> restart;

The series U in the paramatrization

Positivity of U

First we establish that U is the derivative of w tQ1: This is Equation (8) of the paper:

> $wUnu := \frac{1}{32}\frac{1}{(-1+2U)^2\nu^3}\left((U\nu+U-2)\,U\left(8U^3\nu^2+16U^3\nu-11U^2\nu^2+8U^3-24U^2\nu+4U\nu^2-13U^2+14U\nu+6U-4\nu\right)\right);$

$$wUnu := \frac{1}{32(-1+2U)^2\nu^3}\left((U\nu+U-2)\,U\left(8U^3\nu^2+16U^3\nu-11U^2\nu^2+8U^3-24U^2\nu+4U\nu^2-13U^2+14U\nu+6U-4\nu\right)\right) \quad (1.1.1)$$

> $Q1Unu := \frac{1}{2}\left((6U^3\nu^2+12U^3\nu-8U^2\nu^2+6U^3-16U^2\nu+3U\nu^2-8U^2+7U\nu+4U-2\nu)\,U(\nu+1)\right)\Big/\left((8U^3\nu^2+16U^3\nu-11U^2\nu^2+8U^3-24U^2\nu+4U\nu^2-13U^2+14U\nu+6U-4\nu)\,\nu\right);$

$$Q1Unu := \left((6U^3\nu^2+12U^3\nu-8U^2\nu^2+6U^3-16U^2\nu+3U\nu^2-8U^2+7U\nu+4U-2\nu)\,U(\nu+1)\right)\Big/\left(2\left(8U^3\nu^2+16U^3\nu-11U^2\nu^2+8U^3-24U^2\nu+4U\nu^2-13U^2+14U\nu+6U-4\nu\right)\nu\right) \quad (1.1.2)$$

> $wUnu\cdot Q1Unu;$

$$\frac{1}{64(-1+2U)^2\nu^4}\left((U\nu+U-2)\,U^2\left(6U^3\nu^2+12U^3\nu-8U^2\nu^2+6U^3-16U^2\nu+3U\nu^2-8U^2+7U\nu+4U-2\nu\right)(\nu+1)\right) \quad (1.1.3)$$

> $collect\left((6U^3\nu^2+12U^3\nu-8U^2\nu^2+6U^3-16U^2\nu+3U\nu^2-8U^2+7U\nu+4U-2\nu), U, factor\right);$

$$6(\nu+1)^2U^3-8(\nu+1)^2U^2+(3\nu+4)(\nu+1)U-2\nu \quad (1.1.4)$$

> $factor(simplify(diff(wUnu, U)));$

$$\frac{1}{8(-1+2U)^3\nu^3}\left((3U^2\nu+3U^2-3U\nu-3U+\nu)\left(4U^3\nu^2+8U^3\nu-3U^2\nu^2+4U^3-12U^2\nu-9U^2+6U\nu+6U-2\right)\right) \quad (1.1.5)$$

> $simplify\left(\frac{diff(wUnu\cdot Q1Unu, U)}{diff(wUnu, U)}\right);$

$$\frac{U(\nu+1)}{2\nu} \quad (1.1.6)$$

Radius of Convergence of U

The radius of convergence of U is one of the roots of the discriminant of the algebraic equation of U:

```
> algU := numer(wUnu - w);
```

$$algU := 8 U^5 \nu^3 + 24 U^5 \nu^2 - 11 U^4 \nu^3 + 24 U^5 \nu - 51 U^4 \nu^2 + 4 U^3 \nu^3 - 128 U^2 \nu^3 w + 8 U^5 - 69 U^4 \nu + 40 U^3 \nu^2 + 128 U \nu^3 w - 29 U^4 + 68 U^3 \nu - 12 U^2 \nu^2 - 32 w \nu^3 + 32 U^3 - 32 U^2 \nu - 12 U^2 + 8 U \nu \qquad (1.2.1)$$

```
> dis := factor(discrim(algU, U));
```

$$dis := -4096 (\nu - 1) (\nu - 3)^2 (\nu + 1)^6 (27648 \nu^4 w^2 + 864 \nu^4 w + 7 \nu^4 - 2592 w \nu^3 - 42 \nu^3 + 864 \nu^2 w + 75 \nu^2 + 864 \nu w - 20 \nu - 36) (131072 \nu^9 w^3 - 1728 \nu^9 w^2 + 5184 \nu^8 w^2 + 7104 \nu^7 w^2 - 10560 \nu^6 w^2 - 48 \nu^5 w + 96 \nu^4 w - 48 w \nu^3 + 4 \nu^3 - 12 \nu^2 - 15 \nu + 23) \nu^2 \qquad (1.2.2)$$

We have two factors, one of degree 2 and one of degree 3 :

```
> P2 := 27648 ν^4 w^2 + 864 ν^4 w + 7 ν^4 - 2592 w ν^3 - 42 ν^3 + 864 ν^2 w + 75 ν^2
      + 864 ν w - 20 ν - 36;
```

$$P2 := 27648 \nu^4 w^2 + 864 \nu^4 w + 7 \nu^4 - 2592 w \nu^3 - 42 \nu^3 + 864 \nu^2 w + 75 \nu^2 + 864 \nu w - 20 \nu - 36 \qquad (1.2.3)$$

```
> P1 := 131072 ν^9 w^3 - 1728 ν^9 w^2 + 5184 ν^8 w^2 + 7104 ν^7 w^2 - 10560 ν^6 w^2 - 48 ν^5 w
      + 96 ν^4 w - 48 w ν^3 + 4 ν^3 - 12 ν^2 - 15 ν + 23;
```

$$P1 := 131072 \nu^9 w^3 - 1728 \nu^9 w^2 + 5184 \nu^8 w^2 + 7104 \nu^7 w^2 - 10560 \nu^6 w^2 - 48 \nu^5 w + 96 \nu^4 w - 48 w \nu^3 + 4 \nu^3 - 12 \nu^2 - 15 \nu + 23 \qquad (1.2.4)$$

Possible nu's where the two have a common root (to find nu_c)

```
> factor(resultant(P1, P2, w));
```

$$4194304 \nu^{12} (13573 \nu^4 - 54292 \nu^3 + 69811 \nu^2 - 31038 \nu + 67482) (7 \nu^2 - 14 \nu + 6)^3 (\nu + 1)^4 (\nu - 3)^4 \qquad (1.2.1.1)$$

First factor has no positive root and is not relevant for us:

```
> evalf(solve((13573 ν^4 - 54292 ν^3 + 69811 ν^2 - 31038 ν + 67482), nu));
```

$$-0.145791810 + 0.9404920565 I, 2.145791810 - 0.9404920565 I, -0.145791810 - 0.9404920565 I, 2.145791810 + 0.9404920565 I \qquad (1.2.1.2)$$

nu=3 is solution but it gives a negative common root for rho:

```
> solve(subs(nu = 3, algr2), w);
```

Second last factor will give nu_c and another candidate:

```
> solve(6 - 14 ν + 7 ν^2, nu);
```

(1.2.1.3)

$$1+\frac{\sqrt{7}}{7}, 1-\frac{\sqrt{7}}{7} \quad \textbf{(1.2.1.3)}$$

The root with a - gives negative common root for rho:

> $factor\left(simplify\left(subs\left(nu = 1 - \frac{1}{7}\sqrt{7}, P1\right)\right)\right);$

$$\frac{1}{661624362}\left(\left(3553\sqrt{7} - 9415\right)\left(5609520\,w\sqrt{7} - 95551488\,w^2 - 698005\sqrt{7} + 12340944\,w - 1878268\right)\left(864\,w + 55 + 25\sqrt{7}\right)\right) \quad \textbf{(1.2.1.4)}$$

> $factor\left(simplify\left(subs\left(nu = 1 - \frac{1}{7}\sqrt{7}, P2\right)\right)\right);$

$$\frac{\left(8\sqrt{7} - 23\right)\left(-3456\,w + 77 + 35\sqrt{7}\right)\left(864\,w + 55 + 25\sqrt{7}\right)}{1323} \quad \textbf{(1.2.1.5)}$$

This leaves nu_c, the common root is rho_nu_c

> $factor\left(subs\left(nu = 1 + \frac{1}{7}\sqrt{7}, P1\right)\right); evalf(solve(\%, w));$

$$-\frac{1}{661624362}\left(\left(9415 + 3553\sqrt{7}\right)\left(5609520\,w\sqrt{7} + 95551488\,w^2 - 698005\sqrt{7} - 12340944\,w + 1878268\right)\left(-864\,w - 55 + 25\sqrt{7}\right)\right)$$

$$0.01289789674, -0.01308431164 + 0.01259678620\,I, -0.01308431164 - 0.01259678620\,I \quad \textbf{(1.2.1.6)}$$

> $factor\left(subs\left(nu = 1 + \frac{1}{7}\sqrt{7}, P2\right)\right); evalf(solve(\%, w));$

$$-\frac{\left(23 + 8\sqrt{7}\right)\left(-864\,w - 55 + 25\sqrt{7}\right)\left(3456\,w - 77 + 35\sqrt{7}\right)}{1323}$$

$$0.01289789674, -0.00451426385 \quad \textbf{(1.2.1.7)}$$

> $\rho_c := solve\left(-864\,w - 55 + 25\sqrt{7}, w\right); \nu_c := 1 + \frac{1}{7}\sqrt{7};$

$$\rho_c := -\frac{55}{864} + \frac{25\sqrt{7}}{864}$$

$$\nu_c := 1 + \frac{\sqrt{7}}{7} \quad \textbf{(1.2.1.8)}$$

Plots of positive roots of the discriminant of algU and exact expressions for rho_nu

P2 has real roots fo nu =< 3

> $factor(discrim(P2, w));$

$$-27648\,\nu^2\,(\nu + 1)^3\,(\nu - 3)^3 \quad \textbf{(1.2.2.1)}$$

P1 has three real roots for nu >= 3 and double real roots for nu=1,3,$1 + 2\sqrt{2}$

(1.2.2.2)

$$1+2\sqrt{2}$$ **(1.2.2.2)**

> *factor*(*discrim*(*P1*, *w*)); *solve*(%); *evalf*(%);

$$82556485632\,\nu^{18}\,(\nu-1)^2\,(\nu^2-2\,\nu-7)^2\,(\nu+1)^3\,(\nu-3)^3$$

$1, 1, -1, -1, -1, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1+2\sqrt{2}, 1-2\sqrt{2}, 1+2\sqrt{2}, 1-2\sqrt{2}$

$1., 1., -1., -1., -1., 3., 3., 3., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 3.828427124, -1.828427124, 3.828427124, -1.828427124$ **(1.2.2.3)**

Plots of the positive roots of P2 (red) and P1 (blue):

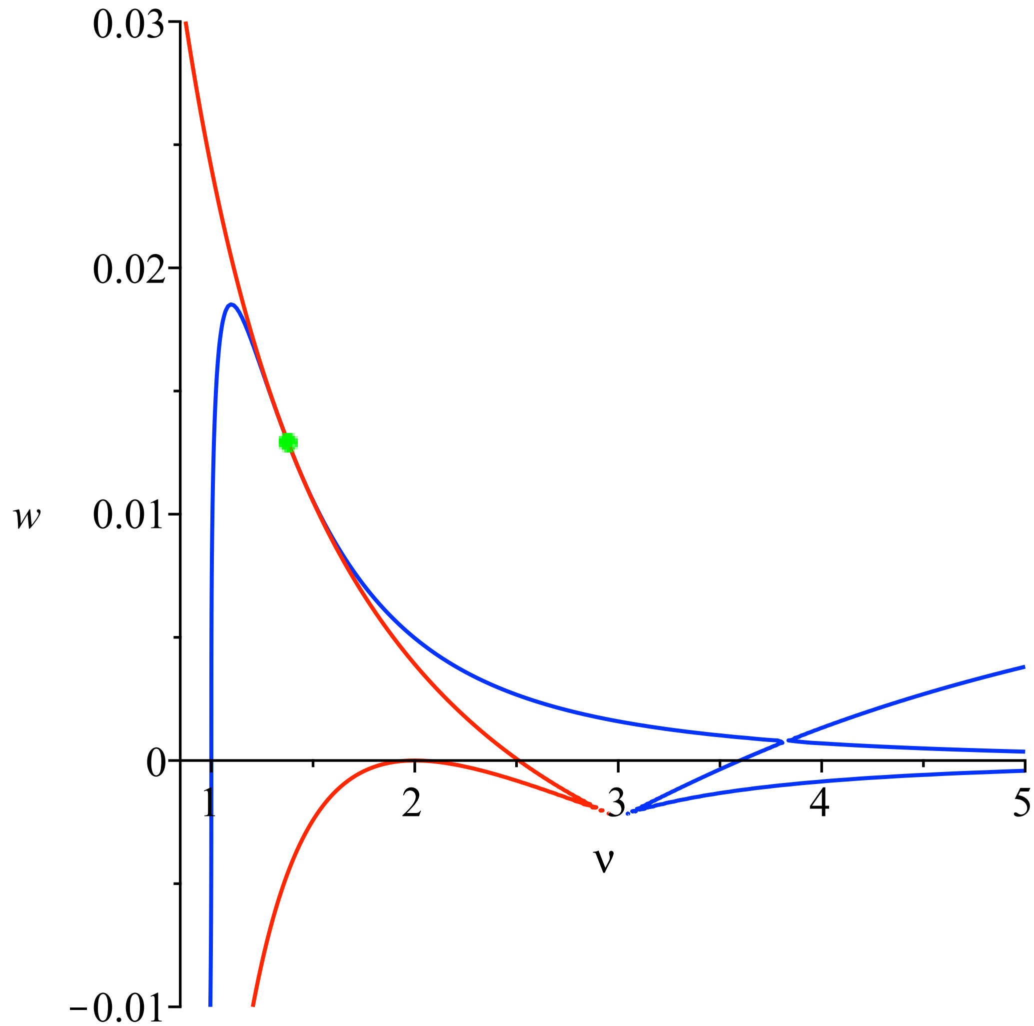
> with(plots, implicitplot) :

> *plotrho2* := *implicitplot*(*P2*, nu = 0 ..5, *w* = -0.01 ..0.03, *numpoints* = 100000, *color* = *red*) :

> *plotrho1* := *implicitplot*(*P1*, nu = 0 ..5, *w* = -0.01 ..0.03, *numpoints* = 100000, *color* = *blue*) :

> *critpoint* := *plot*$\left(\langle\langle \nu_c\rangle | \langle \rho_c\rangle\rangle\right)$, *style* = *point*, *symbol* = *solidcircle*, *color* = *green*, *symbolsize* = 15) :

> *plots*[*display*]({*plotrho2*, *plotrho1*, *critpoint*});



Roots of P2

> $w21, w22 := solve(P2, w);$

$$w21, w22 := \frac{1}{576\,\nu^3}\left(-9\,\nu^3 + 27\,\nu^2 + \sqrt{-3\,\nu^6 + 18\,\nu^5 - 9\,\nu^4 - 84\,\nu^3 + 27\,\nu^2 + 162\,\nu + 81} - 9\,\nu - 9\right),$$
$$-\frac{1}{576\,\nu^3}\left(9\,\nu^3 - 27\,\nu^2 + \sqrt{-3\,\nu^6 + 18\,\nu^5 - 9\,\nu^4 - 84\,\nu^3 + 27\,\nu^2 + 162\,\nu + 81} + 9\,\nu + 9\right) \tag{1.2.2.4}$$

> $factor(-9\,\nu - 9 - 9\,\nu^3 + 27\,\nu^2); factor(27\,\nu^2 - 84\,\nu^3 - 9\,\nu^4 + 18\,\nu^5 - 3\,\nu^6 + 162\,\nu + 81);$

$$-9\,(\nu - 1)\,(\nu^2 - 2\,\nu - 1)$$
$$-3\,(\nu + 1)^3\,(\nu - 3)^3 \tag{1.2.2.5}$$

Roots of P2:

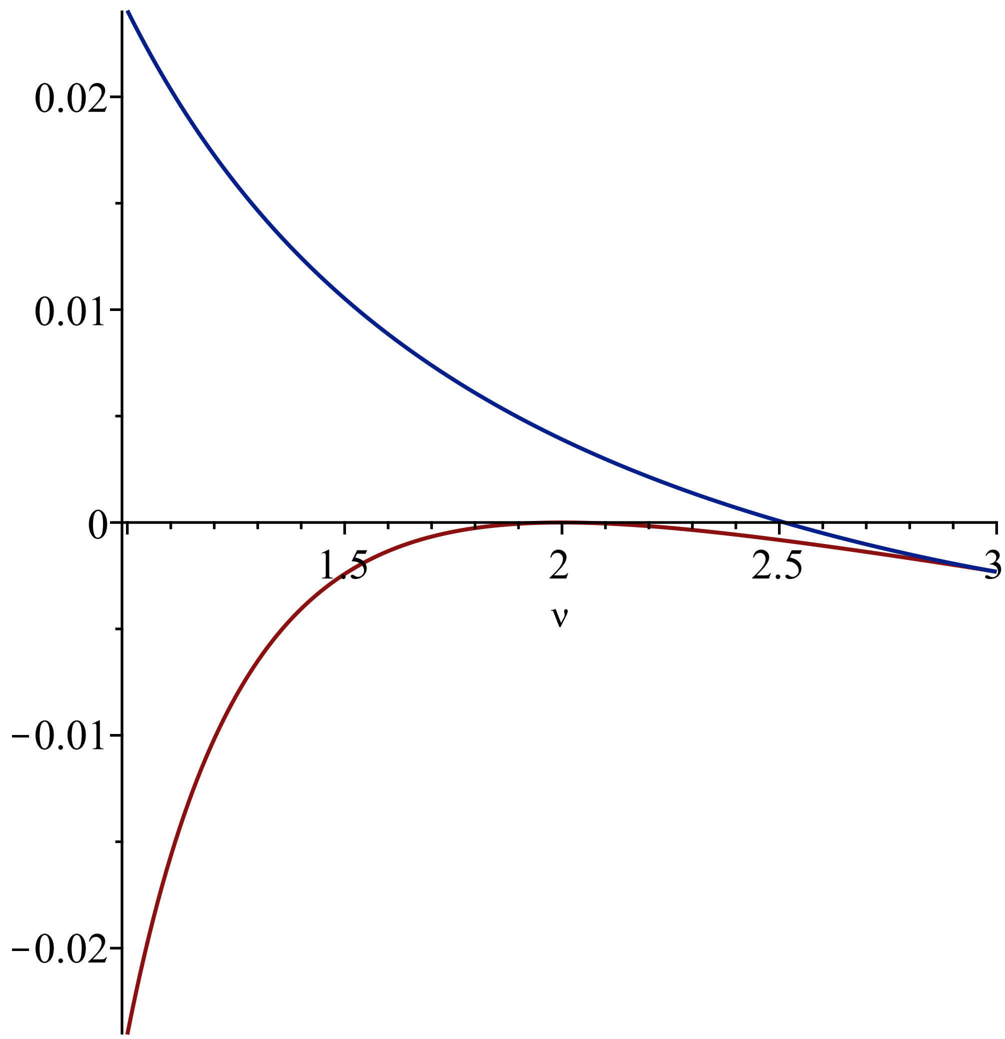
> $w21 :=$

$$\frac{1}{576}\,\frac{1}{\nu^{3}}\left(-9\,(\mathrm{nu}-1)\cdot\left(\nu^{2}-2\cdot\mathrm{nu}-1\right)+(\mathrm{nu}+1)\cdot(3-\mathrm{nu})\cdot\sqrt{3\cdot(\mathrm{nu}+1)\cdot(3-\mathrm{nu})}\right):$$

$$w22 := \frac{1}{576}\,\frac{1}{\nu^{3}}\left(-9\,(\mathrm{nu}-1)\cdot\left(\nu^{2}-2\cdot\mathrm{nu}-1\right)-(\mathrm{nu}+1)\cdot(3-\mathrm{nu})\cdot\sqrt{3\cdot(\mathrm{nu}+1)\cdot(3-\mathrm{nu})}\right):$$

