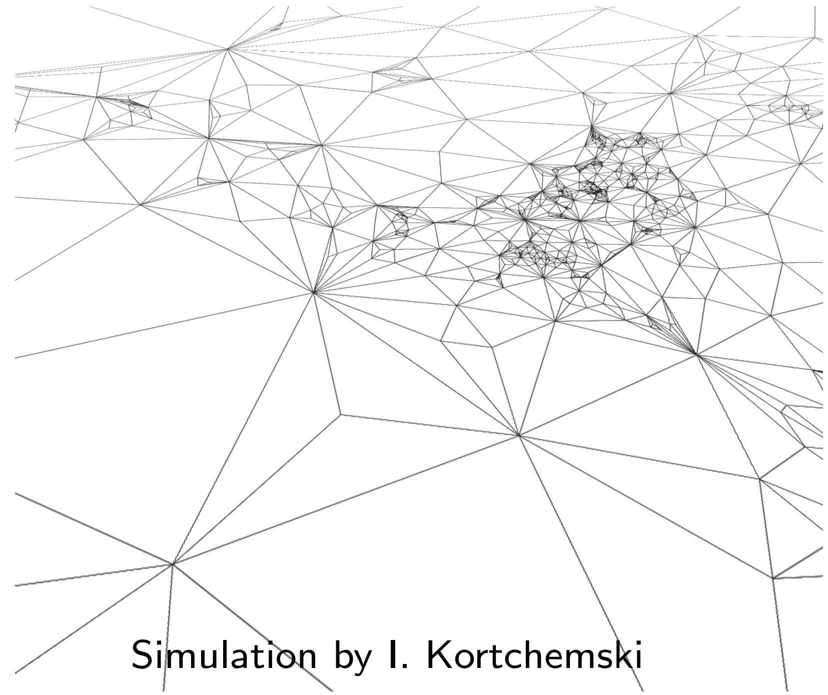
Local point of view :

Look at neighborhoods of the root



Simulation by I. Kortchemski

The **local topology** (= **Benjamini–Schramm topology**) on finite maps is induced by the distance:

$$d_{loc}(m, m') = \frac{1}{1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}}$$

where $B_r(m)$ = ball of radius r centered at the root vertex of m.

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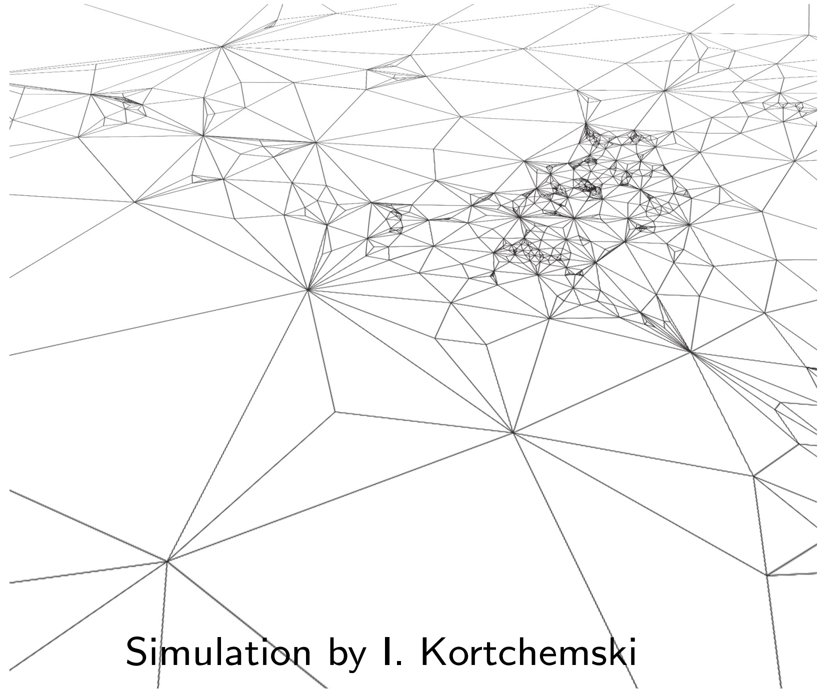

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Theorem [Angel – Schramm, '03]