The positive one is w21

> $plot(\{w21, w22\}, \mathrm{nu} = 1..3);$



w22 is always negative

> $solve(w22, \mathrm{nu});$

$$2,\ 1-\frac{4\sqrt{7}}{7} \qquad \mathbf{(1.2.2.6)}$$

An exact expression for the roots of P1 (denoted rho11, rho12, rho13)

> $Delta0 := factor\left(coeff(P1, w, 2)^{2}-3\cdot coeff(P1, w, 3)\cdot coeff(P1, w, 1)\right);$

$$\Delta0 := 331776\,\nu^{12}\left(9\,\nu^{4}-36\,\nu^{3}-74\,\nu^{2}+220\,\nu+393\right)(\nu-1)^{2} \qquad \mathbf{(1.2.2.7)}$$

> $Delta1 := factor\left(2\cdot coeff(P1, w, 2)^3 - 9\cdot coeff(P1, w, 3)\cdot coeff(P1, w, 2)\cdot coeff(P1, w, 1) + 27\cdot coeff(P1, w, 3)^2\cdot coeff(P1, w, 0)\right);$

$$\Delta1 := -382205952\,\nu^{18}\,(\nu-1)\,\left(27\,\nu^8 - 216\,\nu^7 + 180\,\nu^6 + 1944\,\nu^5 - 2398\,\nu^4 - 7400\,\nu^3 + 1844\,\nu^2 + 18600\,\nu + 20187\right) \quad \textbf{(1.2.2.8)}$$

> $factor\left(\Delta1^2 - 4\cdot\Delta0^3\right);$

$$-38294359833110460235776\,\nu^{36}\,(\nu-1)^2\,\left(\nu^2 - 2\,\nu - 7\right)^2\,(\nu+1)^3\,(\nu-3)^3 \quad \textbf{(1.2.2.9)}$$

> sqrt(38294359833110460235776);

$$195689447424 \quad \textbf{(1.2.2.10)}$$

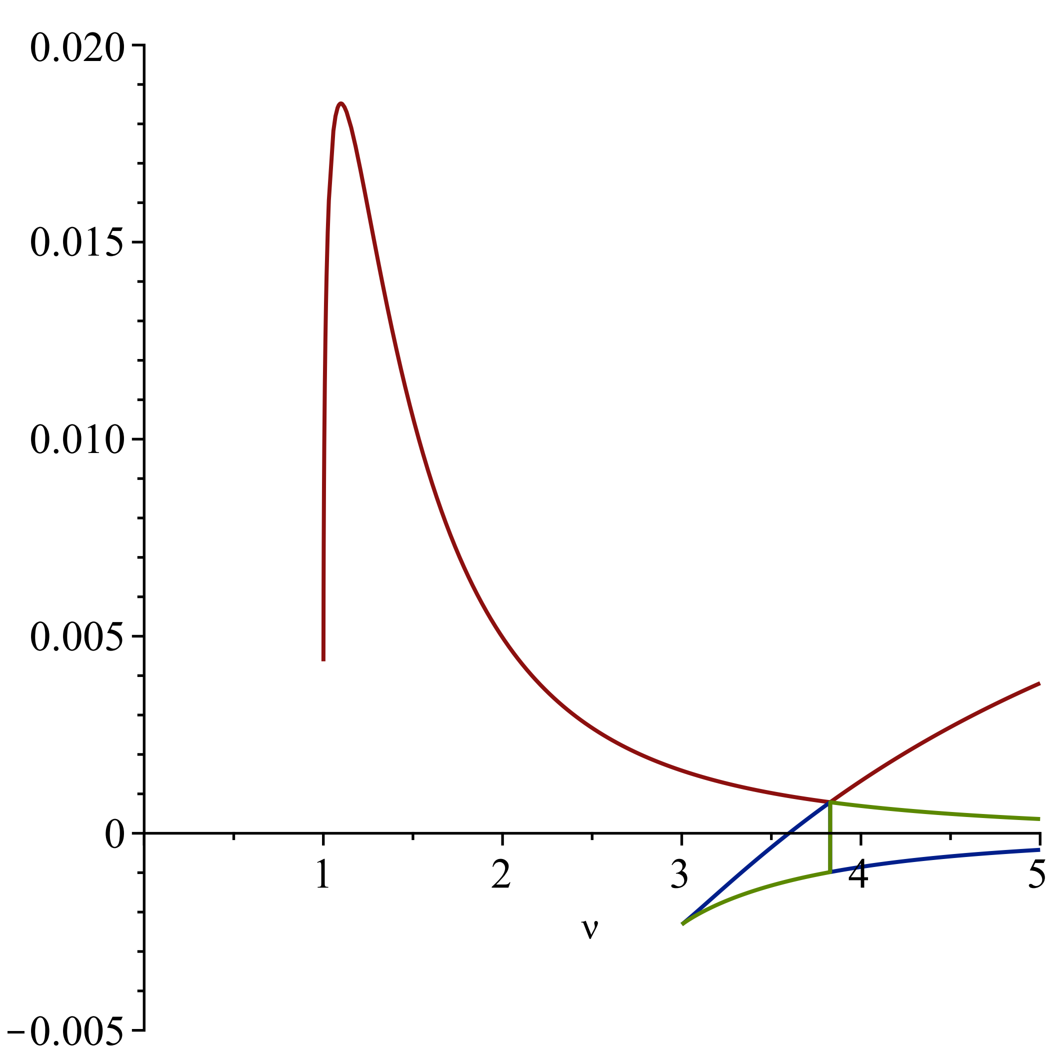
> $Delta2 := \mathrm{sqrt}(38294359833110460235776)\cdot\nu^{18}\cdot(\mathrm{nu}-1)\cdot\left(\nu^2 - 2\,\nu - 7\right)\,(\nu+1)\,(\nu-3)\cdot\mathrm{sqrt}((\mathrm{nu}+1)\cdot(3-\mathrm{nu}));$

$$\Delta2 := 195689447424\,\nu^{18}\,(\nu-1)\,\left(\nu^2 - 2\,\nu - 7\right)\,(\nu+1)\,(\nu-3)\,\sqrt{(\nu+1)\,(3-\nu)} \quad \textbf{(1.2.2.11)}$$

> $Cm := factor\left(\frac{(Delta1 - Delta2)}{2}\right) : Cp := \frac{(Delta1 + Delta2)}{2} :$

> $rho11 := \frac{1}{3\cdot coeff(P1, w, 3)}\cdot(-coeff(P1, w, 2) + \mathrm{root}(-Cp, 3) + \mathrm{root}(-Cm, 3)) :$
$rho12 := \frac{1}{3\cdot coeff(P1, w, 3)}\cdot\left(-coeff(P1, w, 2) + \left(\frac{-1 + \mathrm{sqrt}(3)\cdot I}{2}\cdot root(-Cp, 3) + \frac{-1 - \mathrm{sqrt}(3)\cdot I}{2}\,\mathrm{root}(-\mathrm{Cm}, 3)\right)\right) : rho13 := \frac{1}{3\cdot coeff(P1, w, 3)}\cdot\left(-coeff(P1, w, 2) + \left(\frac{-1 - \mathrm{sqrt}(3)\cdot I}{2}\cdot root(-Cp, 3) + \frac{-1 + \mathrm{sqrt}(3)\cdot I}{2}\,\mathrm{root}(-\mathrm{Cm}, 3)\right)\right) :$

> $plot(\{rho11, rho12, rho13\}, \mathrm{nu} = 0\,..5, view = [0\,..5, -0.005\,..0.02]);$



Computation of U(rho)

The caracteristic equation:

> $PhiU := \frac{U}{wUnu}$; $eqUrho := factor(numer(PhiU - U \cdot diff(PhiU, U)))$;

$$PhiU := (32\,(-1+2\,U)^2\,\nu^3)\,/\,((U\,\nu+U-2)\,(8\,U^3\,\nu^2+16\,U^3\,\nu-11\,U^2\,\nu^2+8\,U^3-24\,U^2\,\nu+4\,U\,\nu^2-13\,U^2+14\,U\,\nu+6\,U-4\,\nu))$$

$$eqUrho := 128\,(-1+2\,U)\,\nu^3\,(3\,U^2\,\nu+3\,U^2-3\,U\,\nu-3\,U+\nu)\,(4\,U^3\,\nu^2+8\,U^3\,\nu-3\,U^2\,\nu^2+4\,U^3-12\,U^2\,\nu-9\,U^2+6\,U\,\nu+6\,U-2) \quad \textbf{(1.3.1)}$$

> $eqUrho3 := 4\,U^3\,\nu^2+8\,U^3\,\nu-3\,U^2\,\nu^2+4\,U^3-12\,U^2\,\nu-9\,U^2+6\,U\,\nu+6\,U-2$:
> $eqUrho2 := (3\,U^2\,\nu+3\,U^2-3\,U\,\nu-3\,U+\nu)$:

> $collect(eqUrho3, U, factor)$;

$$-2+4\,(\nu+1)^2\,U^3-3\,(\nu+3)\,(\nu+1)\,U^2+(6\,\nu+6)\,U \quad \textbf{(1.3.2)}$$

> *solve*(*eqUrho2*, *U*);

$$\frac{3\,\nu+3+\sqrt{-3\,\nu^2+6\,\nu+9}}{6\,(\nu+1)},\ -\frac{-3\,\nu-3+\sqrt{-3\,\nu^2+6\,\nu+9}}{6\,(\nu+1)} \quad \textbf{(1.3.3)}$$

> *factor*$(-3\,\nu^2+6\,\nu+9)$;

$$-3\,(\nu+1)\,(\nu-3) \quad \textbf{(1.3.4)}$$

> *factor*$\left(\textit{subs}\left(\mathrm{nu}=1+\frac{\mathrm{sqrt}(7)}{7},\ \textit{eqUrho2}\right)\right)$; *fsolve*(%);

$$-\frac{(14+\sqrt{7})\,(-9\,U+4+\sqrt{7})\,(9\,U-5+\sqrt{7})}{189}$$

$$0.2615831877,\ 0.7384168123 \quad \textbf{(1.3.5)}$$

U=1/2 is a problem only when nu=1 or 3 which we will deal with later :

> *factor*(*resultant*(*eqUrho3*, 2 *U* − 1, *U*)); *factor*(*resultant*(*eqUrho2*, 2 *U* − 1, *U*));

$$2\,(\nu-1)\,(\nu-3)$$

$$\nu-3 \quad \textbf{(1.3.6)}$$

> *factor*(*resultant*(*eqUrho3*, *eqUrho2*, *U*)); *solve*(%);

$$(7\,\nu^2-14\,\nu+6)\,(\nu-3)^2\,(\nu+1)^3$$

$$3,\ 3,\ -1,\ -1,\ -1,\ 1+\frac{\sqrt{7}}{7},\ 1-\frac{\sqrt{7}}{7} \quad \textbf{(1.3.7)}$$

For nu=3, the common root is not the smallest positive root and U(rho) = 1/8 is a root of the factor of degree 3

> *factor*(*subs*(nu = 3, *eqUrho3*)); *factor*(*subs*(nu = 3, *eqUrho2*))

$$2\,(8\,U-1)\,(-1+2\,U)^2$$

$$3\,(-1+2\,U)^2 \quad \textbf{(1.3.8)}$$

At nu_c the common root is U(rho)

> *factor*$\left(\textit{subs}\left(\mathrm{nu}=1+\frac{1}{7}\sqrt{7},\ \textit{eqUrho3}\right)\right)$; *fsolve*(%); *factor*$\left(\textit{subs}\left(\mathrm{nu}=1+\frac{1}{7}\sqrt{7},\ \textit{eqUrho2}\right)\right)$; *fsolve*(%);

$$-\frac{(29+4\sqrt{7})\,(18\,U\sqrt{7}-324\,U^2-7\sqrt{7}+315\,U-91)\,(9\,U-5+\sqrt{7})}{5103}$$

$$0.2615831877$$

$$-\frac{(14+\sqrt{7})\,(-9\,U+4+\sqrt{7})\,(9\,U-5+\sqrt{7})}{189}$$

$$0.2615831877,\ 0.7384168123 \quad \textbf{(1.3.9)}$$

At nu=1-sqrt(7)/7, the common root is not the smallest and U(rho) is a root of the factor of degree 2

> $factor\left(subs\left(\mathrm{nu}=1-\frac{1}{7}\sqrt{7}, eqUrho3\right)\right); fsolve(\%); factor\left(subs\left(\mathrm{nu}=1-\frac{1}{7}\sqrt{7}, eqUrho2\right)\right); fsolve(\%);$

$$\frac{\left(-29+4\sqrt{7}\right)\left(18\,U\sqrt{7}+324\,U^2-7\sqrt{7}-315\,U+91\right)\left(-9\,U+5+\sqrt{7}\right)}{5103}$$

$$0.8495279234$$

$$\frac{\left(-14+\sqrt{7}\right)\left(9\,U-4+\sqrt{7}\right)\left(-9\,U+5+\sqrt{7}\right)}{189}$$

$$0.1504720766, 0.8495279234 \qquad \textbf{(1.3.10)}$$

At nu=1, the right factor is also the factor of degree 2

> $factor(subs(\mathrm{nu}=1, eqUrho3)); fsolve(\%); factor(subs(\mathrm{nu}=1, eqUrho2)); fsolve(\%);$

$$2\left(-1+2\,U\right)^3$$

$$0.5000000000, 0.5000000000, 0.5000000000$$

$$6\,U^2-6\,U+1$$

$$0.2113248654, 0.7886751346 \qquad \textbf{(1.3.11)}$$

> $plotUrho1 := implicitplot(2\,U-1, \mathrm{nu}=0..5, U=0..1, color=yellow):$

> $plotUrho2 := implicitplot(eqUrho2, \mathrm{nu}=0..5, U=0..1, color=red):$

> $plotUrho3 := implicitplot(eqUrho3, \mathrm{nu}=0..5, U=0..1, color=blue):$

> $plotint1 := plot([1, t, t=0..1], color=black, linestyle=spacedash):$

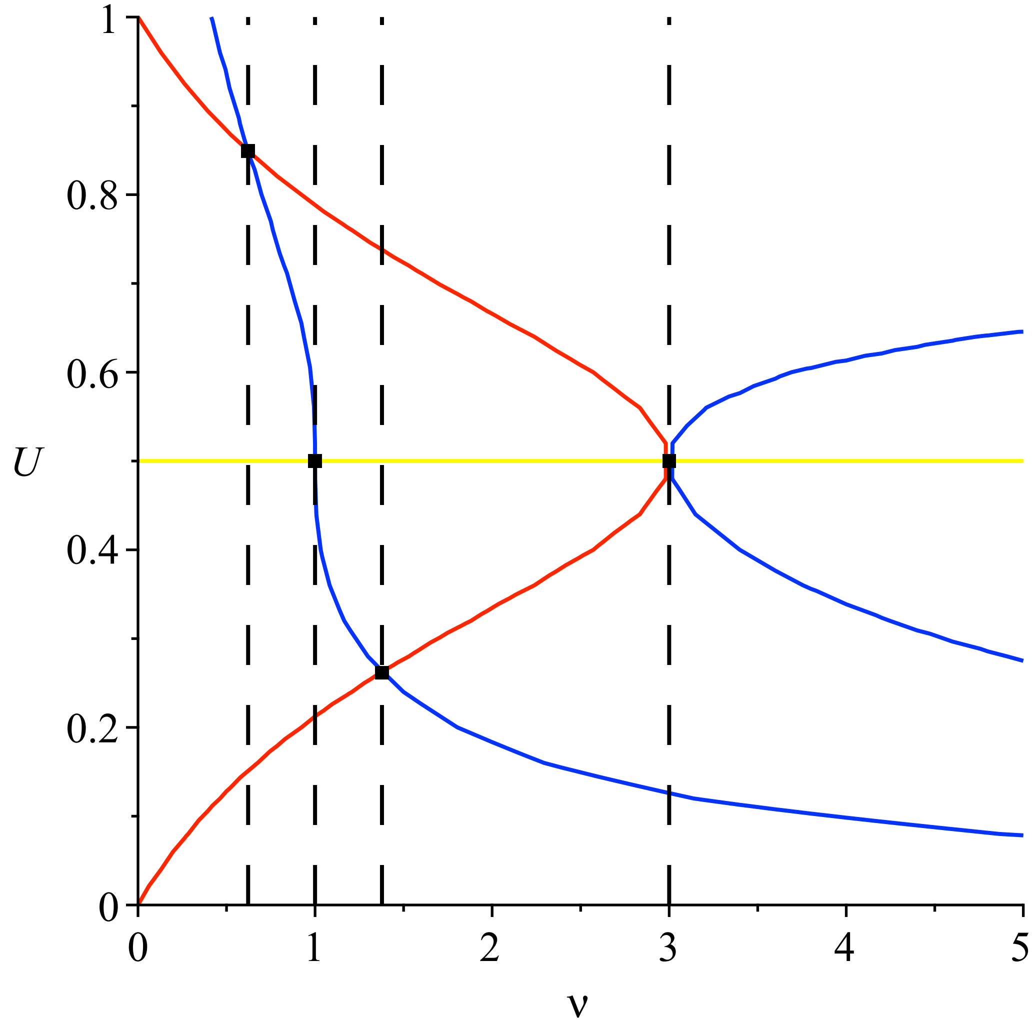
$plotint2 := plot\left(\left[1-\frac{\mathrm{sqrt}(7)}{7}, t, t=0..1\right], color=black, linestyle=spacedash\right):$

$plotint3 := plot\left(\left[1+\frac{\mathrm{sqrt}(7)}{7}, t, t=0..1\right], color=black, linestyle=spacedash\right):$

$plotint4 := plot([3, t, t=0..1], color=black, linestyle=spacedash):$

$pts := pointplot\left(\left[\left[1, \frac{1}{2}\right], \left[1+\frac{\mathrm{sqrt}(7)}{7}, 0.2615\right], \left[3, \frac{1}{2}\right], \left[1-\frac{\mathrm{sqrt}(7)}{7}, 0.849\right]\right], color=[black, black, black, black], symbol=solidbox\right):$

> $plots[display](\{pts, plotint1, plotint2, plotint3, plotint4, plotUrho1, plotUrho2, plotUrho3\});$



>

>

> ?pointplot

>

Unique dominant singularity

Imaginary roots of P2 for nu >= 3

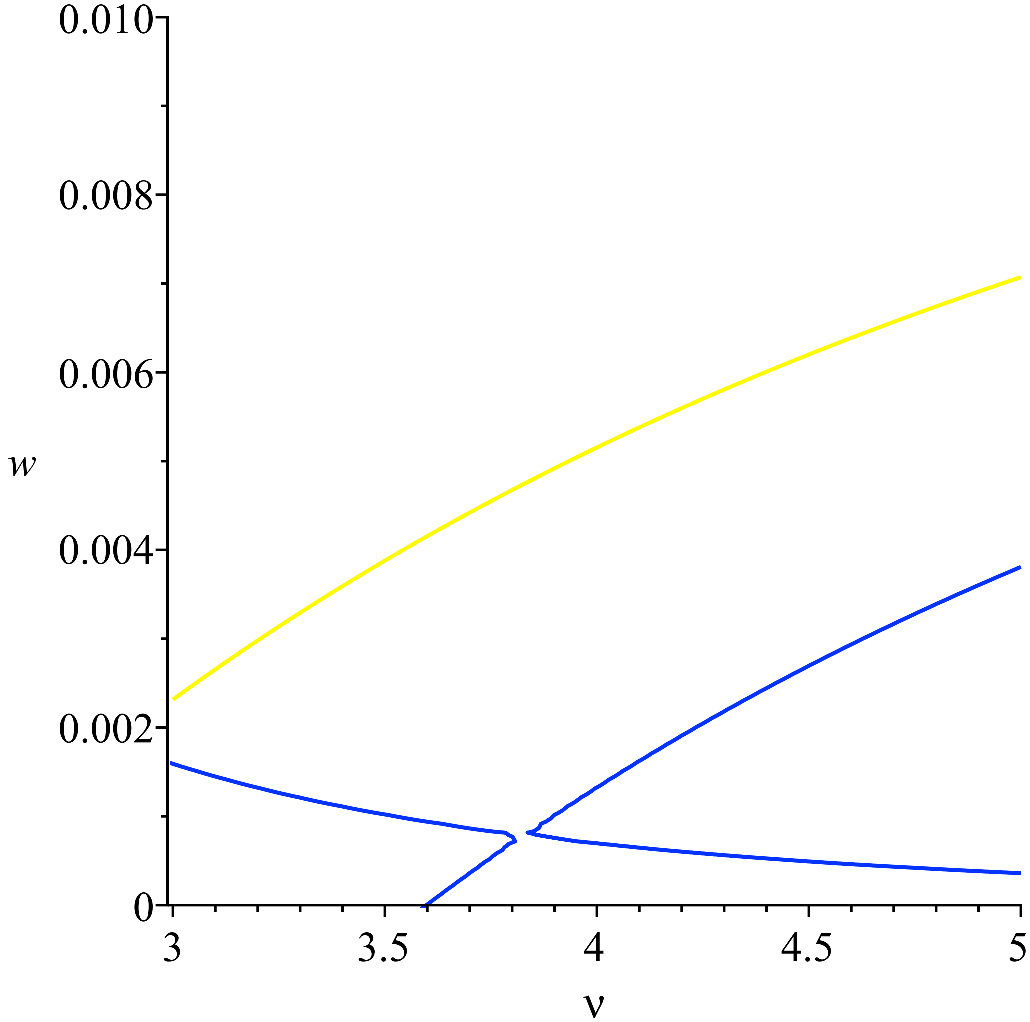
When nu>3, P2 has two imaginary roots, we check that their modulus is not a root of P1 and therefore are not rho_nu

> $w2mod := factor\left(simplify\left(\frac{subs(w=0, P2)}{coeff(P2, w, 2)}\right)\right);$

(1.4.1.1)

$$w2mod := \frac{(-2+\nu)^2 (7\nu^2 - 14\nu - 9)}{27648\,\nu^4} \quad \textbf{(1.4.1.1)}$$

> *plotw2mod* := *plot*(sqrt(*w2mod*), nu = 3 ..5, *color* = *yellow*) :
> *plots*[*display*]({*plotrho1*, *plotw2mod*}, *view* = [3 ..5, 0 ..0.01]);



The modulus w2mod is increasing after nu=3 and is too large

> *factor*(*diff*(*w2mod*, nu)); *evalf*(*solve*(%));

$$\frac{(-2+\nu)(7\nu^2 - 11\nu - 12)}{4608\,\nu^5}$$

$$2.,\ 2.312682738,\ -0.7412541663 \quad \textbf{(1.4.1.2)}$$

Root w22 for nu < nu_c

The radius of convergence of U is w21 for nu =< nu_c and w22 is negative. We check if w21 = -w22 for these values of nu:

> *factor*(*simplify*(*w21* + *w22*)); *solve*(%); *evalf*(%); *evalf*(ν_c);

$$-\frac{(\nu-1)\left(\nu^2-2\nu-1\right)}{32\,\nu^3}$$

$$1,\ 1+\sqrt{2},\ 1-\sqrt{2}$$

$$1.,\ 2.414213562,\ -0.414213562$$

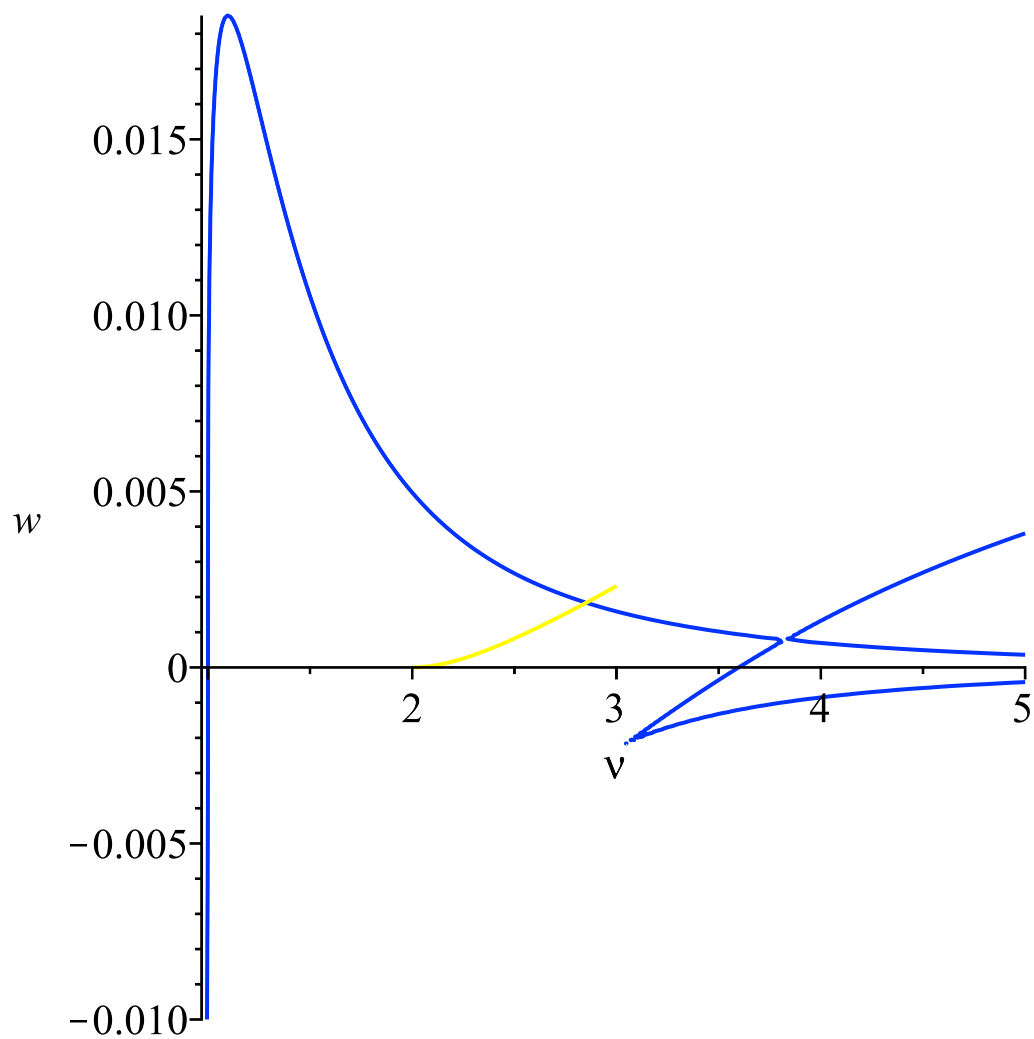
$$1.377964473 \qquad \textbf{(1.4.2.1)}$$

Only possibility is nu=1, which is not a problem since it corresponds to uniform triangulations, and the result is known to be true in this case.

Root w22 for nu_c < nu < 3

The radius of convergence of U is a root of algrho for nu > nu_c and w22 is negative. We check if w22 = -rho for these values of nu:

```
> plots[display]({plotrho1, plot(-w22, nu = 2..3, color = yellow)});
```



There is a candidate !

```
> with(algcurves) : puiseux(algU, w = w22, U, 0);
```

$$\left\{\frac{\sqrt{-(3\nu+3)(\nu-3)}+3\nu+3}{6\nu+6}\right. \tag{1.4.3.1}$$

$$+ 1\Big/\Big(21\nu^4 - 45\nu^2 - 6\nu$$

$$+ 18\Big)$$

$$\Bigg(\Bigg(\Bigg(\Bigg(w$$

$$- \frac{1}{576\nu^3}\Big(-(9\nu-9)\left(\nu^2-2\nu-1\right) - (\nu+1)(3$$

$$- \nu)\sqrt{-(3\nu+3)(\nu-3)}\Big)\Bigg)\left(21\nu^4-45\nu^2-6\nu+18\right)\Bigg)\Big/\Big($$

$$-8\sqrt{-(3\nu+3)(\nu-3)}\,\nu^3+72\nu^4-72\nu^3\Big)^{1/2}\quad\Big($$

$$-8\sqrt{-(3\nu+3)(\nu-3)}\,\nu^3+72\nu^4-72\nu^3\Big)\Bigg), RootOf\Big(\Big(48\nu^2+96\nu$$

$$+48\Big)\_Z^3+\Big(16\sqrt{-3(\nu+1)(\nu-3)}\,\nu-18\nu^2$$

$$+16\sqrt{-3(\nu+1)(\nu-3)}-144\nu-126\Big)\_Z^2$$

$$+\Big(2\sqrt{-3(\nu+1)(\nu-3)}\,\nu-18\nu^2-34\sqrt{-3(\nu+1)(\nu-3)}+72\nu$$

$$+90\Big)\_Z-5\sqrt{-3(\nu+1)(\nu-3)}\,\nu+3\nu^2+13\sqrt{-3(\nu+1)(\nu-3)}$$

$+ 6\nu - 33)\}$

Only one branch is singular and the value at w=w22 is

> $Uw22sing := \frac{3\nu + 3 + \sqrt{3}\sqrt{-(\nu+1)(\nu-3)}}{6\nu+6}$;

$$Uw22sing := \frac{3\nu + 3 + \sqrt{3}\sqrt{-(\nu+1)(\nu-3)}}{6\nu+6} \quad (1.4.3.2)$$

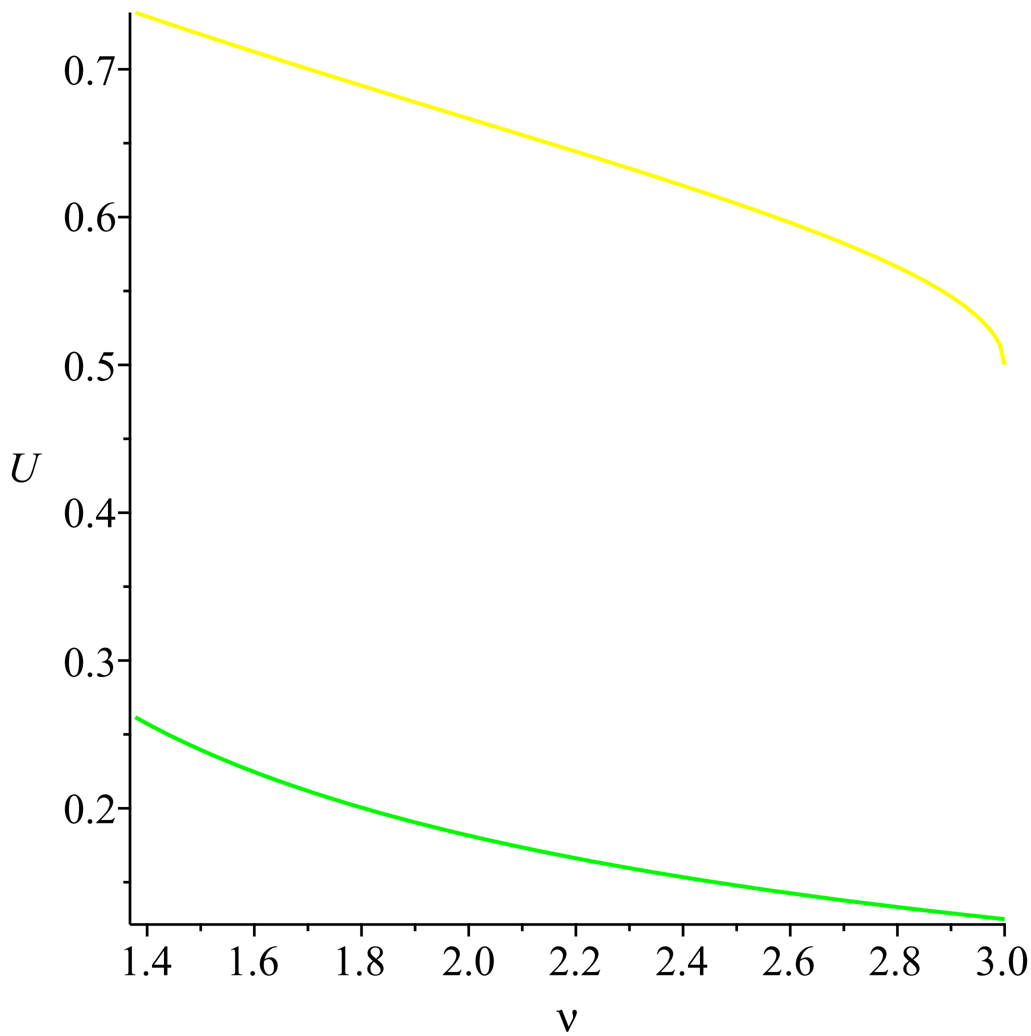
We have to compare this with U(rho_c) who is solution of eqUrho3

>

> $plotUrho3 := implicitplot(eqUrho3, \mathrm{nu} = \nu_c..3, U = 0..0.5, color = green)$:

> $plotUw22sing := plot(Uw22sing, \mathrm{nu} = \nu_c..3, color = yellow)$:

> $plots[display](\{plotUrho3, plotUw22sing\})$;