Let $\mathbb{P}_n^\Delta$ = uniform distribution on triangulations of size n.

$$\mathbb{P}_n^\Delta \xrightarrow{(d)} \text{UIPT}, \qquad \text{for the local topology}$$

UIPT = Uniform Infinite Planar Triangulation
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Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, 03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.

$$\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \frac{2}{7} r^4 \qquad \text{[Angel 04, Curien – Le Gall 12]}$$

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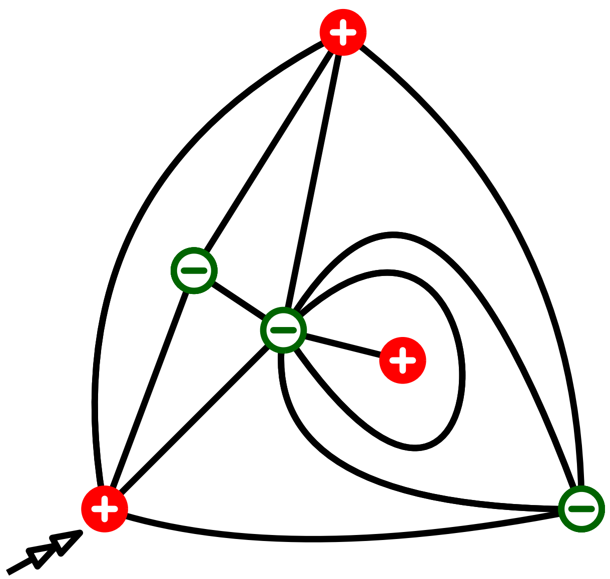
Universality: we expect the **same behavior** for other "reasonable" models of maps.
In particular, we expect the volume growth to be 4.
(proved for quadrangulations [Krikun 05], simple triangulations [Angel 04])

Escaping universality: adding matter

First, **Ising model** on a finite deterministic planar triangulation T:



Spin configuration on T:

$$\sigma : V(T) \to \{-1, +1\} = \{\ominus, \oplus\}.$$

Ising model on T: take a random spin configuration with probability:

$$P(\sigma) \propto e^{\beta J \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v)=\sigma(v')\}}}$$

$\beta > 0$: inverse temperature.
$J = \pm 1$: coupling constant.
$h = 0$: no magnetic field.

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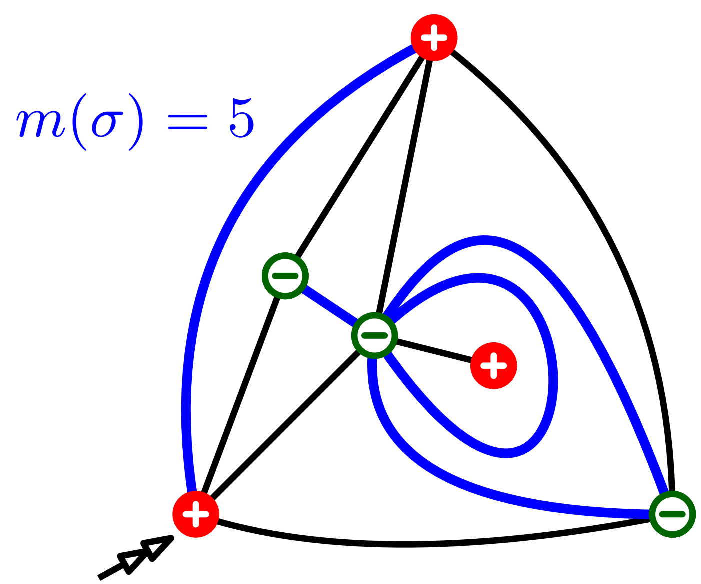