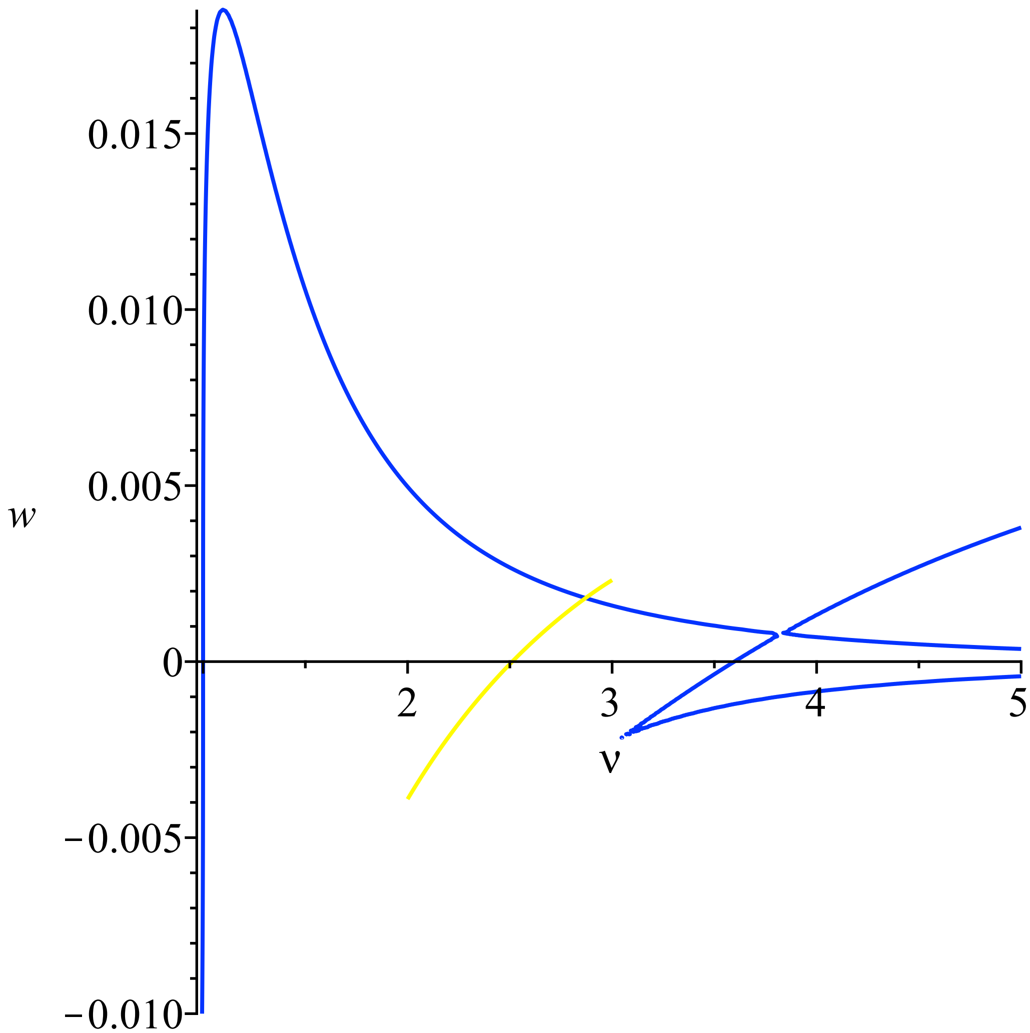


This is not possible, since w22<0 we should have U(w22)<U(rho_nu)

Root w21 for nu_c < nu < 3

This is a lot like the previous case: the radius of convergence of U is a root of algrho for nu > nu_c and w21 can be negative. We check if w21 = -rho for these values of nu:

> *plots*[*display*]({*plotrho1*, *plot*(-*w21*, nu = 2 ..3, *color* = *yellow*)});



There is also a candidate.

> *puiseux*(*algU*, *w* = *w21*, *U*, 0);

$$\left\{ \frac{-\sqrt{-(3\nu+3)(\nu-3)}+3\nu+3}{6\nu+6} \right. \qquad \textbf{(1.4.4.1)}$$

$$+ 1 \Big/ \left(21\,\nu^4 - 45\,\nu^2 - 6\,\nu\right.$$

$+ 18)$

$$\Bigg(\Big(\big(\big(\big(w$$

$$- \frac{1}{576\,\nu^3}\big(-(9\,\nu-9)\,(\nu^2-2\,\nu-1)+(\nu+1)\,(3$$

$$-\nu)\,\sqrt{-(3\,\nu+3)\,(\nu-3)}\,\big)\Big)\,(21\,\nu^4-45\,\nu^2-6\,\nu+18)\Big)\Big/$$

$$\left(8\,\sqrt{-(3\,\nu+3)\,(\nu-3)}\,\nu^3+72\,\nu^4-72\,\nu^3\right)\Big)$$

$$^{1/2}$$

$$\left(8\,\sqrt{-(3\,\nu+3)\,(\nu-3)}\,\nu^3+72\,\nu^4-72\,\nu^3\right)\Bigg), \mathit{RootOf}\big((48\,\nu^2+96\,\nu$$

$$+48)\,\_Z^3+\big(-16\,\sqrt{-3\,(\nu+1)\,(\nu-3)}\,\nu-18\,\nu^2$$

$$-16\,\sqrt{-3\,(\nu+1)\,(\nu-3)}-144\,\nu-126\big)\,\_Z^2+\big($$

$$-2\,\sqrt{-3\,(\nu+1)\,(\nu-3)}\,\nu-18\,\nu^2+34\,\sqrt{-3\,(\nu+1)\,(\nu-3)}+72\,\nu$$

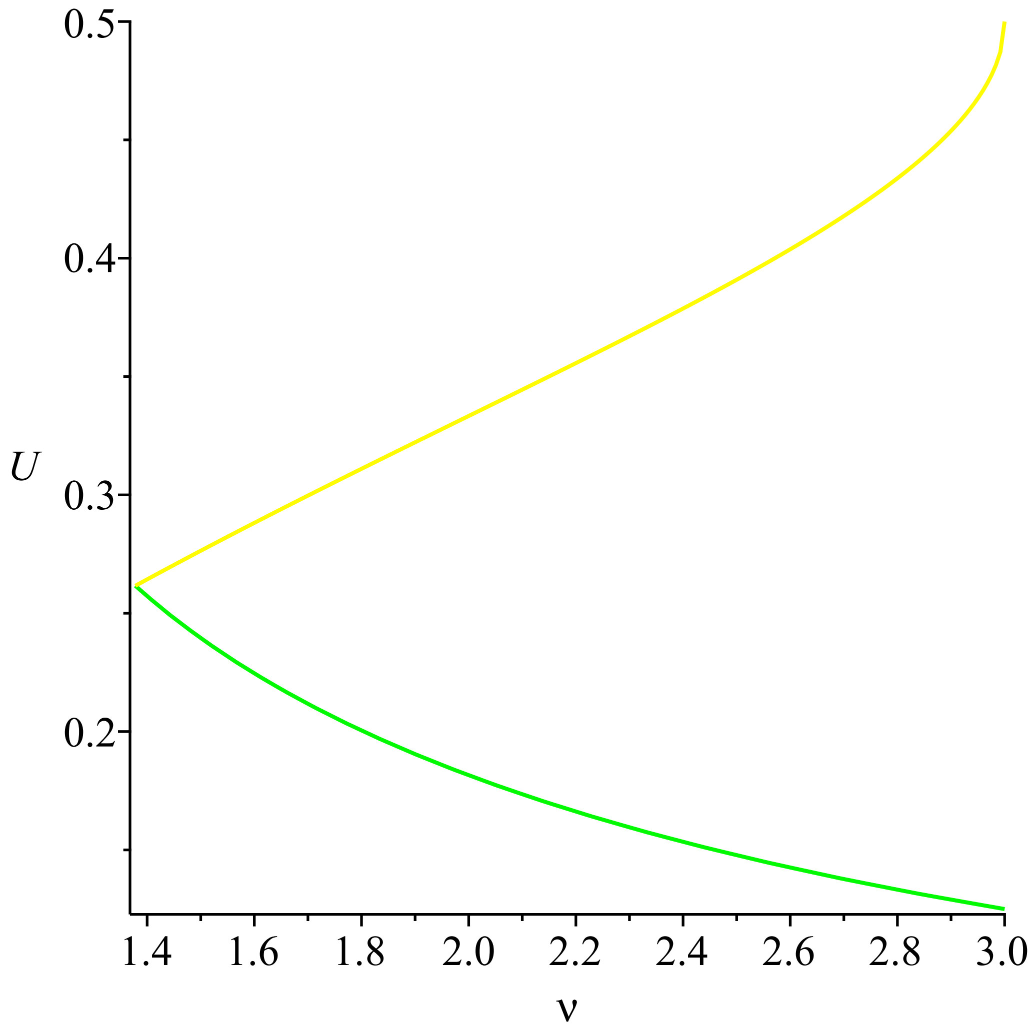
$$+90\big)\,\_Z+5\,\sqrt{-3\,(\nu+1)\,(\nu-3)}\,\nu+3\,\nu^2-13\,\sqrt{-3\,(\nu+1)\,(\nu-3)}$$

$$+6\,\nu-33\big)\}$$

$$> \mathit{Uw21sing} := \frac{3+3\,\nu-\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}}{6\,\nu+6}:$$

$$> \mathit{plotUw21sing} := \mathit{plot}\big(\mathit{Uw21sing}, \mathrm{nu}=\nu_c..3, \mathit{color}=\mathit{yellow}\big):$$

```
> plots[display]({plotUrho3, plotUw21sing});
```



Same impossibility as before, except at nu_c where w21=rho.

>

Real roots of P1 for nu >=3

We check when rho and -rho are roots of P1:

```
> factor(resultant(P1, subs(w=-w, P1), w)); solve(%); evalf(%);
```

$$1099511627776\,\nu^{27}\,\left(4\,\nu^2-8\,\nu-23\right)\left(81\,\nu^4-324\,\nu^3-602\,\nu^2+1852\,\nu+2449\right)^2(\nu-1)^3$$

$$1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1+\frac{3\sqrt{3}}{2}, 1-\frac{3\sqrt{3}}{2}, 1-\frac{2\sqrt{136-10\sqrt{10}}}{9}, 1+\frac{2\sqrt{136-10\sqrt{10}}}{9}, 1-\frac{2\sqrt{136+10\sqrt{10}}}{9}, 1$$

$$+\frac{2\sqrt{136+10\sqrt{10}}}{9},\ 1-\frac{2\sqrt{136-10\sqrt{10}}}{9},\ 1$$

$$+\frac{2\sqrt{136-10\sqrt{10}}}{9},\ 1-\frac{2\sqrt{136+10\sqrt{10}}}{9},\ 1+\frac{2\sqrt{136+10\sqrt{10}}}{9}$$

1., 1., 1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 3.598076212, −1.598076212, −1.270337153, 3.270337153, −1.877093669, 3.877093669, −1.270337153, 3.270337153, −1.877093669, 3.877093669 **(1.4.5.1)**

We have three possible values for nu:

> $nu1 := 1+\frac{3}{2}\sqrt{3}$: $evalf(\%)$; $nu2 := 1+\frac{2}{9}\sqrt{136-10\sqrt{10}}$: $evalf(\%)$;
> $nu3 := 1+\frac{2}{9}\sqrt{136+10\sqrt{10}}$: $evalf(\%)$;

3.598076212

3.270337153

3.877093669 **(1.4.5.2)**

First value is when one of the roots of P1 is 0, which is not singular for U

> $evalf(solve(simplify(subs(\mathrm{nu}=nu1, P1))))$;

0., 0.0009428090128, −0.001208340163 **(1.4.5.3)**

nu3 does not work either:

> $evalf(solve(simplify(subs(\mathrm{nu}=nu3, P1))))$;

0.0009447149241, −0.0009447149241, 0.000759232603 **(1.4.5.4)**

When do the roots of P1 meet (to know if nu3 is before or after)

> $factor(discrim(P1, w))$; $fsolve(\%)$;

$$82556485632\,\nu^{18}\,(\nu-1)^2\,(\nu^2-2\,\nu-7)^2\,(\nu+1)^3\,(\nu-3)^3$$

−1.828427125, −1.828427125, −1., −1., −1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 3., 3., 3., 3.828427125, 3.828427125 **(1.4.5.5)**

nu3 is after, so rho_nu3 is the smallest positive root of P1, .000759... and the negative root is outside the circle of convergence

>

>

>

For nu2 we have to check by hand since a negative root of P1 is on the circle of convergence

> $factor(simplify(subs(\mathrm{nu}=nu2, P1)))$; $fsolve(\%)$;

$$\Big(\big(340668182080\sqrt{136-10\sqrt{10}}\,\sqrt{10}-1888752594322\sqrt{136-10\sqrt{10}}$$
$$+4743730980000\sqrt{10}-23288005045449\big)$$
$$\big(672918721623116460\sqrt{136-10\sqrt{10}}\,\sqrt{10}$$

$$+ 22091879585772674334822 4\, w^2$$

$$+ 2711575949339267856 \sqrt{136 - 10\sqrt{10}} - 6645649759606551105 \sqrt{10}$$

$$- 28768640509499485500\Big) \Big(1378055795 \sqrt{136 - 10\sqrt{10}}\, \sqrt{10}$$

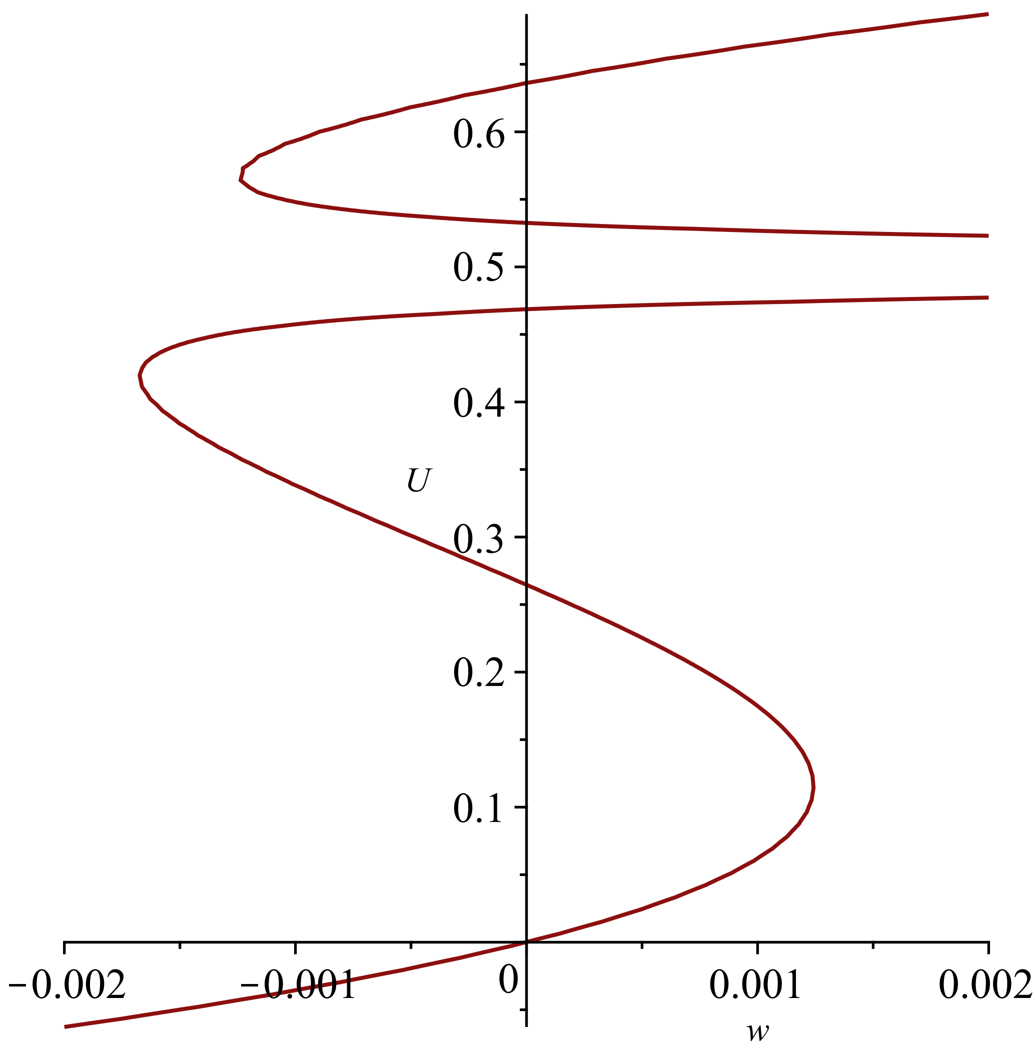
$$+ 3065912636 \sqrt{136 - 10\sqrt{10}} - 16806994560 \sqrt{10} - 5640239942016\, w$$

$$- 32140229112\Big)\Big) \Big/ 36830100657738663418262152583755330558 83$$

$$-0.001674427933, -0.001242162516, 0.001242162516 \qquad \textbf{(1.4.5.6)}$$

We could try to check the singular behavior at -0.0012... with puiseux or algeqtoseries but Maple does not handle it well ... Instead we can see that our branch of U is not singular directly:

> *implicitplot*(*factor*(*subs*(nu = *nu2*, *algU*)), *w* = -0.002 ..0.002, *U* = -0.5 ..1, *numpoints* = 10000);



At w=-0.0012 there is a double root for U but its modulus is too large to be our branch. The other roots are simple and not singular.

> *factor*(*subs*(nu = *nu2*, *eqUrho*)); *fsolve*(%);

$$-\frac{1}{2179240250625}\Big(64\Big(1340550\sqrt{136-10\sqrt{10}}\sqrt{10} - 13094217\sqrt{136-10\sqrt{10}} + 21225290\sqrt{10} - 157590473\Big)\Big(\sqrt{136-10\sqrt{10}}\sqrt{10} + 540\,U - 54\sqrt{10} - 35\sqrt{136-10\sqrt{10}} + 270\Big)(-1+2\,U)\Big(2\sqrt{136-10\sqrt{10}}\sqrt{10} - 270\,U^2 + 11\sqrt{136-10\sqrt{10}} - 18\sqrt{10} + 270\,U - 189\Big)\Big(190\,U\sqrt{136-10\sqrt{10}}\sqrt{10} - 347\sqrt{136-10\sqrt{10}}\sqrt{10} + 640\,U\sqrt{136-10\sqrt{10}} - 2160\,U\sqrt{10} - 5400\,U^2 - 1220\sqrt{136-10\sqrt{10}} + 3618\sqrt{10} - 2160\,U + 11880\Big)\Big)$$

0.1154879305, 0.4185807983, 0.5000000000, 0.5671915934 **(1.4.5.7)**

Puiseux (and algeqtoseries) mishandle approximations :

> $puiseux\Big(subs(\mathrm{nu} = nu2, algU), w = -\frac{3}{470019995168}\Big(319651561216609500 - 747687468470129 40\sqrt{136-10\sqrt{10}}\sqrt{10} + 738405528845172345\sqrt{10} - 301286216593251984\sqrt{136-10\sqrt{10}}\Big)^{1/2}, U, 0\Big) : evalf(allvalues(\%));$

$$\{0.3584510760 + 0.0001152166492\,\mathrm{I}, 0.5671915904 + 2.21141033\sqrt{0.4522001125\,w + 0.0005617060241}\}, \{0.4519238598 - 0.0000934506\,\mathrm{I}, 0.5671915904 + 2.21141033\sqrt{0.4522001125\,w + 0.0005617060241}\}, \{-0.0428676330 - 0.00002176594915\,\mathrm{I}, 0.5671915904 + 2.21141033\sqrt{0.4522001125\,w + 0.0005617060241}\}$$ **(1.4.5.8)**

> $puiseux\Big(subs(\mathrm{nu} = nu2, algU), w = \frac{3}{470019995168}\Big(319651561216609500 - 74768746847012940\sqrt{136-10\sqrt{10}}\sqrt{10} + 738405528845172345\sqrt{10} - 301286216593251984\sqrt{136-10\sqrt{10}}\Big)^{1/2}, U, 0\Big) : evalf(allvalues(\%));$

(1.4.5.9)

$$\{0.4881285348 - 0.008985460244\,I, 0.1154879334 - 12.99194739\sqrt{-0.07697075504\,w + 0.00009561018582}\}, \{0.5085661962 + 0.01009404887\,I, 0.1154879334 - 12.99194739\sqrt{-0.07697075504\,w + 0.00009561018582}\}, \{0.6742198859 - 0.001108588617\,I, 0.1154879334 - 12.99194739\sqrt{-0.07697075504\,w + 0.00009561018582}\} \quad \textbf{(1.4.5.9)}$$

> *with*(*gfun*) :

> $algeqtoseries\Big(factor\Big(simplify\Big(subs\Big(\mathrm{nu} = nu2, w = -\frac{3}{470019995168}\big(3196515612166609500 - 74768746847012940\sqrt{136 - 10\sqrt{10}}\sqrt{10} + 738405528845172345\sqrt{10} - 301286216593251984\sqrt{136 - 10\sqrt{10}}\big)^{1/2} \cdot (1 - x), algU\Big)\Big)\Big), x, U, 2\Big) : evalf(allvalues(\%));$

$$[0.3584510760 + 0.0001152166492\,I + (-0.1089496895 + 0.000134093385\,I)\,x + O(x^2), 0.5671915900 + 0.05241117254\sqrt{x} + O(x)], [0.4519238598 - 0.0000934506\,I + (0.03302134033 - 0.00017511338\,I)\,x + O(x^2), 0.5671915900 + 0.05241117254\sqrt{x} + O(x)], [-0.0428676330 - 0.00002176594915\,I + (0.03567788714 + 0.000041020070\,I)\,x + O(x^2), 0.5671915900 + 0.05241117254\sqrt{x} + O(x)], [0.3584510760 + 0.0001152166492\,I + (-0.1089496895 + 0.000134093385\,I)\,x + O(x^2), 0.5671915900 - 0.05241117254\sqrt{x} + O(x)], [0.4519238598 - 0.0000934506\,I + (0.03302134033 - 0.00017511338\,I)\,x + O(x^2), 0.5671915900 - 0.05241117254\sqrt{x} + O(x)], [-0.0428676330 - 0.00002176594915\,I + (0.03567788714 + 0.000041020070\,I)\,x + O(x^2), 0.5671915900 - 0.05241117254\sqrt{x} + O(x)] \quad \textbf{(1.4.5.10)}$$

> $algeqtoseries\Big(factor\Big(simplify\Big(subs\Big(\mathrm{nu} = nu2, w = \frac{3}{470019995168}\big(3196515612166609500 - 74768746847012940\sqrt{136 - 10\sqrt{10}}\sqrt{10}$

$$+ 738405528845172345\sqrt{10} - 301286216593251984\sqrt{136-10\sqrt{10}}\Big)^{1/2}$$

$$\cdot(1-x), algU\Big)\Big)\Big)\Big], x, U, 2\Big) : evalf(allvalues(\%));$$

$$[0.4881285348 - 0.008985460244\,I + (-0.00074145229 - 0.002082456881\,I)\,x + O(x^2), 0.1154879334 + 0.1270358602\sqrt{x} + O(x)], [0.5085661962 + 0.01009404887\,I + (0.0031474438 + 0.001457363226\,I)\,x + O(x^2), 0.1154879334 + 0.1270358602\sqrt{x} + O(x)], [0.6742198859 - 0.001108588617\,I + (-0.0318124358 + 0.000625093629\,I)\,x + O(x^2), 0.1154879334 + 0.1270358602\sqrt{x} + O(x)], [0.4881285348 - 0.008985460244\,I + (-0.00074145229 - 0.002082456881\,I)\,x + O(x^2), 0.1154879334 - 0.1270358602\sqrt{x} + O(x)], [0.5085661962 + 0.01009404887\,I + (0.0031474438 + 0.001457363226\,I)\,x + O(x^2), 0.1154879334 - 0.1270358602\sqrt{x} + O(x)], [0.6742198859 - 0.001108588617\,I + (-0.0318124358 + 0.000625093629\,I)\,x + O(x^2), 0.1154879334 - 0.1270358602\sqrt{x} + O(x)] \quad \textbf{(1.4.5.11)}$$

Complex roots of P1 for nu_c =< nu =< 3

the product of the roots is rho^3 (since rho is a root of P1 in this domain):

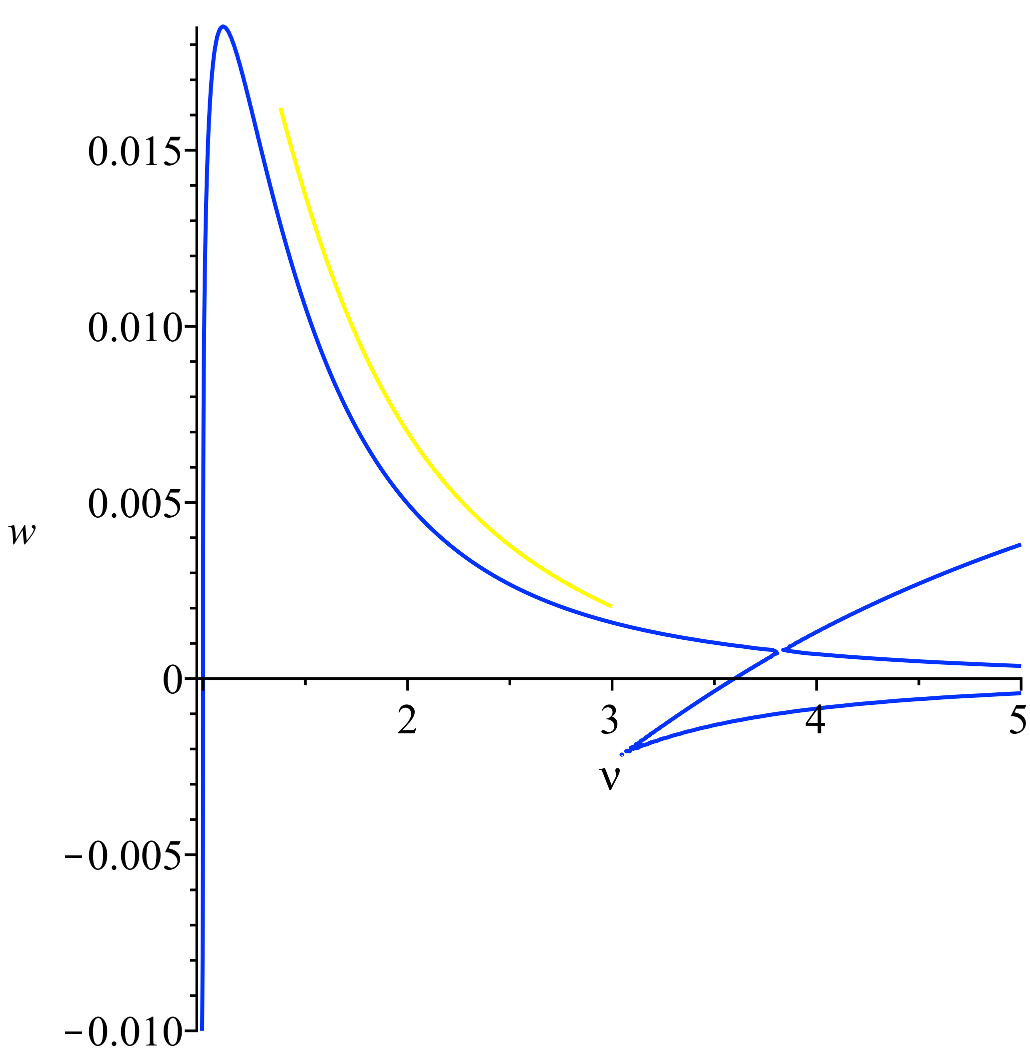
$$> w3mod := factor\left(-\frac{(subs(w=0, P1))}{coeff(P1, w, 3)}\right);$$

$$w3mod := -\frac{(\nu-1)\,(4\nu^2-8\nu-23)}{131072\,\nu^9} \quad \textbf{(1.4.6.1)}$$

$$> plotw3mod := plot\left((w3mod)^{\frac{1}{3}}, \text{nu} = \nu_c..3, color = yellow\right) :$$

In this range of nu P1 does not have 3 roots with the same modulus:

$$> plots[display](\{plotw3mod, plotrho1\});$$



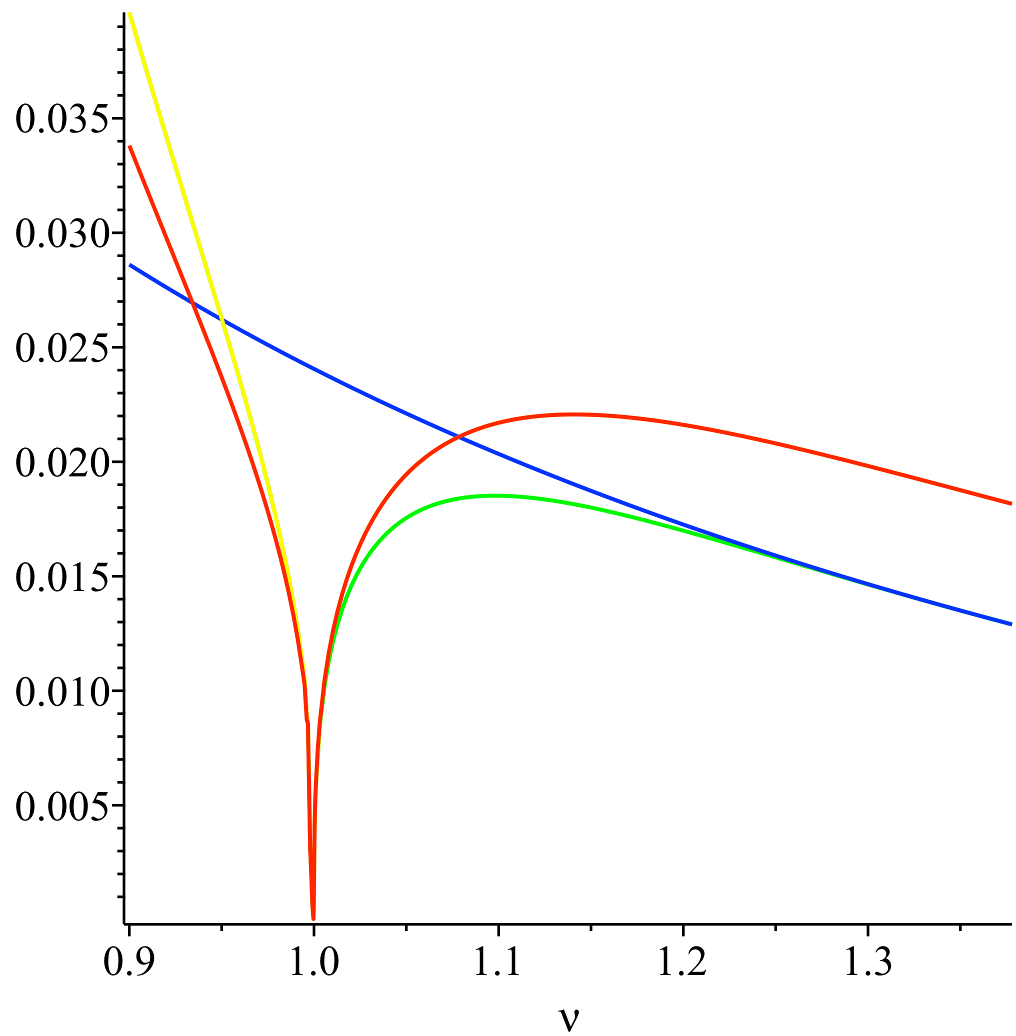
Roots of P1 for nu<nu_c

> $w11, w12, w13 := solve(P1, w)$:

> $Pl1 := plot\left(|w11|, \mathrm{nu} = 0.9\,..\nu_c, color = green\right): Pl2 := plot\left(|w12|, \mathrm{nu} = 0.9\,..\nu_c, color = red\right): Pl3 := plot\left(|w13|, \mathrm{nu} = 0.9\,..\nu_c, color = yellow\right):$

> $PlT := plot\left(w21, \mathrm{nu} = 0.9\,..\nu_c, color = blue\right)$:

> $plots[display](\{PlT, Pl1, Pl2, Pl3\});$



We have 3 candidates for nu ! (the green curve meets the blue one at nu_c. Maple cannot handle the expressions of w1i correctly so we will check that they are never singular before nu_c with Newton polygon method.

Important : 0 is a triple root at nu=1

```
> subs(nu = 1, P1);
```

$$131072\, w^3 \quad \textbf{(1.4.7.1)}$$

We have to write an algebraic equation for (w-w1i) and (U-U(w1i))

First, an equation for U(w1i)

```
> eqUw1i := factor(resultant(algU, P1, w));
```

$$eqUw1i := -2048\,\nu^9\,\big(2048\,U^9\,\nu^5 + 10240\,U^9\,\nu^4 - 5376\,U^8\,\nu^5 + 20480\,U^9\,\nu^3 - 34560\,U^8\,\nu^4 + 5472\,U^7\,\nu^5 + 20480\,U^9\,\nu^2 - 84480\,U^8\,\nu^3 + 48480\,U^7\,\nu^4 - 2972\,U^6\,\nu^5 + 10240\,U^9\,\nu - 99840\,U^8\,\nu^2 + 142656\,U^7\,\nu^3 - 35332\,U^6\,\nu^4 + 1428\,U^5\,\nu^5 + 2048\,U^9 - 57600\,U^8\,\nu + 191808\,U^7\,\nu^2 - 127176\,U^6\,\nu^3 + 13548\,U^5\,\nu^4 - 843\,U^4\,\nu^5 - 13056\,U^8 + 122208\,U^7\,\nu - 191480\,U^6\,\nu^2 + 61656\,U^5\,\nu^3 - 2925\,U^4\,\nu^4 + 328\,U^3\,\nu^5 + 30048\,U^7 - 127900\,U^6\,\nu \quad \textbf{(1.4.7.2)}$$

$$+ 105000 U^5 \nu^2 - 11610 U^4 \nu^3 + 1076 U^3 \nu^4 - 48 U^2 \nu^5 - 31236 U^6 + 72084 U^5 \nu - 28470 U^4 \nu^2 - 3760 U^3 \nu^3 - 552 U^2 \nu^4 + 16620 U^5 - 24411 U^4 \nu + 1976 U^3 \nu^2 + 2592 U^2 \nu^3 + 96 U \nu^4 - 5469 U^4 + 7528 U^3 \nu + 360 U^2 \nu^2 - 432 U \nu^3 + 1044 U^3 - 2976 U^2 \nu + 24 U \nu^2 + 16 \nu^3 + 624 U^2 + 864 U \nu - 48 \nu^2 - 552 U - 60 \nu + 92\big) \big(4 U^3 \nu^2 + 8 U^3 \nu - 3 U^2 \nu^2 + 4 U^3 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U - 2\big)^2$$

There are two factors. We will see that the right one is the first before nu_c and the second after nu_c

> *eqUw1i1* := *op*(3, *eqUw1i*) : *eqUw1i2* := *op*(1, *op*(4, *eqUw1i*)) :

they do meet at nu_c

> *factor*(*subs*(nu = ν_c, *eqUw1i1*));

$$-\frac{1}{4921675101}\Big(\big(14966 + 4201\sqrt{7}\big)\big(4199040 U^5 \sqrt{7} + 15116544 U^6 - 9716112 U^4 \sqrt{7} - 47449152 U^5 + 7910136 U^3 \sqrt{7} + 56270052 U^4 - 2959524 U^2 \sqrt{7} - 30304044 U^3 + 649746 U \sqrt{7} + 7914645 U^2 - 108262 \sqrt{7} - 1479366 U + 312872\big)\big(54 U \sqrt{7} - 216 U^2 - 25\sqrt{7} + 189 U - 55\big)\big(9 U - 5 + \sqrt{7}\big)\Big) \quad \textbf{(1.4.7.3)}$$