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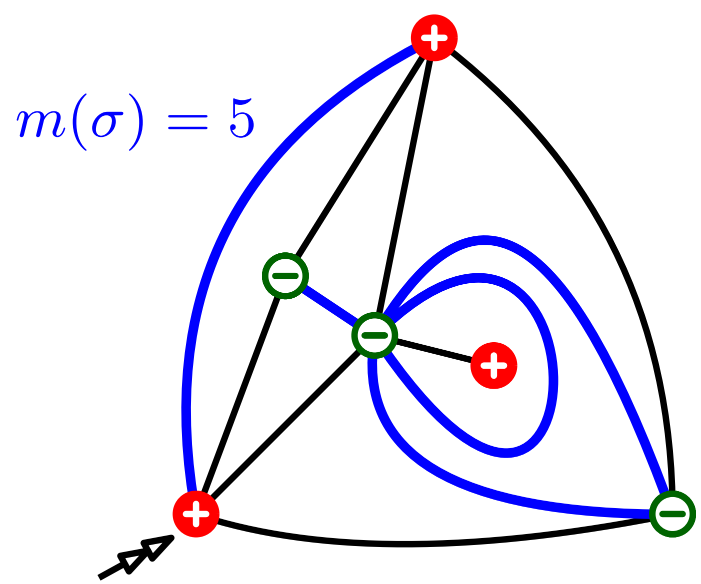

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Next step: Sample a triangulation of size n **together** with a spin configuration, with probability $\propto \nu^{m(T,\sigma)}$.

$$\boxed{\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}\delta_{|e(T)|=3n}}{\mathcal{Z}_n}.}$$

$\mathcal{Z}_n =$ normalizing constant.

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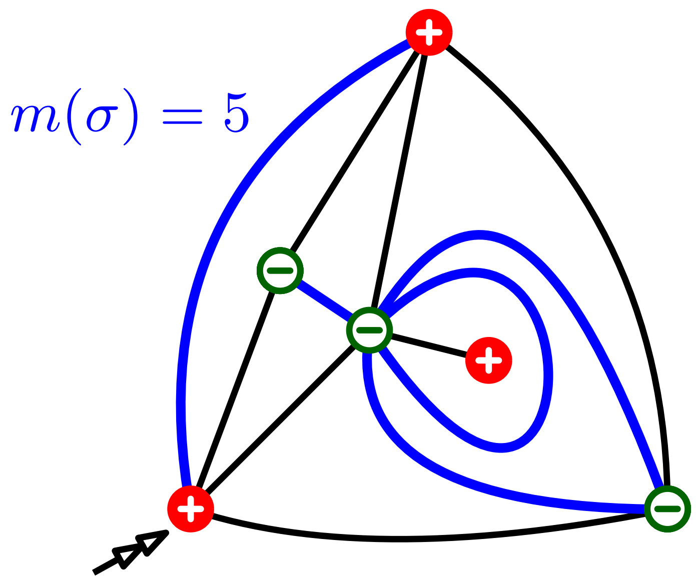

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Remark: This is a probability distribution on triangulations **with** spins. But, forgetting the spins gives a probability a distribution on triangulations **without** spins **different from the uniform distribution**.

Escaping universality: new asymptotic behavior

Counting exponent for undecorated maps:

coeff $[t^n]$ of generating series of (undecorated) maps $\sim \kappa \rho^{-n} n^{-\mathbf{5/2}}$

(e.g.: triangulations, quadrangulations, general maps, simple maps,...)

where κ and ρ depend on the combinatorics of the model.

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Generating series of Ising-weighted triangulations:

$$Q(\nu, t) = \sum_{T\in\mathcal{T}_f} \sum_{\sigma:V(T)\to\{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

Theorem [Bernardi – Bousquet-Mélou 11]
For every $\nu > 0$, $Q(\nu, t)$ is algebraic and satisfies

$$[t^{3n}]Q(\nu, t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\,\rho_{\nu_c}^{-n}\,n^{\mathbf{-7/3}} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\ \kappa\,\rho_{\nu}^{-n}\,n^{\mathbf{-5/2}} & \text{if } \nu \neq \nu_c. \end{cases}$$

See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

This suggests a **different behavior** of the underlying maps for $\nu = \nu_c$.

Local convergence of triangulations with spins

Theorem [A. – Ménard – Schaeffer, 21]

Let $\mathbb{P}_n^\nu = \nu$–Ising weighted probability distribution:

$$\mathbb{P}^\nu \xrightarrow{(d)} \nu\text{-IIPT}, \qquad \text{for the local topology}$$

ν-IIPT $= \nu$-Ising Infinite Planar Triangulation
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Moreover, simple random walk is **recurrent** on the ν_c-IIPT.