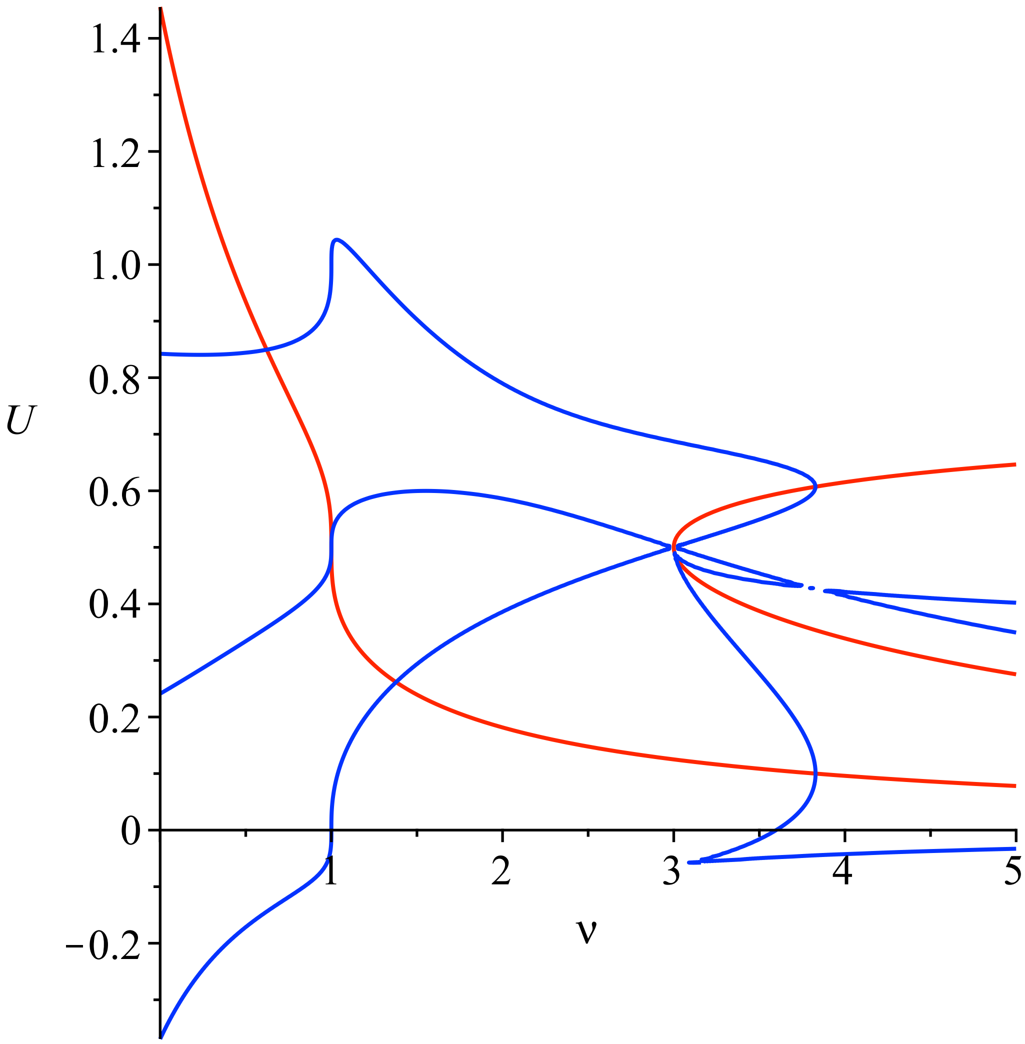
> *factor*(*subs*(nu = ν_c, *eqUw1i2*));

$$-\frac{1}{5103}\Big(\big(29 + 4\sqrt{7}\big)\big(18 U \sqrt{7} - 324 U^2 - 7\sqrt{7} + 315 U - 91\big)\big(9 U - 5 + \sqrt{7}\big)\Big) \quad \textbf{(1.4.7.4)}$$

When else ?

> *aa1* := *implicitplot*(*eqUw1i1*, nu = 0 ..5, *U* = -0.5 ..2, *numpoints* = 100000, *color* = *blue*) : *aa2* := *implicitplot*(*eqUw1i2*, nu = 0 ..5, *U* = -0.5 ..2, *numpoints* = 100000, *color* = *red*) :

> *plots*[*display*]({*aa1*, *aa2*});



> *factor*(*resultant*(*eqUw1i1*, *eqUw1i2*, *U*)); *solve*(%); *evalf*(%);

$$-47775744\,(7\nu^2-14\nu+6)\,(\nu^2-2\nu-7)^2\,(\nu-1)^3\,(\nu-3)^7\,(\nu+1)^{13}$$

$$1, 1, 1, 3, 3, 3, 3, 3, 3, 3, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 1+2\sqrt{2}, 1-2\sqrt{2}, 1+2\sqrt{2}, 1-2\sqrt{2}, 1+\frac{\sqrt{7}}{7}, 1-\frac{\sqrt{7}}{7}$$

1., 1., 1., 3., 3., 3., 3., 3., 3., 3., −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., 3.828427124, −1.828427124, 3.828427124, −1.828427124, 1.377964473, 0.6220355269 **(1.4.7.5)**

Before nu_c, there is 1 and 0.622,

When nu = 1 the meeting point is 1/2 but w3i=0

> *factor*(*subs*(nu = 1 , *eqUw1i2*)); *solve*(%); *factor*(*subs*(nu = 1, *eqUw1i1*)); *solve*(%);

$$2\,(-1+2\,U)^3$$

$$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$$

$$8192\, U^3\, (-1+2\,U)^3\, (U-1)^3$$

$$0, 0, 0, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \quad \textbf{(1.4.7.6)}$$

Meeting point for nu=0.622

> $factor\left(subs\left(\mathrm{nu}=1-\frac{1}{7}\sqrt{7}, eqUw1i2\right)\right); fsolve(\%); factor\left(subs\left(\mathrm{nu}=1-\frac{1}{7}\sqrt{7}, eqUw1i1\right)\right); fsolve(\%);$

$$\frac{1}{5103}\left(\left(-29+4\sqrt{7}\right)\left(18\,U\sqrt{7}+324\,U^2-7\sqrt{7}-315\,U+91\right)\left(-9\,U+5+\sqrt{7}\right)\right)$$

$$0.8495279234$$

$$-\frac{1}{4921675101}\left(\left(-14966+4201\sqrt{7}\right)\left(4199040\,U^5\sqrt{7}-15116544\,U^6-9716112\,U^4\sqrt{7}+47449152\,U^5+7910136\,U^3\sqrt{7}-56270052\,U^4-2959524\,U^2\sqrt{7}+30304044\,U^3+649746\,U\sqrt{7}-7914645\,U^2-108262\sqrt{7}+1479366\,U-312872\right)\left(54\,U\sqrt{7}+216\,U^2-25\sqrt{7}-189\,U+55\right)\left(-9\,U+5+\sqrt{7}\right)\right)$$

$$-0.1442045455, 0.3577667178, 0.8495279234 \quad \textbf{(1.4.7.7)}$$

> $resultant\left(\left(18\,U\sqrt{7}+324\,U^2-7\sqrt{7}-315\,U+91\right), \left(4199040\,U^5\sqrt{7}-15116544\,U^6-9716112\,U^4\sqrt{7}+47449152\,U^5+7910136\,U^3\sqrt{7}-56270052\,U^4-2959524\,U^2\sqrt{7}+30304044\,U^3+649746\,U\sqrt{7}-7914645\,U^2-108262\sqrt{7}+1479366\,U-312872\right)\left(54\,U\sqrt{7}+216\,U^2-25\sqrt{7}-189\,U+55\right), U\right);$

$$\left(1416180860939914471987200\sqrt{7}+3823666794178547212350720\right)\left(14883264\sqrt{7}+61136856\right) \quad \textbf{(1.4.7.8)}$$

Value of U(rho_nu)

> $factor\left(subs\left(w=w21, \mathrm{nu}=1-\frac{1}{7}\sqrt{7}, algU\right)\right); fsolve(\%);$

$$\frac{1}{964467}\left(\left(-434+85\sqrt{7}\right)\left(594\,U^2\sqrt{7}-1944\,U^3-630\,U\sqrt{7}+3213\,U^2+182\sqrt{7}-2016\,U+469\right)\left(9\,U-4+\sqrt{7}\right)^2\right)$$

$$0.1504720766, 0.1504720766, 1.278680597 \quad \textbf{(1.4.7.9)}$$

The meeting point is larger than U(rho_c) and corresponds to wrong branches or values of w1i outside the circle of convergence. Therefore, for nu<nu_c and values of w1i inside the disk of convergence, U(w1i) satisfies eqUw1i1 and not eqUw1i2

the factor of degree 3 was also in the caracteristic equation of U(rho)

> $eqUrho3;$

$$4 U^3 \nu^2 + 8 U^3 \nu - 3 U^2 \nu^2 + 4 U^3 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U - 2 \quad \textbf{(1.4.7.10)}$$

> *eqUw1i2*;

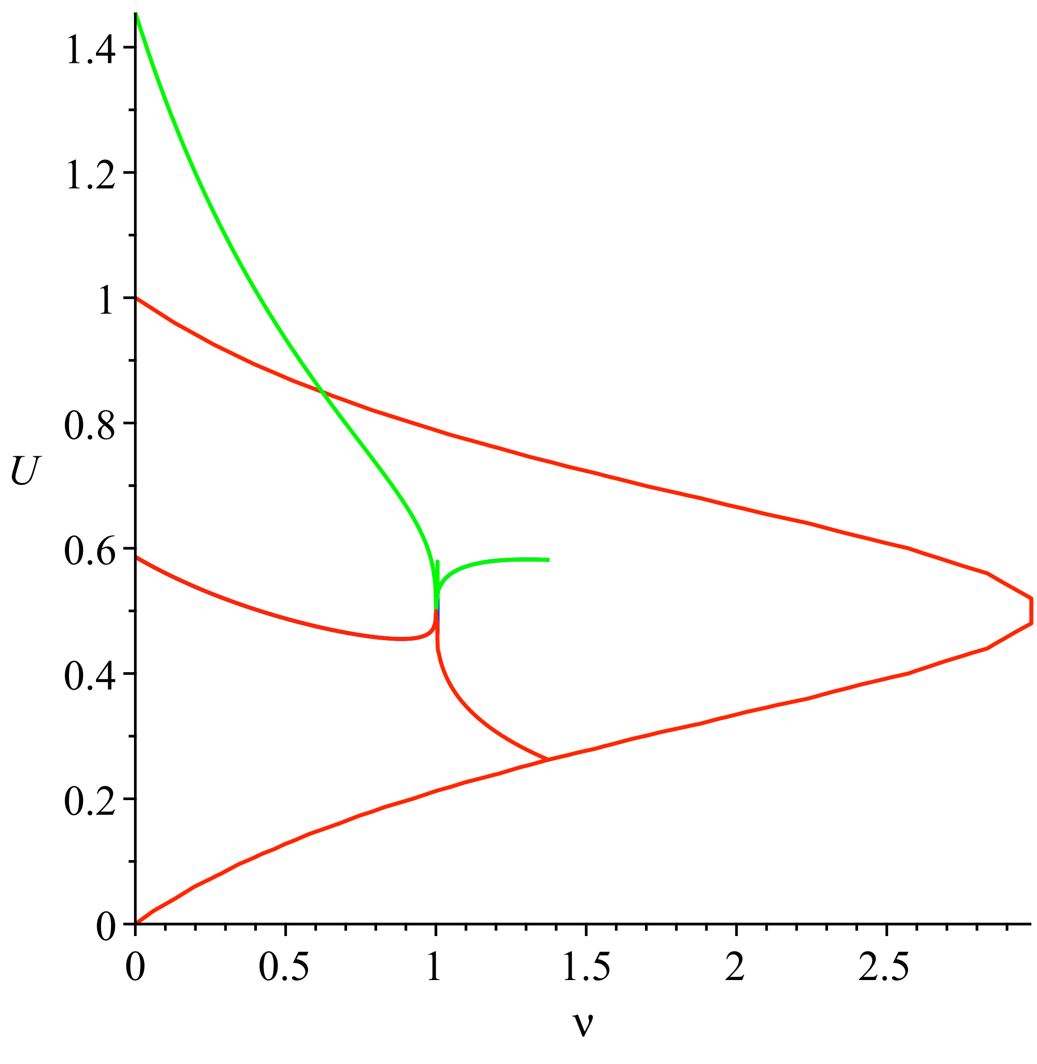
$$4 U^3 \nu^2 + 8 U^3 \nu - 3 U^2 \nu^2 + 4 U^3 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U - 2 \quad \textbf{(1.4.7.11)}$$

For nu < nu_c, we already know that the real root of this equation is larger than U(rho). We also know that if w1i is on the circle of convergence, |U(w1i)| < U(rho), we will see that is is never the case for this factor if nu < nu_c

> $u31, u32, u33 := solve(eqUrho3, U)$:

> $U1 := plot\left(|u31|, \text{nu} = 0..\nu_c, color = green\right): U2 := plot\left(|u32|, \text{nu} = 0..\nu_c, color = red\right): U3 := plot\left(|u33|, \text{nu} = 0..\nu_c, color = blue\right):$

> $plots[display](\{plotUrho2, U1, U2, U3\})$;



EqUrho3 has two complex conjuguate roots :

> $factor(discrim(eqUrho3, U))$;

$$108 (\nu - 3) (\nu - 1)^2 (\nu + 1)^3 \quad \textbf{(1.4.7.12)}$$

Equation for the modulus of these roots:

> $eqmodUw1i := 6 (\text{nu} + 1) \cdot 4 (\text{nu} + 1)^2 \cdot U^4 - \left(4 (\text{nu} + 1)^2\right)^2 U^6 + 4 - 2 \cdot 3 (\text{nu}$

$+ 3) \cdot (\mathrm{nu} + 1)\, U^2;$

$$eqmodUw1i := 4\,(6\,\nu + 6)\,(\nu + 1)^2\,U^4 - 16\,(\nu + 1)^4\,U^6 + 4 - 6\,(\nu + 3)\,(\nu + 1)\,U^2 \quad \textbf{(1.4.7.13)}$$

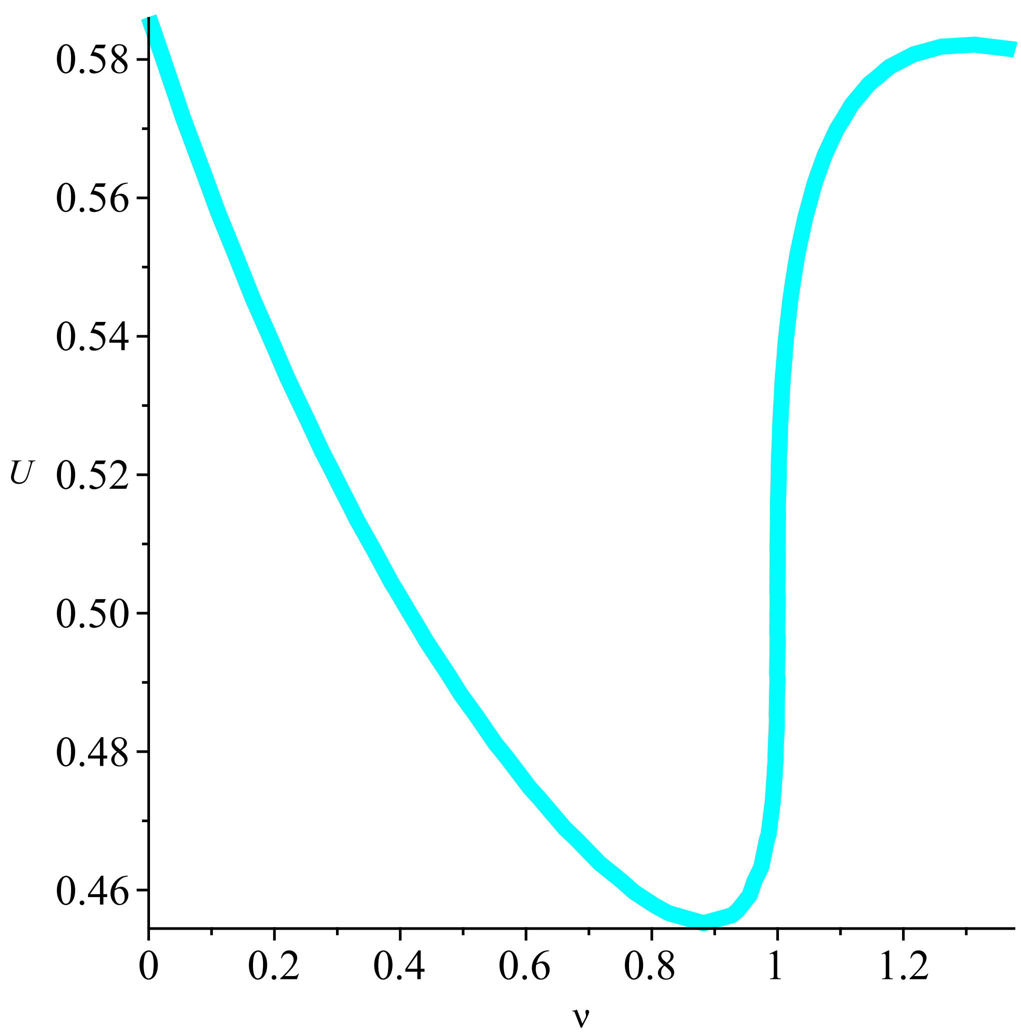
When can the modulus become smaller than U(rho) : never !