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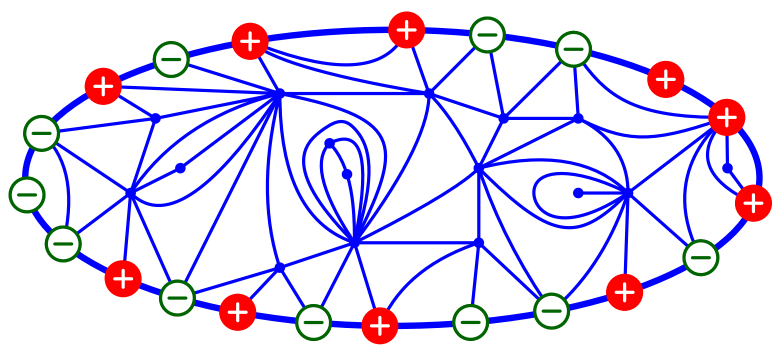

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with $Z_\omega(\nu,\,t)$ generating series of Ising-weighted triangulations with boundary condition given by ω.

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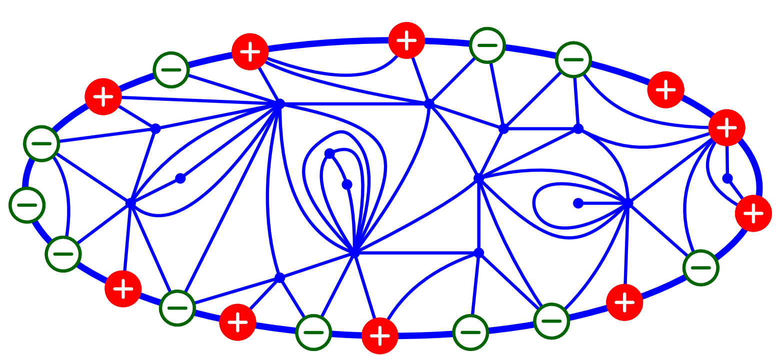

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We use the blossoming bijection of [Bousquet-Mélou, Schaeffer 02] to prove that !

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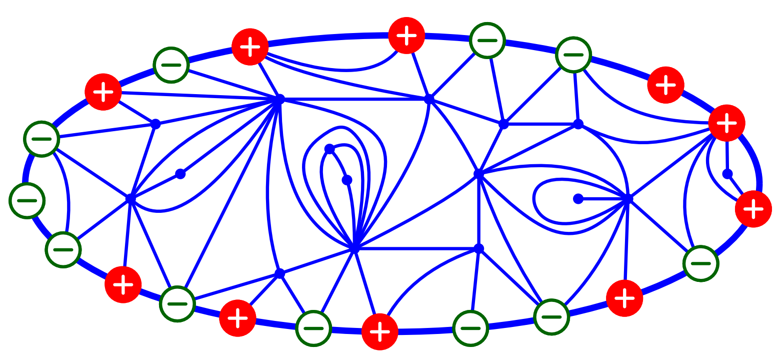

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- Proof of the tightness: combinatorial proof by a double counting argument.

Perspectives

- Blossoming bijections in higher genus ? Other rationality schemes to investigate ?

- Track distances in blossoming bijections to study more constrained models.
  e.g. scaling limit of planar graphs ?

- Extend bootstrapping principle for the convergence of trees to more general models.
  e.g. α-stable trees ?

- Study of the clusters of the ν-IIPT, following [Bernardi,Curien,Miermont, 15]

- Bijections for the Ising model, blossoming bijection by [Bousquet-Mélou, Schaeffer 02].
  Can we find a "mating-of-tree" type bijection ?

- Can we say anything about the growth volume of the ν-IIPT ?

Thank you for your attention !

Un énorme merci à tous mes collaborateurs et collaboratrices :

Louigi Addario-Berry, Jérémie Bouttier, Éric Fusy, Lucas Gerin, Christina Goldschmidt, Nina Holden, Kolja Knauer, Mathias Lepoutre, Jean-François Marckert, Laurent Ménard, Philippe Nadeau, Dominique Poulalhon, Gilles Schaeffer, Xin Sun.