> *factor*(*resultant*(*eqmodUw1i*, *eqUrho2*, *U*)); *fsolve*(%);

$$4\,(\nu - 3)\,(64\,\nu^6 - 32\,\nu^5 - 168\,\nu^4 - 396\,\nu^3 + 405\,\nu^2 + 1458)\,(\nu + 1)^7$$

$$-1., -1., -1., -1., -1., -1., -1., 3. \quad \textbf{(1.4.7.14)}$$

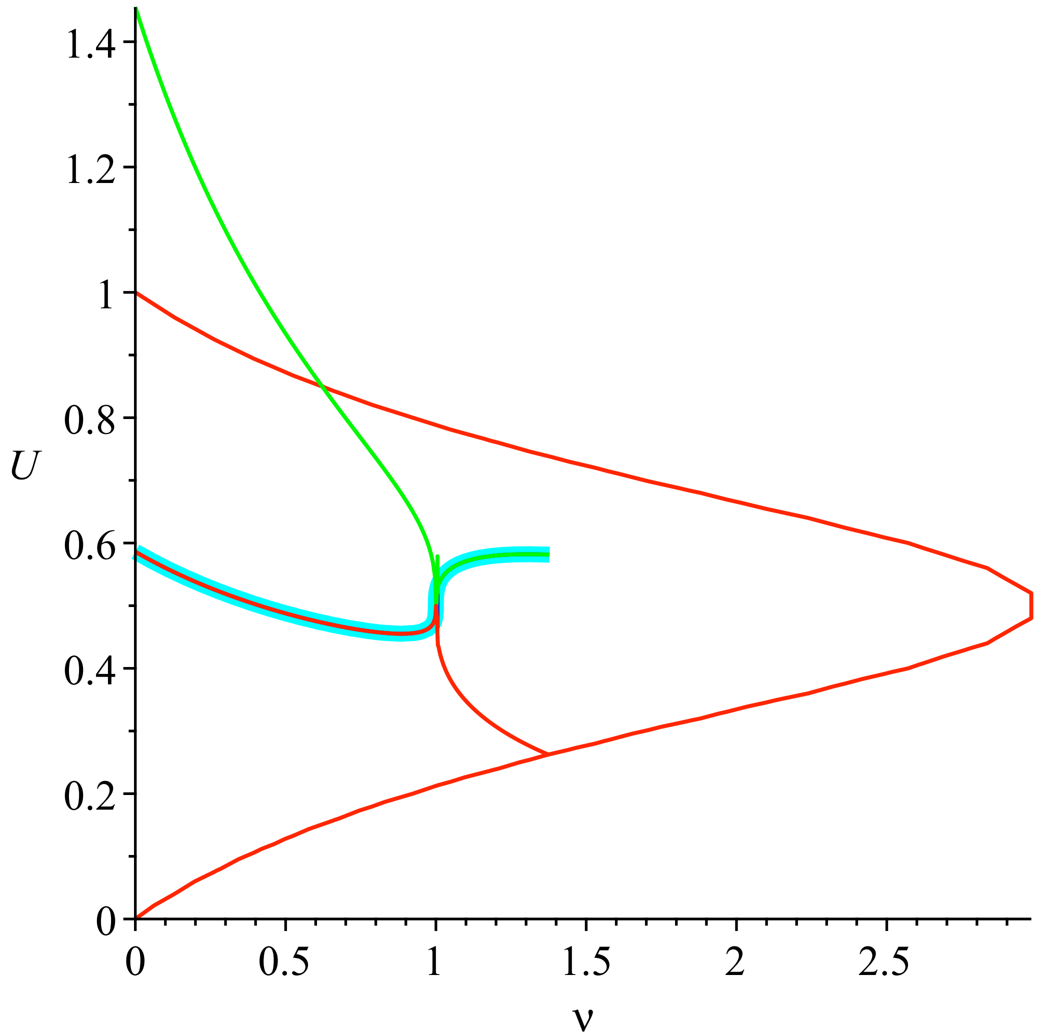
> $MU := implicitplot\left(eqmodUw1i, \mathrm{nu} = 0..\nu_c, U = 0..2, color = cyan, thickness = 6\right);$



> *factor*(*subs*(nu = 1, *eqmodUw1i*));

$$-4\,(-1 + 2\,U)^3\,(2\,U + 1)^3 \quad \textbf{(1.4.7.15)}$$

> *plots*[*display*]({*plotUrho2*, *U1*, *U2*, *U3*, *MU*});



>

Now we know that U(w1i) is given by the big factor. Starting form *eqUw1i1*, we write an equation satisfied by w and UU with U=U(w1i) - UU (Maple does not factorize it)

$$2048\,U^9\nu^5+10240\,U^9\nu^4-5376\,U^8\nu^5+20480\,U^9\nu^3-34560\,U^8\nu^4+5472\,U^7\nu^5+20480\,U^9\nu^2-84480\,U^8\nu^3+48480\,U^7\nu^4-2972\,U^6\nu^5+10240\,U^9\nu-99840\,U^8\nu^2+142656\,U^7\nu^3-35332\,U^6\nu^4+1428\,U^5\nu^5+2048\,U^9-57600\,U^8\nu+191808\,U^7\nu^2-127176\,U^6\nu^3+13548\,U^5\nu^4-843\,U^4\nu^5-13056\,U^8+122208\,U^7\nu-191480\,U^6\nu^2+61656\,U^5\nu^3-2925\,U^4\nu^4+328\,U^3\nu^5+30048\,U^7-127900\,U^6\nu+105000\,U^5\nu^2-11610\,U^4\nu^3+1076\,U^3\nu^4-48\,U^2\nu^5-31236\,U^6+72084\,U^5\nu-28470\,U^4\nu^2-3760\,U^3\nu^3-552\,U^2\nu^4+16620\,U^5-24411\,U^4\nu+1976\,U^3\nu^2+2592\,U^2\nu^3+96\,U\nu^4-5469\,U^4+7528\,U^3\nu+360\,U^2\nu^2-432\,U\nu^3+1044\,U^3-2976\,U^2\nu+24\,U\nu^2+16\,\nu^3 \quad \textbf{(1.4.7.16)}$$

$$+\,624\,U^2 + 864\,U\,\nu - 48\,\nu^2 - 552\,U - 60\,\nu + 92$$

```
> eqUUw1i1 := resultant(eqUw1i1, subs(U = U - UU, algU), U) : indets(%);
```

$$\{UU, \nu, w\} \qquad \textbf{(1.4.7.17)}$$

Then we write an equation for UU and WW with w=w1i - WW et U= U(w1i) - UU, w1i being a root of P1 (2 min computing time)

```
> eqWW1UU1 := resultant(P1, subs(w = w - WW, eqUUw1i1), w) :
```

Maple can factorize it by it can take a few minutes !!! (20 on my laptop):

```
> eqWW1UU1 := factor(eqWW1UU1) :
> nops(%);
```

$$5 \qquad \textbf{(1.4.7.18)}$$

```
> op(1, eqWW1UU1); op(2, eqWW1UU1); op(3, eqWW1UU1);
```

$$-4903985730770844346746710486880989387579965190987526963 2$$

$$(\nu + 1)^{45}$$

$$\nu^{81} \qquad \textbf{(1.4.7.19)}$$

```
> subs(UU = 0, WW = 0, op(4, eqWW1UU1));
```

$$0 \qquad \textbf{(1.4.7.20)}$$

```
> factor(subs(UU = 0, WW = 0, op(5, eqWW1UU1))); fsolve(%); degree(op(5,
     eqWW1UU1), WW)
```

$$-1310067672419160639406 08\,(\nu^2 - 2\,\nu - 7)^6\,(\nu + 1)^9\,(\nu - 1)^{10}\,(\nu - 3)^{25}$$

$-1.828427125, -1.828427125, -1.828427125, -1.828427125, -1.828427125, -1.828427125, -1., -1., -1., -1., -1., -1., -1., -1., -1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3.828427125, 3.828427125, 3.828427125, 3.828427125, 3.828427125, 3.828427125$

$$18 \qquad \textbf{(1.4.7.21)}$$

The first factor is always the right one before nu_c ! We can do Newton's method

```
> eqWW1UU1good := collect(op(4, eqWW1UU1), {WW, UU}, factor) :
     degree(eqWW1UU1good, WW);
```

$$9 \qquad \textbf{(1.4.7.22)}$$

```
> for i from 0 to 9 do
  ldegree(coeff(eqWW1UU1good, WW, i), UU);
  coeff(coeff(eqWW1UU1good, WW, i), UU, %); fsolve(%); od;
```

$$9$$

$$191102976\,(13573\,\nu^4 - 54292\,\nu^3 + 69811\,\nu^2 - 31038\,\nu + 67482)\,(\nu - 1)^2\,(\nu^2 - 2\,\nu - 7)^2\,(7\,\nu^2 - 14\,\nu + 6)^2\,(\nu - 3)^9\,(\nu + 1)^{10}$$

$-1.828427125, -1.828427125, -1., -1., -1., -1., -1., -1., -1., -1., -1., -1., 0.6220355270, 0.6220355270, 1., 1., 1.377964473, 1.377964473, 3., 3., 3., 3., 3., 3., 3., 3., 3., 3.828427125, 3.828427125$

8

$$-1019215872\, \nu^3 \left(49\, \nu^4 - 196\, \nu^3 + 339\, \nu^2 - 286\, \nu + 102\right) \left(13573\, \nu^4 - 54292\, \nu^3 + 69811\, \nu^2 - 31038\, \nu + 67482\right) (\nu - 1)^2 \left(\nu^2 - 2\, \nu - 7\right)^2 (\nu + 1)^9 (\nu - 3)^9$$

−1.828427125, −1.828427125, −1., −1., −1., −1., −1., −1., −1., −1., −1., 0., 0., 0., 1., 1., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3.828427125, 3.828427125

7

$$56623104\, \nu^6 \left(\nu^2 - 2\, \nu - 7\right) \left(15961848\, \nu^{12} - 191542176\, \nu^{11} + 957162619\, \nu^{10} - 2548413070\, \nu^9 + 3210739098\, \nu^8 + 1764577920\, \nu^7 - 10717931194\, \nu^6 + 3195829884\, \nu^5 + 25842319124\, \nu^4 - 35230532064\, \nu^3 + 23728532391\, \nu^2 - 19564752366\, \nu + 14587990002\right) (\nu - 1)^2 (\nu + 1)^7 (\nu - 3)^8$$

−1.828427125, −1., −1., −1., −1., −1., −1., −1., 0., 0., 0., 0., 0., 0., 1., 1., 3., 3., 3., 3., 3., 3., 3., 3., 3.828427125

6

$$-16777216\, \nu^9 \left(\nu^2 - 2\, \nu - 7\right) \left(808177139\, \nu^{12} - 9698125668\, \nu^{11} + 46556200397\, \nu^{10} - 109964062810\, \nu^9 + 159720819568\, \nu^8 - 345499166672\, \nu^7 + 990817163826\, \nu^6 - 1670019200108\, \nu^5 + 1266409702955\, \nu^4 + 38943843492\, \nu^3 - 530185174623\, \nu^2 + 44916452694\, \nu - 414979921710\right) (\nu - 1)^2 (\nu + 1)^5 (\nu - 3)^7$$

−1.828427125, −1., −1., −1., −1., −1., −0.6940748849, 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 2.694074885, 3., 3., 3., 3., 3., 3., 3., 3.828427125

5

$$12582912\, \nu^{12} \left(17823292487\, \nu^{14} - 249526094818\, \nu^{13} + 1407043935773\, \nu^{12} - 3909170298740\, \nu^{11} + 4957978223199\, \nu^{10} - 1396155435454\, \nu^9 - 4219025596163\, \nu^8 + 31721054006056\, \nu^7 - 103825901192355\, \nu^6 + 96699512390594\, \nu^5 + 54116108603583\, \nu^4 - 58253103431028\, \nu^3 - 15238553021955\, \nu^2 - 67969070267490\, \nu + 2300590998567\right) (\nu - 1)^2 (\nu + 1)^3 (\nu - 3)^6$$

−1.800513327, −1., −1., −1., −0.9085016737, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.03356371495, 1., 1., 1.966436285, 2.908501674, 3., 3., 3., 3., 3., 3., 3.800513327

4

$$-12884901888\,\nu^{15}\,(277982901\,\nu^{12} - 3335794812\,\nu^{11} + 16122301322\,\nu^{10} - 38910536780\,\nu^{9} + 36257967195\,\nu^{8} + 56689258248\,\nu^{7} - 220515526388\,\nu^{6} + 263055018888\,\nu^{5} - 116228389445\,\nu^{4} + 53162531700\,\nu^{3} - 81758241270\,\nu^{2} + 12201729732\,\nu + 179748005013)\,(\nu-1)^2\,(\nu+1)^2\,(\nu-3)^6$$

−1.738685063, −1., −1., −0.7154269072, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 2.715426907, 3., 3., 3., 3., 3., 3., 3.738685063

3

$$35184372088832\,\nu^{18}\,(\nu+1)\,(1467508\,\nu^{10} - 14675080\,\nu^{9} + 51504055\,\nu^{8} - 59830520\,\nu^{7} - 68361826\,\nu^{6} + 261632860\,\nu^{5} - 219224808\,\nu^{4} - 105745968\,\nu^{3} + 2509110\,\nu^{2} + 415307412\,\nu + 116574633)\,(\nu-1)^2\,(\nu-3)^6$$

−1.643450035, −1., −0.2823650610, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 2.282365061, 3., 3., 3., 3., 3., 3., 3.643450035

2

$$-108086391056891904\,\nu^{21}\,(4077\,\nu^{8} - 32616\,\nu^{7} + 71231\,\nu^{6} + 29238\,\nu^{5} - 218739\,\nu^{4} + 71204\,\nu^{3} + 137493\,\nu^{2} + 8478\,\nu - 127710)\,(\nu-1)^2\,(\nu-3)^6$$

−1.469961881, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 3., 3., 3., 3., 3., 3., 3.469961881

1

$$138350580552821637 12\,\nu^{24}\,(\nu+1)\,(161\,\nu^{2} - 322\,\nu - 159)\,(\nu-1)^2\,(\nu-3)^8$$

−1., −0.4098147537, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 2.409814754, 3., 3., 3., 3., 3., 3., 3., 3.

0

$$-472236648286964521369 6\,\nu^{27}\,(\nu-1)^2\,(\nu-3)^8$$

0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 3., 3., 3., 3., 3., 3., 3., 3. **(1.4.7.23)**

The behavior is non singular for a generic nu < nu_c, the only possible change is for nu=0.622... where the coef UU^9WW^0 vanishes

> $evalf\left(1 - \frac{1}{7}\sqrt{7}\right);$

$$0.6220355269 \qquad \textbf{(1.4.7.24)}$$

It is the meeting point we saw earlier and we know that it corresponds to vales w1i outside the disk of convergence of U so it does not concerns us

Singular behavior at the radius of convergence

We apply Newton polygon method before and after nu_c

After nu_c

For nu > nu_c, the radius of convergence is a root w3i of P1. Recall the two factors of the equation for U(w1i)

```
> eqUw1i1; eqUw1i2;
```

$$2048 U^9 \nu^5 + 10240 U^9 \nu^4 - 5376 U^8 \nu^5 + 20480 U^9 \nu^3 - 34560 U^8 \nu^4 + 5472 U^7 \nu^5 + 20480 U^9 \nu^2 - 84480 U^8 \nu^3 + 48480 U^7 \nu^4 - 2972 U^6 \nu^5 + 10240 U^9 \nu - 99840 U^8 \nu^2 + 142656 U^7 \nu^3 - 35332 U^6 \nu^4 + 1428 U^5 \nu^5 + 2048 U^9 - 57600 U^8 \nu + 191808 U^7 \nu^2 - 127176 U^6 \nu^3 + 13548 U^5 \nu^4 - 843 U^4 \nu^5 - 13056 U^8 + 122208 U^7 \nu - 191480 U^6 \nu^2 + 61656 U^5 \nu^3 - 2925 U^4 \nu^4 + 328 U^3 \nu^5 + 30048 U^7 - 127900 U^6 \nu + 105000 U^5 \nu^2 - 11610 U^4 \nu^3 + 1076 U^3 \nu^4 - 48 U^2 \nu^5 - 31236 U^6 + 72084 U^5 \nu - 28470 U^4 \nu^2 - 3760 U^3 \nu^3 - 552 U^2 \nu^4 + 16620 U^5 - 24411 U^4 \nu + 1976 U^3 \nu^2 + 2592 U^2 \nu^3 + 96 U \nu^4 - 5469 U^4 + 7528 U^3 \nu + 360 U^2 \nu^2 - 432 U \nu^3 + 1044 U^3 - 2976 U^2 \nu + 24 U \nu^2 + 16 \nu^3 + 624 U^2 + 864 U \nu - 48 \nu^2 - 552 U - 60 \nu + 92$$

$$4 U^3 \nu^2 + 8 U^3 \nu - 3 U^2 \nu^2 + 4 U^3 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U - 2 \quad \textbf{(1.5.1.1)}$$

We also have an equation for U(rho_nu):

```
> eqUrho;
```

$$128 (-1 + 2 U) \nu^3 (3 U^2 \nu + 3 U^2 - 3 U \nu - 3 U + \nu) (4 U^3 \nu^2 + 8 U^3 \nu - 3 U^2 \nu^2 + 4 U^3 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U - 2) \quad \textbf{(1.5.1.2)}$$

```
> factor(resultant(eqUw1i1, (-1 + 2 U)·(3 U^2 ν + 3 U^2 - 3 U ν - 3 U + ν), U));
    fsolve(%);
```

$$-128 (\nu - 1) (\nu - 3)^{10} (7 \nu^2 - 14 \nu + 6) (13573 \nu^4 - 54292 \nu^3 + 69811 \nu^2 - 31038 \nu + 67482) (\nu + 1)^7$$

$-1., -1., -1., -1., -1., -1., -1., 0.6220355270, 1., 1.377964473, 3., 3., 3., 3., 3., 3., 3., 3., 3., 3.$ **(1.5.1.3)**

For nu>nu_c, the first branch can be the good one only for nu=3, but in this case U(rho) is also a root of the second factor

```
> factor(subs(nu = 3, eqUw1i1)); factor(subs(nu = 3, eqUw1i2)); factor(subs(nu = 3, eqUrho));
```

$$8\,(-11 + 16\,U)\,(16\,U + 1)^2\,(-1 + 2\,U)^6$$

$$2\,(8\,U - 1)\,(-1 + 2\,U)^2$$

$$20736\,(-1 + 2\,U)^5\,(8\,U - 1) \quad \textbf{(1.5.1.4)}$$

Starting from eqUw1i2, we write an equation for w and UU with U=Uw1i-UU (Maple does not factorize it)

```
> eqUUw1i2 := resultant(eqUw1i2, subs(U = U - UU, algU), U) : indets(%);
```

$$\{UU, \nu, w\} \quad \textbf{(1.5.1.5)}$$

Then an equation for UU and WW with w=w1i - WW

```
> eqWW1iUU2 := resultant(P1, subs(w = w - WW, eqUUw1i2), w) :
> eqWW1iUU2 := factor(eqWW1iUU2) :
> nops(%);
```

$$5 \quad \textbf{(1.5.1.6)}$$

There are two factors

```
> op(1, eqWW1iUU2); op(2, eqWW1iUU2); op(3, eqWW1iUU2)
```

$$-562949953421312$$

$$(\nu + 1)^{18}$$

$$\nu^{27} \quad \textbf{(1.5.1.7)}$$

```
> eqWW1iUU21 := collect(op(4, eqWW1iUU2), {UU, WW}, factor) :
  eqWW1iUU22 := collect(op(5, eqWW1iUU2), {UU, WW}, factor) :
> subs(UU = 0, WW = 0, eqWW1iUU21); subs(UU = 0, WW = 0, eqWW1iUU22);
```

$$0$$

$$-5038848\,(\nu^2 - 2\,\nu - 7)^2\,(\nu + 1)^3\,(\nu - 1)^6\,(\nu - 3)^7 \quad \textbf{(1.5.1.8)}$$

The first factor is the good one

```
> degree(eqWW1iUU21, WW); degree(eqWW1iUU22, WW);
```

$$3$$

$$6 \quad \textbf{(1.5.1.9)}$$

```
> for i from 0 to 3 do
  ldegree(coeff(eqWW1iUU21, WW, i), UU);
  coeff(coeff(eqWW1iUU21, WW, i), UU, %); od;
```

$$6$$

$$432\,(\nu - 1)\,(7\,\nu^2 - 14\,\nu + 6)\,(\nu - 3)^2\,(\nu + 1)^6$$

$$4$$

$$-20736\,\nu^3\,(\nu - 1)^2\,(\nu - 3)^2\,(\nu + 1)^4$$

$$2$$

$$-64512\,\nu^6\,(\nu+1)^2\,(\nu-3)^2\,(\nu-1)^3$$

$$0$$

$$-131072\,\nu^9\,(\nu-1)^2\,(\nu-3)^2 \qquad \textbf{(1.5.1.10)}$$

For a generic nu we have a square root singularity, we need to check 3 and nu_c

```
> eqWW1iUU21badnu3 := factor(subs(nu = 3, eqWW1iUU21)) :
> degree(eqWW1iUU21badnu3, WW);
```

$$3 \qquad \textbf{(1.5.1.11)}$$

```
> for i from 0 to 3 do
  ldegree(coeff(eqWW1iUU21badnu3, WW, i), UU);
  coeff(coeff(eqWW1iUU21badnu3, WW, i), UU, %); od;
```

$$10$$

$$13759414272$$

$$8$$

$$165112971264$$

$$6$$

$$-1981355655168$$

$$4$$

$$-23776267862016 \qquad \textbf{(1.5.1.12)}$$

Also a square root singularity

```
>
```

Finally at nu_c

```
> eqWW1iUU21nuc := factor(subs(nu = 1 + sqrt(7)/7, eqWW1iUU21)) :
> degree(eqWW1iUU21nuc, WW);
```

$$3 \qquad \textbf{(1.5.1.13)}$$

```
> for i from 0 to 3 do
  ldegree(coeff(eqWW1iUU21nuc, WW, i), UU);
  coeff(coeff(eqWW1iUU21nuc, WW, i), UU, %); od;
```

$$7$$

$$-\frac{1}{3337453428382706771853981}\Big(\big(12016033849\sqrt{7}+32234505926\big)\big(-629856\sqrt{7}+4566456\big)\big(-82281568762008+25407896603040\sqrt{7}\big)\Big)$$

$$4$$

$$-\frac{1}{3337453428382706771853981}\Big(\big(12016033849\sqrt{7}+32234505926\big)\big(-160832\sqrt{7}-1831616\big)\big(-82281568762008+25407896603040\sqrt{7}\big)\Big)$$

$$2$$

$$-\frac{1}{3337453428382706771853981}\Big(\big(12016033849\sqrt{7}+32234505926\big)\big($$

$$-160832\sqrt{7} - 1831616)\,(3944498814144 - 4587623780544\sqrt{7}))$$

$$0$$

$$-\frac{1}{3337453428382706771853981}\left((12016033849\sqrt{7} + 32234505926)\,(-160832\sqrt{7} - 1831616)\,(-2356659716096\sqrt{7} - 14143542788096)\right) \quad \textbf{(1.5.1.14)}$$

We have a possible 1/3 singularity to check (see later)

>

Before nu_c

For nu < nu_c, the radius of convergence is w21 the root of P2.

We have to write an algebraic equation for (w-w21) and (U-U(w21))

First, an equation for U(w2i)

```
> eqUw2i := factor(resultant(algU, P2, w));
```

$$eqUw2i := 1024\,\nu^4\,(192\,U^6\,\nu^4 + 768\,U^6\,\nu^3 - 144\,U^5\,\nu^4 + 1152\,U^6\,\nu^2 - 1440\,U^5\,\nu^3 - 53\,U^4\,\nu^4 + 768\,U^6\,\nu - 3456\,U^5\,\nu^2 + 720\,U^4\,\nu^3 + 94\,U^3\,\nu^4 + 192\,U^6 - 3168\,U^5\,\nu + 3450\,U^4\,\nu^2 + 24\,U^3\,\nu^3 - 21\,U^2\,\nu^4 - 1008\,U^5 + 4528\,U^4\,\nu - 1572\,U^3\,\nu^2 - 198\,U^2\,\nu^3 - 14\,U\,\nu^4 + 1851\,U^4 - 2840\,U^3\,\nu + 504\,U^2\,\nu^2 + 126\,U\,\nu^3 + 7\,\nu^4 - 1338\,U^3 + 870\,U^2\,\nu - 186\,U\,\nu^2 - 42\,\nu^3 + 189\,U^2 - 158\,U\,\nu + 75\,\nu^2 + 168\,U - 20\,\nu - 36)\,(3\,U^2\,\nu + 3\,U^2 - 3\,U\,\nu - 3\,U + \nu)^2 \quad \textbf{(1.5.2.1)}$$

```
> eqUrho;
```

$$128\,(-1 + 2\,U)\,\nu^3\,(3\,U^2\,\nu + 3\,U^2 - 3\,U\,\nu - 3\,U + \nu)\,(4\,U^3\,\nu^2 + 8\,U^3\,\nu - 3\,U^2\,\nu^2 + 4\,U^3 - 12\,U^2\,\nu - 9\,U^2 + 6\,U\,\nu + 6\,U - 2) \quad \textbf{(1.5.2.2)}$$

```
> factor(resultant((192 U^6 ν^4 + 768 U^6 ν^3 − 144 U^5 ν^4 + 1152 U^6 ν^2 − 1440 U^5 ν^3
    − 53 U^4 ν^4 + 768 U^6 ν − 3456 U^5 ν^2 + 720 U^4 ν^3 + 94 U^3 ν^4 + 192 U^6
    − 3168 U^5 ν + 3450 U^4 ν^2 + 24 U^3 ν^3 − 21 U^2 ν^4 − 1008 U^5 + 4528 U^4 ν
    − 1572 U^3 ν^2 − 198 U^2 ν^3 − 14 U ν^4 + 1851 U^4 − 2840 U^3 ν + 504 U^2 ν^2
    + 126 U ν^3 + 7 ν^4 − 1338 U^3 + 870 U^2 ν − 186 U ν^2 − 42 ν^3 + 189 U^2
    − 158 U ν + 75 ν^2 + 168 U − 20 ν − 36), (-1 + 2 U) ν^3 (4 U^3 ν^2 + 8 U^3 ν
    − 3 U^2 ν^2 + 4 U^3 − 12 U^2 ν − 9 U^2 + 6 U ν + 6 U − 2), U)); fsolve(%);
```

$$1728\,(\nu - 1)^6\,(\nu - 3)^6\,\nu^{18}\,(7\,\nu^2 - 14\,\nu + 6)\,(13573\,\nu^4 - 54292\,\nu^3 + 69811\,\nu^2 - 31038\,\nu + 67482)\,(\nu + 1)^{10}$$

(1.5.2.3)

$-1., -1., -1., -1., -1., -1., -1., -1., -1., -1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.6220355270, 1., 1., 1., 1., 1., 1., 1.377964473, 3., 3., 3., 3., 3., 3., 3.$ **(1.5.2.3)**

> $factor\left(subs\left(\mathrm{nu} = 1 - \frac{\mathrm{sqrt}(7)}{7}, \left(192\, U^6 \nu^4 + 768\, U^6 \nu^3 - 144\, U^5 \nu^4 + 1152\, U^6 \nu^2 - 1440\, U^5 \nu^3 - 53\, U^4 \nu^4 + 768\, U^6 \nu - 3456\, U^5 \nu^2 + 720\, U^4 \nu^3 + 94\, U^3 \nu^4 + 192\, U^6 - 3168\, U^5 \nu + 3450\, U^4 \nu^2 + 24\, U^3 \nu^3 - 21\, U^2 \nu^4 - 1008\, U^5 + 4528\, U^4 \nu - 1572\, U^3 \nu^2 - 198\, U^2 \nu^3 - 14\, U \nu^4 + 1851\, U^4 - 2840\, U^3 \nu + 504\, U^2 \nu^2 + 126\, U \nu^3 + 7 \nu^4 - 1338\, U^3 + 870\, U^2 \nu - 186\, U \nu^2 - 42 \nu^3 + 189\, U^2 - 158\, U \nu + 75 \nu^2 + 168\, U - 20 \nu - 36\right)\right)\right);$

$$-\frac{1}{964467}\left(\left(-953 + 232\sqrt{7}\right)\left(594\, U^2 \sqrt{7} - 1944\, U^3 - 630\, U \sqrt{7} + 3213\, U^2 + 182\sqrt{7} - 2016\, U + 469\right)\left(54\, U\sqrt{7} + 216\, U^2 - 25\sqrt{7} - 189\, U + 55\right)\left(-9\, U + 5 + \sqrt{7}\right)\right)$$ **(1.5.2.4)**

> $factor\left(subs\left(\mathrm{nu} = 1 - \frac{\mathrm{sqrt}(7)}{7}, 3\, U^2 \nu + 3\, U^2 - 3\, U \nu - 3\, U + \nu\right)\right);$

$$\frac{\left(-14 + \sqrt{7}\right)\left(9\, U - 4 + \sqrt{7}\right)\left(-9\, U + 5 + \sqrt{7}\right)}{189}$$ **(1.5.2.5)**

The rightfactor is always the second one

> $eqUw2i2 := \left(3\, U^2 \nu + 3\, U^2 - 3\, U \nu - 3\, U + \nu\right);$

$$eqUw2i2 := 3\, U^2 \nu + 3\, U^2 - 3\, U \nu - 3\, U + \nu$$ **(1.5.2.6)**

Starting from it, we write an equation for w and UU with U=Uw2i-UU

> $eqUUw2i2 := factor(resultant(eqUw2i2, subs(U = U - UU, algU), U)) :$
> $indets(\%);$

$$\{UU, \nu, w\}$$ **(1.5.2.7)**

Then an equation for UU and WW with w=w2i - WW

> $eqWW2iUU2 := resultant(P2, subs(w = w - WW, eqUUw2i2), w) :$

> $eqWW2iUU2 := factor(eqWW2iUU2) :$

> $nops(\%);$

$$5$$ **(1.5.2.8)**

There are two factors

> $op(1, eqWW2iUU2); op(2, eqWW2iUU2); op(3, eqWW2iUU2)$

$$20639121408$$

$$(\nu + 1)^6$$

$$\nu^8$$ **(1.5.2.9)**

> $eqWW2iUU21 := collect(op(4, eqWW2iUU2), \{UU, WW\}, factor) :$
> $eqWW2iUU22 := collect(op(5, eqWW2iUU2), \{UU, WW\}, factor) :$

> $subs(UU = 0, WW = 0, eqWW2iUU21); subs(UU = 0, WW = 0, eqWW2iUU22);$

$$(\nu+1)^3 (\nu-3)^5$$
$$0 \qquad \textbf{(1.5.2.10)}$$

The first factor is the good one

```
> degree(eqWW2iUU21, WW);
```

$$2 \qquad \textbf{(1.5.2.11)}$$

```
> for i from 0 to 2 do
  ldegree(coeff(eqWW2iUU21, WW, i), UU);
  coeff(coeff(eqWW2iUU21, WW, i), UU, %); od;
```

$$0$$
$$(\nu+1)^3 (\nu-3)^5$$
$$2$$
$$15552\,\nu^3 (\nu-1)(\nu+1)^2 (\nu-3)^2$$
$$0$$
$$27648\,\nu^6 (\nu-3)^2 \qquad \textbf{(1.5.2.12)}$$

Again a generic square root singularity except maybe at 1-sqrt(7)/7 and nu_c

$$> eqWW2iUU21bad := collect\left(factor\left(subs\left(\mathrm{nu} = 1 - \frac{\mathrm{sqrt}(7)}{7}, eqWW2iUU22\right)\right), WW\right):$$

```
> degree(eqWW2iUU21bad, WW);
```

$$2 \qquad \textbf{(1.5.2.13)}$$

```
> for i from 0 to 2 do
  ldegree(coeff(eqWW2iUU21bad, WW, i), UU);
  coeff(coeff(eqWW2iUU21bad, WW, i), UU, %); od;
```

$$5$$
$$\frac{1}{678113317090881}\left(\left(-1284977 + 442192\sqrt{7}\right)\left(813564\sqrt{7} + 1115370\right)\left(-629856\sqrt{7} - 4566456\right)\right)$$
$$2$$
$$\frac{1}{678113317090881}\left(\left(-1284977 + 442192\sqrt{7}\right)\left(813564\sqrt{7} + 1115370\right)\left(-160832\sqrt{7} + 1831616\right)\right)$$
$$0$$
$$\frac{1}{678113317090881}\left(\left(-1284977 + 442192\sqrt{7}\right)\left(160832\sqrt{7} - 1831616\right)\left(-160832\sqrt{7} + 1831616\right)\right) \qquad \textbf{(1.5.2.14)}$$

we have to check:

$$> simplify\left(puiseux\left(subs\left(\mathrm{nu} = 1 - \frac{\mathrm{sqrt}(7)}{7}, algU\right), w = subs\left(\mathrm{nu} = 1 - \frac{\mathrm{sqrt}(7)}{7},\right.\right.\right.$$

$w21\Big), U, 0\Big)\Big);$

$$\left\{ \frac{1}{81\,(-7+\sqrt{7})^2\,(101\sqrt{7}-179)} \left(179\sqrt{7-\sqrt{7}}\,\sqrt{101\sqrt{7}-179}\,\Big(\sqrt{7} - \frac{707}{179}\Big)\sqrt{(704\,w+7)\sqrt{7}-2240\,w} + 473130\sqrt{7} - 1231398\right), \right.$$ **(1.5.2.15)**

$$\left. RootOf\big(1944\,\_Z^3 + (-594\sqrt{7}-3213)\,\_Z^2 + (630\sqrt{7}+2016)\,\_Z - 182\sqrt{7} - 469\big) \right\}$$

A square root singularity

>

Finally at nu_c

> $eqWW2iUU22nuc := collect\left(factor\left(subs\left(\mathrm{nu} = 1 + \frac{\mathrm{sqrt}(7)}{7}, eqWW2iUU22\right)\right), WW\right):$

> $degree(eqWW2iUU22nuc, WW);$

$$2$$ **(1.5.2.16)**

> **for** i **from** 0 **to** 2 **do**
$ldegree(coeff(eqWW2iUU22nuc, WW, i), UU);$
$coeff(coeff(eqWW2iUU22nuc, WW, i), UU, \%);$ **od**;

$$5$$

$$-\frac{1}{678113317090881}\big((1284977+442192\sqrt{7})\,(813564\sqrt{7}-1115370)\,(-629856\sqrt{7}+4566456)\big)$$

$$2$$

$$-\frac{1}{678113317090881}\big((1284977+442192\sqrt{7})\,(813564\sqrt{7}-1115370)\,(-160832\sqrt{7}-1831616)\big)$$

$$0$$

$$-\frac{1}{678113317090881}\big((1284977+442192\sqrt{7})\,(160832\sqrt{7}+1831616)\,(-160832\sqrt{7}-1831616)\big)$$ **(1.5.2.17)**

we have to check to be sure that the singularity is 1/3

At nu_c

> $simplify\left(puiseux\left(subs\left(\mathrm{nu} = 1 + \frac{\mathrm{sqrt}(7)}{7}, algU\right), w = subs\left(\mathrm{nu} = 1 + \frac{\mathrm{sqrt}(7)}{7},\right.\right.\right.$

$w21\Big), U, 0\Big)\Big)$; $evalf(allvalues(\%))$;

$$\left\{\frac{1}{-2282+188\sqrt{7}}\left(-14^{1/3}\left(1235-257\sqrt{7}\right)^{2/3}\left(\left(176w-5\right)\sqrt{7}+560w\right)^{1/3}+358\sqrt{7}-1414\right), RootOf\left(216\_Z^2+\left(-54\sqrt{7}-189\right)\_Z+25\sqrt{7}+55\right)\right\}$$

$$\{0.5970188747, 0.09121213316\,(1025.652231\,w-13.22875656)^{1/3}+0.2615831876\}, \{0.9394189531, 0.09121213316\,(1025.652231\,w-13.22875656)^{1/3}+0.2615831876\} \quad \textbf{(1.5.3.1)}$$

A 1/3 singularity !

> $algeqtoseries\left(subs\left(\text{nu}=1+\frac{\text{sqrt}(7)}{7}, w=subs\left(\text{nu}=1+\frac{\text{sqrt}(7)}{7}, w21\right)\cdot(1-x), algU\right), x, U, 2\right)$;

$$\left[RootOf\left(216\_Z^2+\left(-54\sqrt{7}-189\right)\_Z+25\sqrt{7}+55\right)+\left(\frac{1715}{12672}+\frac{515\sqrt{7}}{19008}-\frac{1415\,RootOf\left(216\_Z^2+\left(-54\sqrt{7}-189\right)\_Z+25\sqrt{7}+55\right)}{4752}\right)x+\mathrm{O}(x^2), \frac{5}{9}-\frac{\sqrt{7}}{9}+RootOf\left(39366\_Z^3+310\sqrt{7}-425\right)x^{1/3}+\mathrm{O}(x^{2/3})\right] \quad \textbf{(1.5.3.2)}$$

>